Vanishing theorems on reduced curves *

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Abstract

Let C be a reduced curve contained in a smooth algebraic surface. In this paper we show that for a generic divisor $\mathcal{F}$ on C of degree $\geq p_a(B)$ on each subcurve $B \subseteq C$ we have $h^1(C, \mathcal{F}) = 0$, and if the degree is $\geq p_a(B) + 1$ on each subcurve $B \subseteq C$ then the system $|\mathcal{F}|$ is base point free.

As an application we show that a divisor $\mathcal{H}$ on a reduced connected curve C is normally generated if $\deg \mathcal{H}_{|B} \geq 2p_a(B) + 1$ for all subcurve $B \subseteq C$.

Introduction

Let $C = \Gamma_1 \cup \ldots \cup \Gamma_s$ be a curve contained in a smooth algebraic surface (Γ_i irreducible components) and let $\mathcal{F}$ be an invertible sheaf on C. For each i the natural inclusion map $\epsilon_i : \Gamma_i \to C$ induces a map $\epsilon_i^* : \mathcal{F}_{|C} \to \mathcal{F}_{|\Gamma_i}$.

Following the papers [Ar 1-2] we let $d_i = \deg \mathcal{F}_{|\Gamma_i}$ (for the definition of degree on a curve we refer to the next section) and we define the multidegree of $\mathcal{F}$ on C $\mathbf{d} := (d_1, ..., d_s)$.

By $\mathrm{Pic}^{\mathbf{d}}$ we will denote the Picard scheme which parameterizes the class of invertible sheaves of multidegree $\mathbf{d}$.

Let $\mathbf{d} = (d_1, ..., d_s) \in \mathbb{N}^s$. We say that $\mathbf{d} = (d_1, ..., d_s)$ satisfies:

condition (1) if for any $\mathcal{F}$ invertible sheaf of multidegree $\mathbf{d}$ we have

$$\forall B \subset C, \quad p_a(B) \leq \deg \mathcal{F}_{|B}$$

condition (2) if for any $\mathcal{F}$ invertible sheaf of multidegree $\mathbf{d}$ we have

$$\forall B \subset C, \quad p_a(B) + 1 \leq \deg \mathcal{F}_{|B}$$

Our first results are the following theorems:

Theorem A *Assume $C = \Gamma_1 \cup \ldots \cup \Gamma_s$ to be a reduced curve contained in smooth algebraic surface. If $\mathbf{d} = (d_1, ..., d_s)$ satisfies condition (1) then*

$$W = \{[\mathcal{F}] | \mathcal{F} \textit{has of multidegree } \mathbf{d} \textit{ and } h^1(C, \mathcal{F}) \neq 0\} \subseteq \mathrm{Pic}^{\mathbf{d}}(C)$$

has dimension $< \dim(\mathrm{Pic}^{\mathbf{d}}(C))$, that is, for $[\mathcal{F}]$ generic in $\mathrm{Pic}^{\mathbf{d}}$ $H^1(C, \mathcal{F}) = 0$.

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Theorem B *Assume* $C = \Gamma_1 \cup \ldots \cup \Gamma_s$ *to be a reduced curve contained in smooth algebraic surface. If* $\mathbf{d} = (d_1, ..., d_s)$ *satisfies condition* (2) *then for* $[\mathcal{F}]$ *generic in* $\mathrm{Pic}^{\mathbf{d}}$, $|\mathcal{F}|$ *is a base-point free system.*

Theorem A and B follow essentially from Serre and Grothendieck duality

$$H^1(C,\mathcal{F}) \stackrel{\mathrm{d}}{=} \mathrm{Hom}(\mathcal{F},\omega_C) \quad \text{for } \mathcal{F} \text{ a coherent sheaf,}$$

(where $\stackrel{\mathrm{d}}{=}$ denotes duality of vector spaces) and from an extension to the reduced case of the classical Abel map.

If C is reduced and $\mathcal{F} \cong \mathcal{O}_C(D_1 + \ldots + D_s)$, where $D_i = P_{i,1} + \ldots + P_{i,d_i} \subset \Gamma_i$, $[\mathcal{F}]$ generic corresponds to a generic choice of the points on each component.

We apply the above theorems to study the homogeneous ring

$$R(C,\mathcal{H}) = \bigoplus_{k \geq 0} H^0(C,\mathcal{H}^{\otimes k})$$

where $\mathcal{H}$ is an invertible sheaf of sufficiently positive degree. If $\mathcal{H}$ has degree at least $2p_a(B)+1$ on each subcurve B of C then R is generated in degree 1.

