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Limiti 9

Argomenti: limiti di funzioni e successioni **Difficoltà:** ⋆⋆⋆

Prerequisiti: razionalizzazioni, limiti notevoli

Calcolare i limiti delle seguenti successioni.

	a) Successione	Limite	b) Successione	Limite
1)	$\sqrt{n+1}-\sqrt{n-1}$	0	$n\left(\sqrt{n^2+3}-\sqrt{n^2-2}\right)$	5
2)	$\sqrt{2n+1}-\sqrt{n-1}$	$+\infty$	$\sqrt{n^2+3n+1}-n$	3/2
3)	$\dfrac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{4n+1}-\sqrt{4n-1}}$	4	$\dfrac{\sqrt{4n^2+3}-2n}{\sqrt{9n^2+7}-3n}$	9/14
4)	$\sqrt[3]{n^3+2n^2}-n$	2/3	$\dfrac{\sqrt[3]{n+1}-\sqrt[3]{n-1}}{\sqrt[3]{4n+7}-\sqrt[3]{4n+8}}$	$-4\sqrt[3]{2}$
5)	$\sqrt{n+1}-\sqrt[3]{n-1}$	$+\infty$	$\dfrac{\sqrt{n+1}-\sqrt[3]{n-1}}{\sqrt{4n+7}-\sqrt[3]{4n+7}}$	1/2
6)	$\sqrt[n]{5^n+n^3}-\sqrt[n]{3^n+n^5}$	2	$\left(\sqrt{5^n+n^5}-\sqrt[n]{5^n+n^3}\right)^{1/n}$	$\sqrt{5}$

Calcolare i seguenti limiti di funzione.

	a) Funzione	Limite	b) Funzione	Limite
7)	$\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}$	1/2	$\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+x}-1}{x}$	1/3
8)	$\lim\limits_{x\to 0}\dfrac{\sqrt[4]{1+x^2}-1}{x}$	0	$\lim\limits_{x\to 0}\dfrac{\sqrt[4]{x^2+1}-1}{\sqrt[3]{2x^2+1}-1}$	3/8
9)	$\lim\limits_{x\to 0}\dfrac{\sqrt{1+\sin x}-\cos x}{\tan(3x)}$	1/6	$\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x}-1}{\sqrt[3]{x^2+1}-e^x}$	0
10)	$\lim\limits_{x\to 0}\dfrac{\sqrt{2+x}-\sqrt{2+3x}}{\sqrt{3+x}-\sqrt{3+3x}}$	$\sqrt{3}/\sqrt{2}$	$\lim\limits_{x\to 0}\dfrac{40^x-\sqrt{\cos x}}{\arcsin^4 x+\sin(8x)}$	$\lg 10/8$
11)	$\lim\limits_{x\to+\infty}\sqrt{4^x+3^x}-2^x$	$+\infty$	$\lim\limits_{x\to+\infty}\sqrt{4^x+2^x+x^2}-2^x$	1/2
12)	$\lim\limits_{x\to+\infty}\sqrt[4]{x^2+x^{3/2}+1}-\sqrt{x}$	1/4	$\lim\limits_{x\to-\infty}\sqrt{x^2+x+3}+x$	−1/2
13)	$\lim\limits_{x\to 0^-}\cos x\cdot\cos\dfrac{1}{x}$	N.E.	$\lim\limits_{x\to+\infty}\sin x\cdot\sin\dfrac{1}{x}$	0

1/6

1.a) $$\sqrt{n+1}-\sqrt{n-1} = \left(\sqrt{n+1}-\sqrt{n-1}\right)\frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}} =$$

$$= \frac{n+1-n+1}{\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{\sqrt{n+1}+\sqrt{n-1}} \to 0$$

1.b) $$n\left(\sqrt{n^2+3}-\sqrt{n^2-2}\right) = n\left(\sqrt{n^2+3}-\sqrt{n^2-2}\right)\frac{\sqrt{n^2+3}+\sqrt{n^2-2}}{\sqrt{n^2+3}+\sqrt{n^2-2}} =$$

$$= \frac{n(\cancel{n^2}+3-\cancel{n^2}+2)}{\sqrt{n^2+3}+\sqrt{n^2-2}} = \frac{5\cancel{n}}{\cancel{n}\left(\sqrt{1+3/n^2}+\sqrt{1-2/n^2}\right)} \to 5$$

