Forme canoniche 4

Argomenti: forme canoniche di applicazioni lineari **Difficoltà:** ⋆⋆⋆⋆

Prerequisiti: autovalori, autovettori, forme canoniche, matrici di cambio di base

1. Determinare, per ciascuna delle seguenti matrici, la forma canonica reale e quella complessa (se diversa da quella reale). Determinare anche, nel caso reale ed eventualmente nel caso complesso, una possibile matrice di cambio di base che realizza il passaggio alla forma canonica.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 0 & 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 10 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 7 \\ 1 & 2 & 3 & 7 \\ 1 & 2 & 3 & 7 \\ 1 & 2 & 3 & 7 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 13 & 0 \\ 0 & 5 & 0 & -2 \\ 0 & 0 & 5 & 0 \\ 8 & 1 & 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$

2. Consideriamo le seguenti applicazioni lineari $f : \mathbb{R}^4 \to \mathbb{R}^4$:

$$(x, y, z, w) \to (y, z, w, x) \qquad (x, y, z, w) \to (z, w, x, y).$$

   Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice assume la forma canonica.

3. Consideriamo le seguenti applicazioni lineari $f : \mathbb{R}_{\leq 3}[x] \to \mathbb{R}_{\leq 3}[x]$:

$$p(x) \to p(3x), \qquad p(x) \to xp''(x), \qquad p(x) \to p(3), \qquad p(x) \to p(3) \cdot x^3.$$

   Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice assume la forma canonica.

4. Consideriamo le seguenti applicazioni lineari $f : M_{2\times 2} \to M_{2\times 2}$:

$$A \to A^t, \qquad A \to A - A^t, \qquad A \to A + 3A^t, \qquad A \to \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} A.$$

   Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice (che rappresenta f) assume la forma canonica.

5. Consideriamo l'applicazione lineare $f : M_{2\times 2} \to M_{2\times 2}$ che ad ogni matrice 2×2 associa la matrice ottenuta sostituendo ogni elemento con la somma dei due elementi ad esso adiacenti.

   Determinare la forma canonica di f, ed una base in cui la matrice (che rappresenta f) assume la forma canonica.

SOLUZIONI?

BY GIMUSI 13/01/15

1. Determinare, per ciascuna delle seguenti matrici, la forma canonica reale e quella complessa (se diversa da quella reale). Determinare anche, nel caso reale ed eventualmente nel caso complesso, una possibile matrice di cambio di base che realizza il passaggio alla forma canonica.

(a) $\begin{pmatrix}1&0&0&0\\2&1&0&0\\3&1&2&0\\1&0&1&3\end{pmatrix}$ (b) $\begin{pmatrix}1&0&0&1\\1&1&0&0\\1&1&0&1\\1&0&0&1\end{pmatrix}$ (c) $\begin{pmatrix}1&0&1&0\\2&0&2&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ (d) $\begin{pmatrix}1&0&0&0\\2&0&0&2\\0&0&0&-1\\0&0&1&1\end{pmatrix}$

(e) $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&1&1&-1\\1&0&10&3\end{pmatrix}$ (f) $\begin{pmatrix}1&2&3&7\\1&2&3&7\\1&2&3&7\\1&2&3&7\end{pmatrix}$ (g) $\begin{pmatrix}2&0&13&0\\0&5&0&-2\\0&0&5&0\\8&1&0&2\end{pmatrix}$ (h) $\begin{pmatrix}1&2&1&2\\2&1&2&1\\1&2&1&2\\2&1&2&1\end{pmatrix}$

(a) $\begin{pmatrix}1&0&0&0\\2&1&0&0\\3&1&2&0\\1&0&1&3\end{pmatrix}$ $\begin{cases}\lambda_1=\lambda_2=1\\ \lambda_3=2\\ \lambda_4=3\end{cases}$ $\lambda_{1,2}=1 \rightsquigarrow \begin{pmatrix}0&0&0&0\\2&0&0&0\\3&1&1&0\\1&0&1&2\end{pmatrix}X=0$ RANK=3

$\rightsquigarrow x_1=\begin{pmatrix}0\\2\\-2\\1\end{pmatrix}$ MG=1< MA=2 JORDAN!! 1 BLOCCO 2X2 $\begin{pmatrix}1&1&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&3\end{pmatrix} \rightsquigarrow Ax_2=x_1+x_2$
$\equiv (A-I)x_2=x_1 \rightsquigarrow x_2=\begin{pmatrix}1\\-3\\0\\0\end{pmatrix}$