Theorem C *Let* C *be a reduced, connected curve contained in a smooth algebraic surface and let* $\mathcal{H}$ *be an invertible sheaf on* C *such that*

$$\forall B \subset C, \qquad 2p_a(B) + 1 \leq \deg \mathcal{H}_{|B}$$

Then $\mathcal{H}$ *is normally generated on* C*, that is, the multiplication maps*

$$\rho_k : (H^0(X,\mathcal{H}))^{\otimes k} \longrightarrow H^0(X,\mathcal{H}^{\otimes k})$$

are surjective for all k*.*

This is a generalization of a classical result due to Castelnuovo and Mumford in the case where C is smooth and irreducible.

Under this numerical conditions, in [CFHR] it was proved that $\mathcal{H}$ is very ample. Considering the embedding $\varphi_{|\mathcal{H}|} : C \hookrightarrow \mathbb{P}^n$ associated to the system $|\mathcal{H}|$, in our case it is not true that for the image of C in $\mathbb{P}^n$ the subscheme obtained from a general hyperplane section behaves like d points in "general position" (in the sense of [ACGH]), since for example C may have several irreducible components, each of them contained in a proper subspace.

Anyway, in this case, theorem A simply says that we can find the points in "relative general position", i.e., if $\Gamma \subset C$ is an irreducible component contained in $V \subset \mathbb{P}^n$, the points on Γ are in general position.

The proof of theorem C then will follow by the standard arguments of Castelnuovo theory.

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1 Notation and background results

Notation

For all the paper we will assume C to be a reduced curve contained in a smooth algebraic surface defined over an algebraically closed field of characteristic p.

- $|\mathcal{H}|$ If $\mathcal{H}$ is an invertible sheaf on C, $|\mathcal{H}|$ will denote the linear system of divisors of sections of $H^0(C, \mathcal{H})$.
- $\deg \mathcal{H}_{|C}$ The degree of $\mathcal{H}$ on C; it can be defined for every torsion free sheaf of rank 1 by

$$\deg \mathcal{H}_{|C} = \chi(\mathcal{H}) - \chi(\mathcal{O}_C).$$

- $p_a(C)$ The arithmetic genus of C, $p_a(C) = 1 - \chi(\mathcal{O}_C)$.
- ω_C Dualising sheaf of C (see [Ha], Chap. III, §7).

Notice that from the definition of degree we get $\deg(\omega_C) = 2p_a(C) - 2$.

If $C = C_1 \cup C_2$ scheme theoretically with $\dim C_1 \cap C_2 = 0$ and $x \in C_1 \cap C_2$, we can define (cf. [Ca], p. 54)

$$(C_1.C_2)_x = \text{length}\, \mathcal{O}_{C_1 \cap C_2, x}; \quad C_1.C_2 = \sum_{x \in C_1 \cap C_2} \text{length}\, \mathcal{O}_{C_1 \cap C_2, x}$$

Notice that if $C = C_1 \cup C_2$, with $\dim C_1 \cap C_2 = 0$, then we recover the classical formula

$$p_a(C) = p_a(C_1) + p_a(C_2) + C_1.C_2 - 1$$

Sometimes, with abuse of notation, we will denote the curve C_2 as $C - C_1$.

Definition 1.1 *A (reduced) curve C is* numerically m-connected *if $C_1.C_2 \geq m$ for every decomposition $C = C_1 \cup C_2$*

We recall that in [C-F-H-R] it is introduced a notion of m-connectedness for C Gorenstein, possibly non reduced, in terms of the degree of the dualising sheaf ω_C on each subcurve B.

A *cluster* Z of *degree* $\deg Z = r$ is simply a 0-dimensional subscheme with $\text{length}\, \mathcal{O}_Z = \dim_k \mathcal{O}_Z = r$.

If $C = C_1 \cup C_2$ with $\dim C_1 \cap C_2 = 0$ we will denote by $\mathcal{O}_{C_1}(C_2)$ the cluster on C_1 defined by the ideal $\mathcal{I}_{C_2} \otimes \mathcal{O}_{C_1}$.

1.1 Background results on projective curves

A fundamental instrument in the study of sheaves on projective curves with several components is the following lemma (which appears in [CFHR]). For the reader's benefit we reproduce here the proof.

Lemma 1.2 (Automatic adjunction) *Let $\mathcal{F}$ be a coherent sheaf on C, and $\varphi\colon \mathcal{F} \to \omega_C$ a map of $\mathcal{O}_C$-modules. Set $\mathcal{J} = \operatorname{Ann}\varphi \subset \mathcal{O}_C$, and write $B \subset C$ for the subscheme defined by $\mathcal{J}$. Then B is Cohen–Macaulay and φ has a canonical factorisation of the form*

$$\mathcal{F} \twoheadrightarrow \mathcal{F}_{|B} \to \omega_B = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}_B, \omega_C) \subset \omega_C,$$

where $\mathcal{F}_{|B} \to \omega_B$ is generically onto.