2.a) $$\sqrt{2n+1}-\sqrt{n-1} = \left(\sqrt{2n+1}-\sqrt{n-1}\right)\frac{\sqrt{2n+1}+\sqrt{n-1}}{\sqrt{2n+1}+\sqrt{n-1}} =$$

$$= \frac{2n+1-n+1}{\sqrt{2n+1}+\sqrt{n-1}} = \frac{n+2}{\sqrt{n}\left(\sqrt{2+1/n}+\sqrt{1-1/n}\right)} \to +\infty$$

2.b) $$\sqrt{n^2+3n+1}-n = \left(\sqrt{n^2+3n+1}-n\right)\frac{\sqrt{n^2+3n+1}+n}{\sqrt{n^2+3n+1}+n} =$$

$$= \frac{\cancel{n^2}+3n+1-\cancel{n^2}}{\sqrt{n^2+3n+1}+n} = \frac{3n+1}{n\left(\sqrt{1+3/n+1/n^2}+1\right)} \to 3/2$$

3.a) $$\frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{4n+1}-\sqrt{4n-1}} =$$

$$= \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{4n+1}-\sqrt{4n-1}}\,\frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}}\,\frac{\sqrt{4n+1}+\sqrt{4n-1}}{\sqrt{4n+1}+\sqrt{4n-1}} =$$

$$= \frac{\cancel{n}+1-\cancel{n}+1}{\cancel{4n}+1-\cancel{4n}+1}\,\frac{\sqrt{4n+1}+\sqrt{4n-1}}{\sqrt{n+1}+\sqrt{n-1}} = \frac{\sqrt{n}\left(\sqrt{4+1/n}+\sqrt{4+1/n}\right)}{\sqrt{n}\left(\sqrt{1+1/n}+\sqrt{1+1/n}\right)} \to 4$$

3.b) $$\frac{\sqrt{4n^2+3}-2n}{\sqrt{9n^2+7}-3n} = \frac{\sqrt{4n^2+3}-2n}{\sqrt{9n^2+7}-3n}\,\frac{\sqrt{4n^2+3}+2n}{\sqrt{4n^2+3}+2n}\,\frac{\sqrt{9n^2+7}+3n}{\sqrt{9n^2+7}+3n} =$$

$$= \frac{\cancel{4n^2}+3-\cancel{4n^2}}{\cancel{9n^2}+7-\cancel{9n^2}}\,\frac{\sqrt{9n^2+7}+3n}{\sqrt{4n^2+3}+2n} = \frac{3}{7}\,\frac{n}{n}\,\frac{\sqrt{9+7/n^2}+3}{\sqrt{4+3/n^2}+2} \to \frac{3}{7}\,\frac{6}{4} = \frac{9}{14}$$

2/6

4.a) $\sqrt[3]{n^3+2n^2} - n = \dfrac{\cancel{n^3}+2n^2-\cancel{n^3}}{\sqrt[3]{(n^3+2n^2)^2}+\sqrt[3]{(n^3+2n^2)\cdot n^3}+\sqrt[3]{n^6}} \to \dfrac{2}{3}$

4.b) $\dfrac{\sqrt[3]{n+1}-\sqrt[3]{n-1}}{\sqrt[3]{4n+7}-\sqrt[3]{4n+8}} = \left[\sqrt[3]{A}-\sqrt[3]{B} = \dfrac{A-B}{\sqrt[3]{A^2}+\sqrt[3]{AB}+\sqrt[3]{B^2}}\right]$

$$= \frac{\cancel{n}+1-\cancel{n}+1}{\sqrt[3]{(n+1)^2}+\sqrt[3]{n^2-1}+\sqrt[3]{(n-1)^2}} \cdot \frac{\sqrt[3]{(4n+7)^2}+\sqrt[3]{(4n+7)(4n+8)}+\sqrt[3]{(4n+8)^2}}{\cancel{4n}+7-\cancel{4n}-8} =$$