$\lambda_3=2 \rightsquigarrow \begin{pmatrix}-1&0&0&0\\2&-1&0&0\\3&1&0&0\\1&0&1&1\end{pmatrix}X=0$ $x_3=\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}$ $\lambda_4=3 \rightsquigarrow \begin{pmatrix}-2&0&0&0\\2&-2&0&0\\3&1&-1&0\\1&0&1&0\end{pmatrix}X=0$

$x_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \rightsquigarrow M=\begin{pmatrix}0&1&0&0\\2&-3&0&0\\-2&0&-1&0\\1&0&1&1\end{pmatrix}$

(b) $\begin{pmatrix}1&0&0&1\\1&1&0&0\\1&1&0&1\\1&0&0&1\end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix}1-\lambda&0&0&1\\1&1-\lambda&0&0\\1&1&-\lambda&1\\1&0&0&1-\lambda\end{vmatrix} = \begin{cases} =-\lambda\left[(1-\lambda)^3-(1-\lambda)\right]= \\ =-\lambda(1-\lambda)\left[(1-\lambda)^2-1\right]= \\ =-\lambda(1-\lambda)(\lambda^2-2\lambda)=0 \\ \lambda_{1,2}=0 \quad \lambda_3=1 \quad \lambda_4=2\end{cases}$

$$\lambda_{1,2}=0 \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} X=0 \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} X=0 \quad x_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

MG = 1 < MA = 2

JORDANIZZ.

1 BLOCCO 2×2

$$\rightsquigarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \rightsquigarrow A x_2 = x_1 \quad x_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_3=1 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} X=0 \quad x_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda_4=2 \rightsquigarrow \begin{pmatrix} -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} X=0$$

$$x_4 = \begin{pmatrix} 1 \\ 1 \\ 3/2 \\ 1 \end{pmatrix} \rightsquigarrow M = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3/2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad |A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 & 0 \\ 2 & -\lambda & 2 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} = \begin{cases} = (1-\lambda)\left[-\lambda(1-\lambda)^2\right] = \\ = -\lambda(1-\lambda)^3 = 0 \\ \lambda_1 = 0 \quad \lambda_{2,3,4} = 1 \end{cases}$$

$$\lambda_1=0 \rightsquigarrow x_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \lambda_{2,3,4}=1 \rightsquigarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} X=0$$

MG = 2 < MA = 3

JORDANIZZ.

2 BLOCCHI

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{cases} A x_2 = x_2 \\ A x_3 = x_3 \\ A x_4 = x_3 + x_4 \end{cases} \rightsquigarrow \begin{cases} (A-I)x_2 = 0 \\ (A-I)x_3 = 0 \\ (A-I)x_4 = x_3 \end{cases} \quad x_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad x_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad x_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow M = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

(d) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 2 & -\lambda & 0 & 2 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 1 & 1-\lambda \end{vmatrix} = \begin{cases} (1-\lambda)\left[\lambda^2(1-\lambda) - \lambda\right] = \\ = \lambda(1-\lambda)(-\lambda^2+\lambda-1) = 0 \\ \lambda_1 = 0 \quad \lambda_2 = 1 \\ \lambda_{3,4} = \dfrac{1 \pm \sqrt{-3}}{2} = \dfrac{1 \pm i\sqrt{3}}{2} \end{cases}$$