Proof. First, ω_C is torsion free, because $\operatorname{Hom}(\mathcal{G}, \omega_C) = 0$ for any sheaf $\mathcal{G}$ with 0 dimensional support, hence $\mathcal{J} = \operatorname{Ann}\varphi$ has no embedded primes, and $\mathcal{O}_B = \mathcal{O}_C/\mathcal{J}$ is Cohen–Macaulay. By construction of $\mathcal{J}$, the image of φ is contained in the submodule

$$\left\{ s \in \omega_C \;\middle|\; \mathcal{J}s = 0 \right\} \subset \omega_C.$$

But this clearly coincides with $\mathcal{H}om(\mathcal{O}_B, \omega_C)$. Now the inclusion morphism $B \hookrightarrow C$ is finite, and $\omega_B = \mathcal{H}om_{\mathcal{O}_C}(\mathcal{O}_B, \omega_C)$ is just the adjunction formula for a finite morphism (see, for example, [Ha], Chap. III, §7, Ex. 7.2, or [R], Prop. 2.11).

The factorisation (1.2) goes like this: φ is killed by $\mathcal{J}$, so it factors via the quotient module $\mathcal{F}/\mathcal{J}\mathcal{F} = \mathcal{F}_{|B}$. As just observed, it maps into $\omega_B \subset \omega_C$. Finally, it maps onto every generic stalk of ω_B, again by definition of $\mathcal{J}$: a submodule of the sum of generic stalks $\bigoplus \omega_{B,\eta}$ is the dual to the generic stalk $\bigoplus \mathcal{O}_{B',\eta}$ of a purely 1-dimensional subscheme $B' \subset B$, and φ is not killed by the corresponding ideal sheaf $\mathcal{J}'$. Q.E.D.

1.2 $\operatorname{Pic}^{\mathbf{d}}(C)$ and $\operatorname{Hilb}^{\delta}(C)$

Let $C = \Gamma_1 \cup \ldots \cup \Gamma_s$ be a reduced curve contained in a smooth algebraic surface (Γ_i irreducible components) and let $\mathcal{F}$ be an invertible sheaf on C. For each i the natural inclusion map $\epsilon_i : \Gamma_i \to C$ induces a map $\epsilon_i^* : \mathcal{F}_{|C} \to \mathcal{F}_{|\Gamma_i}$.

Following the papers [Ar 1-2] we let $d_i = \deg \mathcal{F}_{|\Gamma_i}$ and we define the multidegree of $\mathcal{F}$ on C $\mathbf{d} := (d_1, ..., d_s)$.

Let $\operatorname{Pic}^{\mathbf{d}}$ be the Picard scheme which parameterizes the class of invertible sheaves of multidegree $\mathbf{d}$.

It has a natural structure of an extension of an abelian variety (corresponding to the normalization of C) by an affine group (corresponding to the to the singular points of C) and its dimension is $h^1(C, \mathcal{O}_C)$ (cf. e.g. [B-P-V]) .

Mumford and Mayer (cf. [Mu-1], [Me], [A-I-K] and [A-K]) in the irreducible case proposed a natural compactification of the Picard scheme Pic^d, consisting of torsion free coherent sheaf of rank 1 with Euler characteristic $= d - \chi(\mathcal{O}_C)$. It is usually denoted by $\overline{\mathbf{J}}^d(C)$.

Notice that even in the irreducible case the closure of $\mathrm{Pic}^{\mathbf{d}}$ in $\overline{\mathbf{J}}^d(C)$ may be different from $\overline{\mathbf{J}}^d(C)$ (this implies in particular that $\overline{\mathbf{J}}^d(C)$ is reducible). However, as pointed out by Altman and Kleiman in the paper [A-K] if C is irreducible and it is contained in a smooth surface then $\overline{\mathbf{J}}^d(C)$ is irreducible.

$\mathrm{Hilb}^{\delta}(C)$ will denote the Hilbert scheme of clusters on C of degree δ. Notice that if $C = \Gamma_1 \cup \ldots \cup \Gamma_s$, then

$$\mathrm{CaDiv}^{\mathbf{d}}(C) = \{\text{Cartier divisors of multidegree } \mathbf{d} = (d_1, \ldots, d_s)\}$$

is an open subset of $\mathrm{Hilb}^{\sum d_i}(C)$ (cf. e.g. [Ko], §1).