$$= -2\,\frac{\sqrt[3]{16n^2}}{\sqrt[3]{n^2}} \cdot \frac{\sqrt[3]{1+\ldots}+\sqrt[3]{1+\ldots}+\sqrt[3]{1+\ldots}}{\sqrt[3]{1+\ldots}+\sqrt[3]{1+\ldots}+\sqrt[3]{1+\ldots}} \to -4\sqrt[3]{2}$$

5.a) $\sqrt{n+1}-\sqrt[3]{n-1} = \sqrt{n}\left(\sqrt{1+1/n}-\dfrac{\sqrt[3]{n-1}}{\sqrt{n}}\right) =$

$$= \sqrt{n}\left(\sqrt{1+1/n}-\sqrt[3]{\frac{n-1}{n^{3/2}}}\right) \to +\infty$$

5.b) $\dfrac{\sqrt{n+1}-\sqrt[3]{n-1}}{\sqrt{4n+7}-\sqrt[3]{4n+7}} = \dfrac{\sqrt{n}}{\sqrt{n}}\;\dfrac{\sqrt{1+1/n}-\sqrt[3]{(n-1)/n^{3/2}}}{\sqrt{4+7/n}-\sqrt[3]{(4n+7)/n^{3/2}}} \to \dfrac{1}{2}$

6.a) $\sqrt[n]{5^n+n^3}-\sqrt[n]{3^n+n^5} = 5\cdot\sqrt[n]{1+n^3/5^n} - 3\,\sqrt[n]{1+n^5/3^n} \to 2$

6.b) $\left(\sqrt{5^n+n^5}-\sqrt[n]{5^n+n^3}\right)^{1/n} = \left(\sqrt{5^n}\right)^{1/n}\left(\sqrt{1+n^5/5^n}-\dfrac{\sqrt[n]{5^n+n^3}}{\sqrt{5^n}}\right)^{1/n} =$

$$= \sqrt{5}\left(\sqrt{1+n^5/5^n}^{\;\to 1}-\sqrt{\frac{5^n+n^3}{5^{n^2/2}}}\right)^{1/n} \to \sqrt{5}$$

7.a) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x} \to \dfrac{1}{2}$

$$\frac{\sqrt{1+x}-1}{x} = \frac{\sqrt{1+x}-1}{x}\;\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1} = \frac{\cancel{1}+\cancel{x}^{\,1}-\cancel{1}}{\cancel{x}(\sqrt{1+x}+1)} \to \frac{1}{2}$$

7.b) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x}-1}{x} \to \dfrac{1}{3}$ $\qquad \left[\sqrt[3]{A}-\sqrt[3]{B} = \dfrac{A-B}{\sqrt[3]{A^2}+\sqrt[3]{AB}+\sqrt[3]{B^2}}\right]$

$$\frac{\sqrt[3]{1+x}-1}{x} = \frac{\cancel{1}+\overset{1}{\cancel{x}}-\cancel{1}}{\cancel{x}\left(\sqrt[3]{(1+x)^2}+\sqrt[3]{1+x}+1\right)} \to \frac{1}{3}$$

8.a) $\lim\limits_{x\to 0} \dfrac{\sqrt[4]{1+x^2}-1}{x} \to 0$

$$\left[A^4-B^4=(A^2+B^2)(A^2-B^2)=(A^2+B^2)(A+B)(A-B) \quad A-B=\frac{A^4-B^4}{(A^2+B^2)(A+B)}\right]$$

$$\frac{\sqrt[4]{1+x^2}-1}{x} = \frac{\cancel{1}+x^{\cancel{2}}-\cancel{1}}{\cancel{x}\left(\sqrt[4]{(1+x^2)^2}+1\right)\left(\sqrt[4]{1+x^2}+1\right)} \to 0$$