$$\lambda_1 = 0 \rightsquigarrow x_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \lambda_2 = 1 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x = 0 \quad x_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda_3 = \frac{1+i\sqrt{3}}{2} \rightsquigarrow \begin{pmatrix} \frac{1-i\sqrt{3}}{2} & 0 & 0 & 0 \\ 2 & \frac{-1-i\sqrt{3}}{2} & 0 & 2 \\ 0 & 0 & \frac{-1-i\sqrt{3}}{2} & -1 \\ 0 & 0 & 1 & \frac{1-i\sqrt{3}}{2} \end{pmatrix} x = 0 \quad x_3 = \begin{pmatrix} 0 \\ -1+i\sqrt{3} \\ \frac{-1+i\sqrt{3}}{2} \\ -1 \end{pmatrix} \quad x_4 = \bar{x}_3 = \begin{pmatrix} 0 \\ -1-i\sqrt{3} \\ \frac{-1-i\sqrt{3}}{2} \\ -1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{1-i\sqrt{3}}{2} \end{pmatrix} \qquad M_{\mathbb{C}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & -1+i\sqrt{3} & -1-i\sqrt{3} \\ 0 & 0 & \frac{1-i\sqrt{3}}{2} & \frac{1+i\sqrt{3}}{2} \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & \sqrt{3}/2 \\ 0 & 0 & -\sqrt{3}/2 & 1/2 \end{pmatrix} \rightsquigarrow M_{\mathbb{R}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 2 & -1 & \sqrt{3} \\ 0 & 0 & 1/2 & \sqrt{3}/2 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(e) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 10 & 3 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} 1-\lambda & \cancel{0} & \cancel{0} & \cancel{0} \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 1 & 1-\lambda & -1 \\ 1 & 0 & 10 & 3-\lambda \end{vmatrix} = \begin{cases} (1-\lambda)[(1-\lambda)^2(3-\lambda)+10(1-\lambda)] = \\ =(1-\lambda)^2(\lambda^2-4\lambda+13)=0 \\ \lambda_{1,2}=1 \quad \lambda_{3,4}=\dfrac{4\pm\sqrt{16-52}}{2}= \\ =2\pm 3i \end{cases}$

$\lambda_{1,2}=1 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 10 & 2 \end{pmatrix} X=0$

$MG=1 < MA=2$

JORDAN!!

1 BLOCCO 2x2

$\begin{cases} Ax_1=x_1 \\ Ax_2=x_1+x_2 \end{cases}$ $x_1=\begin{pmatrix} 0 \\ 5 \\ -1 \\ 5 \end{pmatrix}$ $x_2=\begin{pmatrix} 5 \\ -1 \\ 0 \\ 0 \end{pmatrix}$

$\lambda_3=2+3i \rightsquigarrow \begin{pmatrix} -1-3i & 0 & 0 & 0 \\ 1 & -1-3i & 0 & 0 \\ 0 & 1 & -1-3i & -1 \\ 1 & 0 & 10 & 1-3i \end{pmatrix} X=0$ $\quad x_3=\begin{pmatrix} 0 \\ 0 \\ -1+3i \\ 10 \end{pmatrix}$ $\quad x_4=\overline{x_3}=\begin{pmatrix} 0 \\ 0 \\ -1-3i \\ 10 \end{pmatrix}$

$\rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2+3i & 0 \\ 0 & 0 & 0 & 2-3i \end{pmatrix}$ $\quad M_{\mathbb{C}}=\begin{pmatrix} 0 & 5 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ -1 & 0 & -1+3i & -1-3i \\ 5 & 0 & 10 & 10 \end{pmatrix}$

$\rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$ $\quad M_{\mathbb{R}}=\begin{pmatrix} 0 & 5 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 \\ 5 & 0 & 10 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & 2 & 3 & 7 \\ 1 & 2 & 3 & 7 \\ 1 & 2 & 3 & 7 \\ 1 & 2 & 3 & 7 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 & 7 \\ 1 & 2-\lambda & 3 & 7 \\ 1 & 2 & 3-\lambda & 7 \\ 1 & 2 & 3 & 7-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & 3 & 7 \\ \lambda & -\lambda & 0 & 0 \\ 0 & \lambda & -\lambda & 0 \\ 0 & 0 & \lambda & -\lambda \end{vmatrix} =$

$=(1-\lambda)[-\lambda^3]-\lambda[2\lambda^2+7\lambda^2+3\lambda^2] = \lambda^3(\lambda-1-12)=\lambda^3(\lambda-13)=0$

$\lambda_{1,2,3}=0 \quad \lambda_4=13$

$\lambda_{1,2,3}=0 \rightsquigarrow x_1=\begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ $\quad x_2=\begin{pmatrix} 0 \\ 3 \\ -2 \\ 0 \end{pmatrix}$ $\quad x_3=\begin{pmatrix} 0 \\ 0 \\ 7 \\ -3 \end{pmatrix}$

$$\lambda_4 = 13 \rightsquigarrow \begin{pmatrix} -12 & 2 & 3 & 7 \\ 1 & -11 & 3 & 7 \\ 1 & 2 & -10 & 7 \\ 1 & 2 & 3 & -6 \end{pmatrix} x = 0 \qquad x_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 13 \end{pmatrix} \qquad M = \begin{pmatrix} 2 & 0 & 0 & 1 \\ -1 & 3 & 0 & 1 \\ 0 & -2 & 7 & 1 \\ 0 & 0 & -3 & 1 \end{pmatrix}$$