We remark that if C is contained in a smooth algebraic surface S then $\dim(\mathrm{Hilb}^{\delta}(C)) = \delta$.

This can be proved as follow (cf. Rego [Re], Iarrobino [Ia]).

For all $x \in C$ we define $\mathrm{Hilb}^n_x(C) = \{\zeta \subset C | \deg(\zeta) = n,\ supp(\zeta) = x\}$.

Now, $\mathrm{Hilb}^n_x(C) \subseteq \mathrm{Hilb}^n_x(S) = \{\zeta \subset S | \deg(\zeta) = n,\ supp(\zeta) = x\}$, and it is easy to prove that $n - 1 \leq \dim(\mathrm{Hilb}^n_x(C)) \leq \dim(\mathrm{Hilb}^n_x(S)) = n - 1$ (cf. [Re] p.221). If we define

$$\mathbb{N}^h(\delta) = \{\underline{n} = (n_1, \ldots, n_h) \in \mathbb{N}^h |\ n_1 \leq \ldots \leq n_h,\ \sum n_i = \delta\}$$

and for $\underline{n} = (n_1, \ldots, n_h) \in \mathbb{N}^h(\delta)$ we set

$$\begin{aligned}\mathrm{Hilb}^{\delta}_{\underline{n}}(C) := \quad & \{(\zeta_1, \ldots, \zeta_h) | \zeta_i \text{ is a cluster s.t.} \\ & supp(\zeta_i) = P_i \in C,\ \deg(\zeta_i) = n_i\}\end{aligned}$$

then

$$\mathrm{Hilb}^{\delta}(C) = \bigcup_{h=1}^{\delta} \{ \bigcup_{\underline{n} \in \mathbb{N}^h(\delta)} \mathrm{Hilb}^{\delta}_{\underline{n}}(C)\}$$

To conclude the proof it is sufficient to remark that for every $\underline{n} \in \mathbb{N}^h(\delta)$ $\dim(\mathrm{Hilb}^{n_i}_{P_i}(C)) = n_i - 1$ implies $\dim(\mathrm{Hilb}^{\delta}_{\underline{n}}(C)) = \delta$.

We remark as pointed out by Altman and Kleiman ([A-K]), that if there exists a point x such that $\dim T_{x,C} \geq 3$ then we have $\dim(\mathrm{Hilb}^n_x(C)) \geq n$.

2 "Generic divisors" in $\mathrm{Pic}^{\mathbf{d}}$ on reduced curves

In this section we will prove theorem A and B.

Proof of theorem A.

By Serre duality, if $\mathcal{F}$ is a coherent sheaf on C then $H^1(C, \mathcal{F}) \stackrel{\mathrm{d}}{=} \mathrm{Hom}(\mathcal{F}, \omega_C)$.

Assume $H^1(C,\mathcal{F}) \neq 0$. Then there exists a non-zero morphism of sheaves $\varphi : \mathcal{F} \to \omega_C$. We will prove the thesis by an induction argument on the number of components of C.

If C is reduced and irreducible then $H^1(C,\mathcal{F}) \neq 0$ if and only if there exists an exact sequence

$$0 \to \mathcal{F} \stackrel{\varphi}{\to} \omega_C \to \mathcal{O}_\Delta \to 0$$

where Δ is a cluster of length $\delta \leq p_a(C) - 2$.

This means that there is a morphism

$$\begin{array}{ccc} \Psi_1 : \mathrm{Hilb}^\delta(C) & \longrightarrow & \overline{\mathbf{J}}^d(C) \\ \bigcup & & \bigcup \\ \mathrm{CaDiv}^\delta(C) & \longrightarrow & \mathrm{Pic}^{\mathbf{d}}(C) \\ \Delta & \mapsto & \omega_C \otimes \mathcal{I}_\Delta \end{array}$$

Since $\dim(\mathrm{Hilb}^\delta(C)) = \delta$ because C is contained in a smooth surface and $\delta \leq p_a(C) - 2$, while $\dim(\overline{\mathbf{J}}^d(C)) = p_a(C)$ then we conclude that the subset W$=\{[\mathcal{F}] | \mathcal{F}$ of multidegree $\mathbf{d}$ and $h^1(C,\mathcal{F}) \neq 0\} \subseteq \Psi_1(\mathrm{CaDiv}^\delta(C))$ has dimension $\leq \delta < \dim \mathrm{Pic}^d(C)$.