8.b) $\lim\limits_{x\to 0} \dfrac{\sqrt[4]{x^2+1}-1}{\sqrt[3]{2x^2+1}-1} \to \dfrac{3}{8}$ $\qquad \dfrac{\sqrt[4]{x^2+1}-1}{\sqrt[3]{2x^2+1}-1} =$

$$= \frac{\cancel{1}+\cancel{x^2}-\cancel{1}}{\left(\sqrt[4]{(1+x^2)^2}+1\right)\left(\sqrt[4]{1+x^2}+1\right)} \cdot \frac{\sqrt[3]{(2x^2+1)^2}+\sqrt[3]{2x^2+1}+1}{2\cancel{x^2}+\cancel{1}-\cancel{1}} \to \frac{3}{8}$$

9.a) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+\sin x}-\cos x}{\tan(3x)} \to \dfrac{1}{6}$

$$\frac{\sqrt{1+\sin x}-\cos x}{\tan(3x)} \cdot \frac{\sqrt{1+\sin x}+\cos x}{\sqrt{1+\sin x}+\cos x} = \frac{1+\sin x-\cos^2 x}{\tan(3x)\left(\sqrt{1+\sin x}+\cos x\right)} =$$

$$= \frac{\sin x+\sin^2 x}{\tan(3x)\left(\sqrt{1+\sin x}+\cos x\right)} = \frac{1}{3}\,\frac{3x}{\tan(3x)}\,\frac{\sin x/x+\sin^2 x/x}{\left(\sqrt{1+\sin x}+\cos x\right)} \to \frac{1}{6}$$

9.b) $\lim\limits_{x\to 0} \dfrac{\sqrt{\cos x}-1}{\sqrt{x^2+1}-e^x} \to 0$

$$\frac{\sqrt{\cos x}-1}{\sqrt{x^2+1}-e^x} = \frac{\sqrt{\cos x}-1}{\sqrt{x^2+1}-e^x}\,\frac{\sqrt{\cos x}+1}{\sqrt{\cos x}+1}\,\frac{\sqrt{x^2+1}+e^x}{\sqrt{x^2+1}+e^x} =$$

$$= \frac{\cos x - 1}{x^2+1-e^{2x}} \, \frac{\sqrt{x^2+1}+e^x}{\sqrt{\cos x}+1} = \frac{\cos x - 1}{x^2} \, \frac{x^2}{x^2+1-e^{2x}} \, \frac{\sqrt{x^2+1}+e^x}{\sqrt{\cos x}+1} =$$

$$= \overset{\to -\frac{1}{2}}{\frac{\cos x - 1}{x^2}} \; \overset{\to 0}{\frac{1}{1 + \frac{1-e^{2x}}{x^2}}} \; \overset{\to 1}{\frac{\sqrt{x^2+1}+e^x}{\sqrt{\cos x}+1}} \to 0$$

10.a) $\lim\limits_{x\to 0} \dfrac{\sqrt{2+x}-\sqrt{2+3x}}{\sqrt{3+x}-\sqrt{3+3x}} \to \dfrac{\sqrt{3}}{\sqrt{2}}$

$$\frac{\sqrt{2+x}-\sqrt{2+3x}}{\sqrt{3+x}-\sqrt{3+3x}} = \overset{\to 1}{\frac{\cancel{2}+x-\cancel{2}-3x}{\cancel{3}+x-\cancel{3}-3x}} \; \overset{\to \sqrt{3}/\sqrt{2}}{\frac{\sqrt{3+x}+\sqrt{3+3x}}{\sqrt{2+x}+\sqrt{2+3x}}} \to \frac{\sqrt{3}}{\sqrt{2}}$$

10.b) $\lim\limits_{x\to 0} \dfrac{50^x-\sqrt{\cos x}}{\arcsin^5 x+\sin 8x} \to \dfrac{\log 10}{8}$

$$\frac{50^x-\sqrt{\cos x}}{\arcsin^5 x+\sin 8x} = \frac{1}{50^x+\sqrt{\cos x}} \; \frac{50^x - \cos x}{\arcsin^5 x+\sin 8x} =$$

$$= \overset{\to 1/2}{\frac{1}{50^x+\sqrt{\cos x}}} \; \frac{\overset{\to \log 50}{\frac{50^x-1}{x}} + \overset{\to 0}{\frac{1-\cos x}{x}}}{\underset{\to 0}{\frac{\arcsin^5 x}{x}} + \underset{\to 8}{\frac{\sin 8x}{x}}} \to \frac{\log 50}{16} = \frac{\lg 10}{8}$$