(g) $$\begin{pmatrix} 2 & 0 & 13 & 0 \\ 0 & 5 & 0 & -2 \\ 0 & 0 & 5 & 0 \\ 8 & 1 & 0 & 2 \end{pmatrix} \quad |A-\lambda I| = \begin{vmatrix} 2-\lambda & 0 & 13 & 0 \\ 0 & 5-\lambda & 0 & -2 \\ 0 & 0 & 5-\lambda & 0 \\ 8 & 1 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)\left[(5-\lambda)^2(2-\lambda) + \right.$$

$$\left. + 2(5-\lambda)\right] =$$

$$= (2-\lambda)(5-\lambda)(\lambda^2 - 7\lambda + 12) = 0 \quad \lambda_1 = 2 \quad \lambda_2 = 5 \quad \lambda_{3,4} = \frac{7 \pm \sqrt{49-48}}{2} \begin{cases} = 4 \\ = 3 \end{cases}$$

$$\lambda_1 = 2 \rightsquigarrow \begin{pmatrix} 0 & 0 & 13 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 3 & 0 \\ 8 & 1 & 0 & 0 \end{pmatrix} x = 0 \quad x_1 = \begin{pmatrix} 1 \\ -8 \\ 0 \\ -12 \end{pmatrix} \quad \lambda_2 = 5 \rightsquigarrow \begin{pmatrix} -3 & 0 & 13 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 8 & 1 & 0 & -3 \end{pmatrix} \quad x_2 = \begin{pmatrix} 13 \\ -104 \\ 3 \\ 0 \end{pmatrix}$$

$$\lambda_3 = 4 \rightsquigarrow \begin{pmatrix} -2 & 0 & 13 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 8 & 1 & 0 & -2 \end{pmatrix} x = 0 \quad x_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} \quad \lambda_4 = 3 \rightsquigarrow \begin{pmatrix} -1 & 0 & 13 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 2 & 0 \\ 8 & 1 & 0 & -1 \end{pmatrix} x = 0 \quad x_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 13 & 0 & 0 \\ -8 & -104 & 2 & 1 \\ 0 & 3 & 0 & 0 \\ -12 & 0 & 1 & 1 \end{pmatrix}$$

(8) $\begin{pmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 2 & 1 & 2 \\ 2 & 1-\lambda & 2 & 1 \\ 1 & 2 & 1-\lambda & 2 \\ 2 & 1 & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 2 & 1 & 2 \\ 2 & 1-\lambda & 2 & 1 \\ \lambda & 0 & -\lambda & 0 \\ 0 & \lambda & 0 & -\lambda \end{vmatrix} =$

$$= \lambda\left[\lambda - 4\lambda - 4\lambda + \lambda(1-\lambda)\right] - \lambda\left[-\lambda(1-\lambda)^2 + 4\lambda - \lambda(1-\lambda) + 4\lambda\right] =$$

$$= \lambda^2(-7+1-\lambda) - \lambda^2(-1-\lambda^2+2\lambda+8-1+\lambda) = \lambda^2(\lambda^2-4\lambda-12)$$

$$\begin{cases} \lambda_{1,2} = 0 \\ \lambda_{3,4} = \dfrac{4 \pm \sqrt{16+48}}{2} \begin{cases} 6 \\ -2 \end{cases} \end{cases}$$

$\lambda_{1,2}=0 \rightsquigarrow x_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$ $x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_3 = 6 \rightsquigarrow \begin{pmatrix} -5 & 2 & 1 & 2 \\ 2 & -5 & 2 & 1 \\ 1 & 2 & -5 & 2 \\ 2 & 1 & 2 & -5 \end{pmatrix} x = 0$ $x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\lambda_4 = -2 \rightsquigarrow \begin{pmatrix} 3 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 3 \end{pmatrix} x = 0$