Now let $C = \Gamma_1 \cup \ldots \cup \Gamma_s$ and let $\varphi : \mathcal{F} \to \omega_C$, $\varphi \neq 0$, our morphism of sheaves. We claim, by induction hypotheses, that for $\mathcal{F}$ generic every nonzero morphism $\varphi : \mathcal{F} \to \omega_C$ is generically onto.

Indeed, if φ was not generically onto, by automatic adjunction, it would factor as $\varphi' : \mathcal{F}_{|B} \hookrightarrow \omega_B$; but by induction we may assume that for $\mathcal{F}$ generic $\varphi' \equiv 0$, that is $\varphi \equiv 0$.

But now we can proceed as in the irreducible case. coker(φ) defines a cluster of deg $= \delta \leq p_a(C) - 2$, which is a Cartier divisor.

This means that we have a morphism

$$\begin{array}{rcl} \Psi : \mathrm{CaDiv}^\delta(C) & \to & \mathrm{Pic}^{\mathbf{d}}(C) \\ \Delta & \mapsto & \omega_C \otimes \mathcal{I}_\Delta \end{array}$$

and $[\mathcal{F}] \in \mathrm{im}(\Psi)$.

Since $\dim(\mathrm{CaDiv}^\delta(C)) = \delta$ because C is contained in a smooth surface and $\delta \leq p_a(C) - 2$, while $\dim(\mathrm{Pic}^d(C)) = h^1(C, \mathcal{O}_C) \geq p_a(C)$ we obtain the thesis.

Q.E.D. for thm. A

Proof of theorem B.

The proof works essentially as in the previous theorem. We restrict ourselves to consider the open, dense, subset

$$\begin{array}{rl}\underline{\mathrm{Pic}^{\mathbf{d}}}(C) := & \{[\mathcal{F}] \in \mathrm{Pic}^{\mathbf{d}} \mid \text{there exists a } \mathcal{F}' \in |\mathcal{F}| \\ & \text{effective Cartier divisor with support on} \\ & C \setminus C_{sing} \text{ and multidegree } = \mathbf{d}\}\end{array}$$

By our restriction, if $x \in C$ is singular, then it is not a base point for $\mathcal{F}$. Thus we need only to consider the case where $x \in C$ is smooth.

As in the above lemma $|\mathcal{F}|$ is not base point free if and only if there exists a point x on C and there exists a non-zero morphism of sheaves $\varphi : \mathcal{F} \otimes \mathcal{M}_x \to \omega_C$.

If C is reduced and irreducible then we obtain an exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{M}_x \xrightarrow{\varphi} \omega_C \to \mathcal{O}_\Delta \to 0$$

where Δ is a cluster of length $\delta \leq p_a(C) - 2$.

This means that there exists a morphism

$$\begin{array}{ccc}\Psi_1 : \mathrm{Hilb}^\delta(C) \times C & \longrightarrow & \overline{\mathbf{J}}^d(C) \\ \bigcup & & \bigcup \\ \mathrm{CaDiv}^\delta(C) \times \{C \setminus C_{sing}\} & \longrightarrow & \underline{\mathrm{Pic}^{\mathbf{d}}(C)} \\ (\Delta, x) & \mapsto & \omega_C \otimes \mathcal{I}_\Delta \otimes \mathcal{M}_x^{-1}\end{array}$$

Thus, for $\mathcal{F}$ such that $[\mathcal{F}] \in \underline{\mathrm{Pic}^{\mathbf{d}}(C)}$, $|\mathcal{F}|$ has some base point if and only if $[\mathcal{F}] \in \Psi_1(\mathrm{CaDiv}^\delta(C) \times \{C \setminus C_{sing}\})$.

Since $\dim(Im(\Psi_1)) \leq \dim(\mathrm{Hilb}^\delta(C)) \times C = \delta + 1 \leq p_a(C) - 1$, while $\dim(\underline{\mathrm{Pic}^{\mathbf{d}}}(C)) = p_a(C)$ we conclude that for $[\mathcal{F}]$ generic in $\underline{\mathrm{Pic}^{\mathbf{d}}}(C))$, $|\mathcal{F}|$ is base point free.

If $C = \Gamma_1 \cup \ldots \cup \Gamma_s$ we proceed as in the above theorem, applying lemma 2.1.

Q.E.D. for thm. B

3 Divisors normally generated on reduced curves

In this section we will prove theorem C.

The proof of theorem C, as in the classical case, is essentially an application of a lemma of Castelnuovo on base point free systems

Proposition 3.1 (Generalized lemma of Castelnuovo) *Let $\mathcal{F}$ and $\mathcal{H}$ be invertible sheaves on C such that*

1. *$H^1(\mathcal{H} \otimes \mathcal{F}^{-1}) = 0$;*
2. *$|\mathcal{F}|$ is a base point free system on C.*

Then the multiplication map

$$H^0(\mathcal{H}) \otimes H^0(\mathcal{F}) \to H^0(\mathcal{H} \otimes \mathcal{F})$$

is surjective.