11.a) $\lim\limits_{x\to +\infty} \sqrt{5^x+3^x}-2^x \to +\infty$

$$\sqrt{5^x+3^x}-2^x = \frac{\cancel{5^x}+3^x-\cancel{5^x}}{\sqrt{5^x+3^x}+2^x} = \frac{3^x}{2^x} \, \frac{1}{\sqrt{1+\left(\frac{3}{5}\right)^x}+1} \to +\infty$$

11.b) $\lim\limits_{x\to +\infty} \sqrt{5^x+2^x+x^2}-2^x \to \dfrac{1}{2}$

$$\sqrt{5^x+2^x+x^2}-2^x = \frac{\cancel{5^x}+2^x+x^2-\cancel{5^x}}{\sqrt{5^x+2^x+x^2}+2^x} = \overset{\to 1}{\frac{2^x}{2^x}} \; \overset{\to 1/2}{\frac{1+\frac{x^2}{2^x}}{\sqrt{1+\left(\frac{1}{2}\right)^x+\frac{x^2}{5^x}}+1}} \to \frac{1}{2}$$

12.a) $\lim\limits_{x \to +\infty} \sqrt[5]{x^2+x^{3/2}+1} - \sqrt{x} \to \frac{1}{5}$ $\qquad \left[A-B = \frac{A^5-B^5}{(A^2+B^2)(A+B)} \right]$

$$\sqrt[5]{x^2+x^{3/2}+1} - \sqrt{x} = \frac{\cancel{x^2}+x^{3/2}+1-\cancel{x^2}}{\left(\sqrt[5]{(x^2+x^{3/2}+1)^2}+\sqrt{x^2}\right)\left(\sqrt[5]{x^2+x^{3/2}+1}+\sqrt{x}\right)} =$$

$$= \frac{x^{3/2}}{x^{3/2}} \frac{1+\frac{1}{x^{3/2}}}{\left(\sqrt[5]{(1+x^{-1/2}+x^{-2})^2}+1\right)\left(\sqrt[5]{1+x^{-1/2}+x^{-2}}+1\right)} \to \frac{1}{5}$$

12.b) $\lim\limits_{x \to -\infty} \sqrt{x^2+x+3} + x = \lim\limits_{x \to +\infty} \sqrt{x^2-x+3} - x \to -\frac{1}{2}$

$$\sqrt{x^2-x+3} - x = \frac{\cancel{x^2}-x+3-\cancel{x^2}}{\sqrt{x^2-x+3}+x} = \frac{x}{x} \frac{-1+3/x}{\sqrt{1-x^{-1}+3/x^2}+1} \to \frac{-1}{2}$$

13.a) $\lim\limits_{x \to 0^-} \cos x \cdot \cos\frac{1}{x}$ N.E. $\qquad a_n = \frac{-1}{2\pi n} \qquad b_n = \frac{-2}{\pi n}$

$$\lim_{x \to 0^-} \cos x \cdot \cos\frac{1}{x} = \lim_{n \to +\infty} \overset{\to 1}{\cos a_n} \cdot \overset{\to 1}{\cos(1/a_n)} \to 1$$

$$= \lim_{n \to +\infty} \overset{\to 1}{\cos b_n} \cdot \overset{\to 0}{\cos(1/b_n)} \to 0$$

13.b) $\lim\limits_{x \to +\infty} \sin x \cdot \sin\frac{1}{x} \to 0$

$$\overset{\to 0}{-\sin\frac{1}{x}} \le \overset{\to 0}{\sin x \cdot \sin\frac{1}{x}} \le \overset{\to 0}{\sin\frac{1}{x}}$$