$\rightsquigarrow \begin{pmatrix} 3 & 2 & 1 & 2 \\ -1 & 1 & 1 & -1 \\ -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \end{pmatrix} x = 0$ $x_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

OSS x_1, x_2, x_3, x_4
BASE ORTOGONALE
$(A = A^T)$

$\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ $M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{pmatrix}$

2. Consideriamo le seguenti applicazioni lineari $f : \mathbb{R}^4 \to \mathbb{R}^4$:

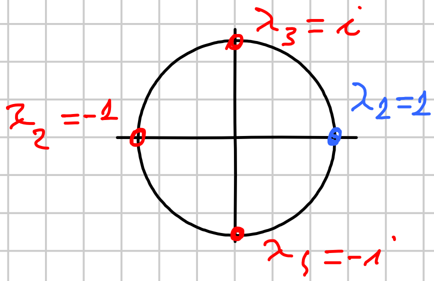
$$(a)\quad (x, y, z, w) \to (y, z, w, x) \qquad (b)\ (x, y, z, w) \to (z, w, x, y).$$

Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice assume la forma canonica.

(a) $(x, y, z, w) \to (y, z, w, x)$ $\qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda(-\lambda^3) - 1 \cdot (1) = \lambda^4 - 1 = 0$$

$\lambda_{1,2} = \pm 1 \quad \lambda_{3,4} = \pm i$



$$\lambda_1 = 1 \rightsquigarrow \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} X = 0 \qquad X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} X = 0 \qquad X_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\lambda_3 = i \rightsquigarrow \begin{pmatrix} -i & 1 & 0 & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -i & 1 \\ 1 & 0 & 0 & -i \end{pmatrix} X = 0 \qquad X_3 = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix} \qquad X_4 = \overline{X_3} = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \rightsquigarrow J_{\mathbb{R}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$M_{\mathbb{C}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{pmatrix} \rightsquigarrow M_{\mathbb{R}} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

(b) $(x,y,z,w) \rightsquigarrow (z,w,x,y)$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{vmatrix} = -\lambda\left[-\lambda^3+\lambda\right] + 1\left[1-\lambda^2\right] =$$

$$\lambda^4 - \lambda^2 + 1 - \lambda^2 = \lambda^4 - 2\lambda^2 + 1 =$$

$$(\lambda^2-1)^2 = (\lambda-1)^2(\lambda+1)^2 = 0$$

$$\lambda_{1,2} = 1 \rightsquigarrow \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} X = 0 \quad x_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_{3,4} = -1 \rightsquigarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} X = 0 \quad x_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \quad x_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

3. Consideriamo le seguenti applicazioni lineari $f : \mathbb{R}_{\leq 3}[x] \to \mathbb{R}_{\leq 3}[x]$:

(a) $p(x) \to p(3x)$, (b) $p(x) \to xp''(x)$, (c) $p(x) \to p(3)$, (d) $p(x) \to p(3) \cdot x^3$.

Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice assume la forma canonica.

(a) $p(x) \rightsquigarrow p(3x)$ BASE: $\{1, x, x^2, x^3\}$

$$\begin{cases} e_1 = 1 \rightsquigarrow 1 = (1,0,0,0) \\ e_2 = x \rightsquigarrow 3x = (0,3,0,0) \\ e_3 = x^2 \rightsquigarrow 9x^2 = (0,0,9,0) \\ e_4 = x^3 \rightsquigarrow 27x^3 = (0,0,0,27) \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 27 \end{pmatrix} \equiv C \quad M = I$$

(b) $p(x) \rightsquigarrow xp''(x)$

$$\begin{cases} e_1 = 1 \rightsquigarrow 0 = (0,0,0,0) \\ e_2 = x \rightsquigarrow 0 = (0,0,0,0) \\ e_3 = x^2 \rightsquigarrow 2x = (0,2,0,0) \\ e_4 = x^3 \rightsquigarrow 6x^2 = (0,0,6,0) \end{cases}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \lambda$$

$\lambda_{1,2,3,4} = 0 \rightsquigarrow x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

MG = 2 < MA = 4

JORDANIZZ.