For the proof we refer to [Mu-2], thm. 2 or [F], lemma 2.1.

The fundamental step in the proof of Theorem C is then to find a base point free system on C satisfying the above conditions.

We will find such invertible sheaf by an inductive argument.

Observe that the condition $\deg \mathcal{H}_{|C} \geq (2p_a(C)+1)$ is equivalent to the inequality $\deg(\omega_C - \mathcal{H})_{|C} \leq -1$, whence there exists an irreducible $\Gamma_1 \subset C$ such that $\deg(\omega_C - \mathcal{H})_{|\Gamma_1} < 0$, (that is $\deg \mathcal{H}_{|\Gamma_1} \geq 2p_a(\Gamma_1) + \Gamma_1.(C - \Gamma_1) - 1$).

Following the paper [Ca-Fr] we define such a Γ_1 to be *$\mathcal{H}$-positive* and we denote by C_2 the curve $C - \Gamma_1$.

Furthermore either Γ_1 is unique and then $\mathcal{H}.\Gamma_1 \geq 2p_a(\Gamma_1) + \Gamma_1.C_2 + 1$ or there exists at least one other irreducible *$\mathcal{H}$-positive* curve.

Such a curve Γ_1 will yield an important role in the next proposition:

Lemma 3.2 *Let C and $\mathcal{H}$ be as in theorem A and let Γ_1 be an irreducible $\mathcal{H}$-positive subcurve of C. Then the exact sequence*

$$0 \to \mathcal{O}_{\Gamma_1}(\mathcal{H}) \otimes \mathcal{I}_{C_2} \to \mathcal{O}_C(\mathcal{H}) \to \mathcal{O}_{C_2}(\mathcal{H}) \to 0$$

is exact on global sections, that is $|\mathcal{H}|_{|C_2} = |\mathcal{H}_{|C_2}|$.

Proof. $\mathcal{O}_{\Gamma_1} \otimes \mathcal{I}_{C_2}$ defines on Γ_1 a cluster of length $C_1.C_2$.

Since $\deg \mathcal{H}_{|\Gamma_1} \geq 2p_a(\Gamma_1) + \Gamma_1.(C - \Gamma_1) - 1$ by thm.1.1 of [CFHR] we get $H^1(\Gamma_1, \mathcal{O}_{\Gamma_1}(\mathcal{H}) \otimes \mathcal{I}_{C_2}) = 0$.

□

Proposition 3.3 *Let C and $\mathcal{H}$ be as in theorem C. Then there exists an invertible subsheaf $\mathcal{F}$ of $\mathcal{H}$ such that*

1. $\deg_B \mathcal{F} \geq p_a(B) + 1$ *for all B subcurve of C.*
2. $H^1(C, \mathcal{F}) = 0$.
3. $|\mathcal{F}|$ *is a base point free system on C.*
4. $H^1(C, \mathcal{H} \otimes \mathcal{F}^{-1}) = 0$.

Proof of proposition 3.3 . We will prove the proposition by induction on the number of components of C.

If C is irreducible then we simply consider a $\mathcal{F}$ generic of degree $p_a(C)+1$. Then By theorem A and theorem B $|\mathcal{F}|$ is a base point free system and $H^1(C, \mathcal{H} \otimes \mathcal{F}^{-1}) = 0$.

Let $C = \Gamma_1 \cup \ldots \cup \Gamma_s$ be a decomposition of C such that Γ_h is $H-positive$ with respect to the curve $C_h = \cup_{i=h}^{s}\Gamma_i$.

For $h = 1, \ldots, s-1$, let $m_h := \Gamma_h.C_{h+1}$

Take $\mathcal{F}$ generic such that $\deg \mathcal{F}_{|\Gamma_h} = \max\{p_a(\Gamma_h) + m_h - 1, p_a(\Gamma_h) + 1\}$.

Since C is reduced (which implies $\Gamma_h.C_{h+1} \geq \Gamma_h.B$ for all $B \subset C_{h+1}$) we immediately obtain $\deg_B \mathcal{F} \geq p_a(B) + 1$ for all B subcurve of C_h and then, by theorem A and B, for $\mathcal{F}$ generic $H^1(C, \mathcal{F}) = 0$ and $|\mathcal{F}|$ is a base point free system on C.

Notice that by our conditions on C we have $\deg \mathcal{F}_{|C_h} \leq h^0(C_h, \mathcal{H})$.

It remains to show that $H^1(C, \mathcal{H} \otimes \mathcal{F}^{-1}) = 0$.

If $\mathcal{F} \cong \mathcal{O}_C(\Sigma)$, where Σ is a cluster of σ smooth points ($\sigma \leq h^0(C, \mathcal{H})$), then it is sufficient to prove that the map $H^0(C, \mathcal{H}) \to H^0(\mathcal{O}_\Sigma)$ is onto.

For this let $C = \Gamma_1 + C_2$, with Γ_1 irreducible $H-positive$ such that

$$\begin{cases} \Sigma_{|\Gamma_1} = \Sigma_1 & \text{consists of } p_1 + m - 1 \text{ smooth general points on } \Gamma_1 \\ \Sigma_{|C_2} = \Sigma_2 & \text{consists of } \sigma_2 \text{ smooth general points on } C_2 \text{ with} \\ & p_2 + 1 \leq \sigma_2 \leq d_2 - p_2 \end{cases}$$

We are done if we show that the points of Σ should be taken in such a way that they impose independent conditions on $|\mathcal{H}|$.

By induction we may assume $H^0(C_2, \mathcal{H}) \twoheadrightarrow H^0(\mathcal{O}_{\Sigma_2})$.

If we consider the embedding $\varphi_{|\mathcal{H}|} : C \to \mathbb{P}^N$, where $N = \deg \mathcal{H}_{|C} - p_a C$ and we identify C and its subcurve with their images in $\mathbb{P}^N$ it is enough to prove that the points of Σ may be taken projectively independent.

To simplify the computations we let $N_1 = h^0(\Gamma_1, \mathcal{H}) - 1$; $N_2 = h^0(C_2, \mathcal{H}) - 1$; $l = h^1(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2} \otimes \mathcal{I}_{\Gamma_1})$; $\Gamma_1.C_2 = m$. Thus we have

$\Gamma_1 \subset V_1$ where V_1 is a linear subspace of dimension $= N_1 - l$
$C_2 \subset V_2$ where V_2 is a linear subspace of dimension $= N_2$
$\dim(V_1 \cap V_2) = (m - l - 1)$

The last equality follows from the exact sequence

$$H^0(C_2, \mathcal{H}) \to H^0(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2}(\Gamma_1)) \to H^1(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2} \otimes \mathcal{I}_{\Gamma_1})$$

Since we can choose $\sigma_2 \leq \dim(V_2)$ by induction and by our choice $p_1 + m - 1 + \sigma_2 \leq N$ it is enough to prove that $p_1 + m - 1 \leq \dim(V_1)$ (since we may assume $\langle \Sigma_1 \rangle \cap \langle \Sigma_2 \rangle = \emptyset$.)

Now

$$p_1 + m - 1 \leq \dim(V_1) \iff d_1 - 2p_1 \geq m + l - 1.$$

We will show this inequality using the fact that $\mathcal{H}.B \geq 2p_a(B) + 1 \;\; \forall \; B \subseteq C$.

If C_2 is irreducible we are done because $\mathcal{H}$ is $(d_2 - 2p_2)$-very ample on C_2 and $d_2 > \max\{2p_2,\; 2p_1 + 2p_2 + 2m - d_1\}$, that is, $l \leq \max\{m - d_2 - 2p_2, 0\} \leq d_1 - 2p_1 - m$.

If C_2 is reducible, let us consider a decomposition $C_2 = A_2 \cup B_2$ with A_2 irreducible s.t. $h^1(A_2, \mathcal{H} \otimes \mathcal{O}_{A_2}(-B_2)) = 0$.

By the following exact sequence

$$H^1(A_2, \mathcal{H} \otimes \mathcal{O}_{A_2} \otimes \mathcal{I}_{B_2 \cup \Gamma_1}) \to H^1(C_2, \mathcal{H} \otimes \mathcal{O}_{C_2} \otimes \mathcal{I}_{\Gamma_1}) \to H^1(B_2, \mathcal{H} \otimes \mathcal{O}_{B_2} \otimes \mathcal{I}_{\Gamma_1}),$$

since by induction

$$h^1(B_2, \mathcal{O}_{B_2}(\mathcal{H} - \Gamma_1)) < B_2.\Gamma_1 - (H.B_2 - 2p_a(B_2))$$

and by the k-very-ampleness of $\mathcal{H}$ on A_2

$$h^1(A_2, \mathcal{O}_{A_2}(\mathcal{H} - B_2 - \Gamma_1)) < \Gamma_1.A_2 - (H.A_2 - A_2.B_2 - 2p_a(A_2))$$

we argue that

$$l \leq \Gamma_1.(A_2 + B_2) - H.(A_2 + B_2) + (2p_a(A_2) + 2p_a(B_2) + A_2.B_2) - 2 \leq$$
$$\leq \Gamma_1.C_2 - (H.C_2 - 2p_a(C_2)) = d_1 - 2p_1 - m.$$

Q.E.D. for proposition 3.3

Notice that in the above proposition no connectedness hypotheses are required

Proof of Theorem C.