2 BLOCCHI

POLINOMIO MINIMO

$$A \cdot A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A \cdot A^2 = 0 \quad J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$Ax_3 = x_2$

$Ax_4 = x_3$

$$x_3 = \begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} \quad x_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/12 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/12 \end{pmatrix}$$

(c) $p(x) \leadsto p(3)$

$$\begin{cases} e_1 = 1 \leadsto 1 = (1,0,0,0) \\ e_2 = x \leadsto 3 = (3,0,0,0) \\ e_3 = x^2 \leadsto 9 = (9,0,0,0) \\ e_4 = x^3 \leadsto 27 = (27,0,0,0) \end{cases}$$

$$A = \begin{pmatrix} 1 & 3 & 9 & 27 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \lambda$$

$\lambda_1 = 1 \quad \lambda_{2,3,4} = 0$

$$\lambda_1 = 1 \leadsto x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \lambda_{2,3,4} = 0 \leadsto x_2 = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} \quad x_4 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -1 \end{pmatrix}$$

$$\leadsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(d) $p(x) \leadsto p(3) \cdot x^3$

$$\begin{cases} e_1 = 1 \leadsto x^3 = (0,0,0,1) \\ e_2 = x \leadsto 3x^3 = (0,0,0,3) \\ e_3 = x^2 \leadsto 9x^3 = (0,0,0,9) \\ e_4 = x^3 \leadsto 27x^3 = (0,0,0,27) \end{cases}$$

λ

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 3 & 9 & 27 \end{pmatrix}$$

$\lambda_1 = 27 \quad \lambda_{2,3,4} = 0$

$$\lambda_1 = 27 \leadsto x_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_{2,3,4} = 0 \leadsto x_2 = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} \quad x_3 = \begin{pmatrix} 0 \\ 3 \\ -1 \\ 0 \end{pmatrix} \quad x_4 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -1 \end{pmatrix}$$

$$\leadsto \begin{pmatrix} 27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

4. Consideriamo le seguenti applicazioni lineari $f : M_{2\times 2} \to M_{2\times 2}$:

$$(a)\ A \to A^t, \quad (b)\ A \to A - A^t, \quad (c)\ A \to A + 3A^t, \quad (d)\ A \to \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} A.$$

Per ciascuna di essere determinare la forma canonica, ed una base in cui la matrice (che rappresenta f) assume la forma canonica.

(a) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ BASE: $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{cases} e_1 \rightsquigarrow (1,0,0,0) \\ e_2 \rightsquigarrow (0,0,1,0) \\ e_3 \rightsquigarrow (0,1,0,0) \\ e_4 \rightsquigarrow (0,0,0,1) \end{cases} \quad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\left[\lambda^2(1-\lambda) - (1-\lambda)\right] = (1-\lambda)^2(\lambda^2 - 1) \quad \lambda_{1,2,3} = 1 \quad \lambda_4 = -1$$

$$\lambda_{1,2,3} = 1 \rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_4 = -1 \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad X_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}$

$$\begin{cases} e_1 \rightsquigarrow (0,0,0,0) \\ e_2 \rightsquigarrow (0,1,-1,0) \\ e_3 \rightsquigarrow (0,-1,1,0) \\ e_4 \rightsquigarrow (0,0,0,0) \end{cases} \quad A=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad |A-\lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & -1 & 0 \\ 0 & -1 & 1-\lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{vmatrix} =$$

$$= -\lambda\left[-\lambda(1-\lambda)^2+\lambda\right] = \lambda^2(1+\lambda^2-2\lambda-1) = \lambda^3(\lambda-2)=0$$

$$\lambda_{1,2,3}=0 \rightsquigarrow \quad X_1=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad X_2=\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad X_3=\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_4=2 \rightsquigarrow \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} X=0 \quad X_4=\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad M=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c) $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow A+3A^{\delta} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + 3\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 4a & b+3c \\ c+3b & 4d \end{pmatrix}$

$$\begin{cases} e_1 \rightsquigarrow (4,0,0,0) \\ e_2 \rightsquigarrow (0,1,3,0) \\ e_3 \rightsquigarrow (0,3,1,0) \\ e_4 \rightsquigarrow (0,0,0,4) \end{cases} \quad A=\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad |A-\lambda I| = \begin{vmatrix} 4-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 3 & 0 \\ 0 & 3 & 1-\lambda & 0 \\ 0 & 0 & 0 & 4-\lambda \end{vmatrix} =$$

$$= (4-\lambda)\left[(4-\lambda)(1-\lambda)^2 - 9(4-\lambda)\right] = (4-\lambda)^2(\lambda^2-2\lambda-8)=0$$

$$\lambda_{1,2}=4 \quad \lambda_{3,4}=\frac{2\pm\sqrt{4+32}}{2} \begin{cases} 4 \\ -2 \end{cases}$$

$$\lambda_{1,2,3}=5 \leadsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad x_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad x_2=\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \quad x_3=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

$$\lambda_4=-2 \leadsto \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \quad x_4=\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}$$

$$\leadsto \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \qquad M=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(d) $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leadsto \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ -a+5c & -b+5d \end{pmatrix}$

$$\begin{cases} e_1 \leadsto (1,0,-1,0) \\ e_2 \leadsto (0,1,0,-1) \\ e_3 \leadsto (2,0,5,0) \\ e_4 \leadsto (0,2,0,5) \end{cases} \quad A=\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ -1 & 0 & 5 & 0 \\ 0 & -1 & 0 & 5 \end{pmatrix} \quad |A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 2 & 0 \\ 0 & 1-\lambda & 0 & 2 \\ -1 & 0 & 5-\lambda & 0 \\ 0 & -1 & 0 & 5-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\left[(1-\lambda)(5-\lambda)^2 + 2(5-\lambda)\right] + 2\left[2+(1-\lambda)(5-\lambda)\right] =$$

$$= (1-\lambda)(5-\lambda)\left[(1-\lambda)(5-\lambda)+2\right] + 2\left[2+(1-\lambda)(5-\lambda)\right] =$$

$$= \left[(1-\lambda)(5-\lambda)+2\right]^2 = (\lambda^2-5\lambda+6)^2 = 0 \qquad \lambda = \frac{5\pm\sqrt{25-24}}{2} \begin{cases} 3 \\ 2 \end{cases}$$

$$\lambda_{1,2}=3 \leadsto \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad x_1=\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \quad x_2=\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

$$\lambda_{3,4}=2 \leadsto \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \quad X_3=\begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad X_4=\begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$\leadsto \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \qquad M=\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

5. Consideriamo l'applicazione lineare $f : M_{2\times 2} \to M_{2\times 2}$ che ad ogni matrice 2×2 associa la matrice ottenuta sostituendo ogni elemento con la somma dei due elementi ad esso adiacenti.

Determinare la forma canonica di f, ed una base in cui la matrice (che rappresenta f) assume la forma canonica.

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leadsto \begin{pmatrix} a+b+c & a+b+d \\ a+c+d & b+c+d \end{pmatrix}$$

$$\begin{cases} e_1 \leadsto (1,1,1,0) \\ e_2 \leadsto (1,1,0,1) \\ e_3 \leadsto (1,0,1,1) \\ e_4 \leadsto (0,1,1,1) \end{cases} \quad A=\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \quad |A-\lambda I|=\begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 0 & 1 \\ 1 & 0 & 1-\lambda & 1 \\ 0 & 1 & 1 & 1-\lambda \end{vmatrix}=$$

$$=\begin{vmatrix} 1-\lambda & 1 & 1 & 0 \\ 1 & 1-\lambda & 0 & 1 \\ 0 & -(1-\lambda) & 1-\lambda & 0 \\ 0 & 1 & 1 & 1-\lambda \end{vmatrix} = \begin{cases} =(1-\lambda)\left[(1-\lambda)^3-(1-\lambda)-(1-\lambda)\right]-1\left[(1-\lambda)^2+(1-\lambda)^2\right]= \\ =(1-\lambda)^2\left[(1-\lambda)^2-2\right]-2(1-\lambda)^2=(1-\lambda)^2(\lambda^2-2\lambda-3)=0 \\ \lambda_{1,2}=1 \quad \lambda_{3,4}=\dfrac{2\pm\sqrt{4+12}}{2} \begin{cases} 3 \\ -1 \end{cases} \end{cases}$$

$$\lambda_{1,2}=1 \leadsto \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} X=0 \quad X_1=\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad X_2=\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\lambda_3=3 \leadsto \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} X=0 \quad X_3=\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_5 = -1 \rightsquigarrow \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix} x = 0 \quad \begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 5 & 5 \end{pmatrix} x = 0$$

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix} x = 0 \qquad x_5 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ -1 & 0 & 1 & -1 \end{pmatrix}$$