For $n = 0, 1$ it is obvious since C 1-connected implies $h^0(C, \mathcal{O}_C) = 1$ (cf. [CFHR], thm. 3.3 for the general case) and we are considering the complete linear system $|\mathcal{H}|$.

To show that $H^0(C, \mathcal{H}) \otimes H^0(C, \mathcal{H}) \twoheadrightarrow H^0(C, \mathcal{H}^{\otimes 2})$we apply the generalized lemma of Castelnuovo. Let $\mathcal{F}$ be as in proposition 3.3. Then we have the following commutative diagram:

$$\begin{array}{ccccc} H^0(C,\mathcal{F}) \otimes H^0(C,\mathcal{H}) & \hookrightarrow & H^0(C,\mathcal{H}) \otimes H^0(C,\mathcal{H}) & \twoheadrightarrow & H^0(C,\mathcal{O}_\Delta \otimes \mathcal{H}) \otimes H^0(C,\mathcal{H}) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(C,\mathcal{F} \otimes \mathcal{H}) & \hookrightarrow & H^0(C,\mathcal{H}^{\otimes 2}) & \twoheadrightarrow & H^0(C,\mathcal{O}_\Delta \otimes \mathcal{H}^{\otimes 2}) \end{array}$$

By theorem 1.1 of [CFHR], $\mathcal{H}$ is very ample on C (in particular it is base point free).

$\mathcal{O}_\Delta \otimes \mathcal{H}^{\otimes 2} \cong \mathcal{O}_\Delta \otimes \mathcal{H} \cong \mathcal{O}_\Delta$ is a skyscraper sheaf of finite length.

We can pick a section $s \in H^0(C, \mathcal{H})$ such that for all $x \in \mathrm{Supp}(\Delta)$ $s(x) \neq 0$. Then

$$H^0(\Delta \otimes \mathcal{H}) \otimes_{\mathbb{K}} \langle s \rangle \xrightarrow{\sim} H^0(\Delta \otimes \mathcal{H}^{\otimes 2}),$$

that is, the third map is onto. Now, the first map $H^0(C, \mathcal{F}) \otimes H^0(C, \mathcal{H}) \to H^0(C, \mathcal{F} \otimes \mathcal{H})$ is surjective by lemma 3.1 and proposition 3.3, that is the required map is onto.

For $n \geq 3$ we use induction applying the generalized lemma of Castelnuovo to the sheaves $\mathcal{H}^{\otimes(n-1)}$ and $\mathcal{H}$ since, by lemma 2.1 of [Ca-Fr], if $\deg \mathcal{H}_{|B} \geq 2p_a(B) - 1$ for all subcurve $B \subseteq C$ then $H^1(C, \mathcal{H}) = 0$.

Q.E.D. for Theorem C

References

[A-C-G-H] E.Arbarello, M.Cornalba, P.A. Griffiths, J.Harris "*Geometry of algebraic curves, I*", Springer (1984).

[A-I-K] A. Altman, A. Iarrobino, S. Kleiman "*Irreducibility of the Compactified Jacobian*" Nordic Summer School NAVF, Oslo 1976, Noordhoff (1977)

[A-K] A. Altman, S. Kleiman "*Compactifying the Picard scheme*", Advvances in Mathematics **32** (1980), 50–112

[Ar1] M. Artin, "*Some numerical criteria for contractibility of curves on algebraic surfaces*", Amer. J. Math. **84** (1962), 485–496.

[Ar2] M. Artin, "*On isolated rational singularities of surfaces*", Amer. J. Math. **88** (1966), 129–136.

[B–P–V] W. Barth, C. Peters and A. Van de Ven "*Compact complex surfaces*", Springer (1984).

[C] G. Castelnuovo, "*Sui multipli di una serie lineare di gruppi di punti appartenenti ad una curva algebrica*" Rend. Circ. Mat. Palermo **7** (1893), 89–110

[Ca] F. Catanese, "*Pluricanonical Gorenstein curves*", in 'Enumerative Geometry and Classical Algebraic Geometry', Nice, Prog. in Math. **24** (1981), Birkhäuser, 51–95.

[Ca-Fr] F. Catanese and M. Franciosi, "*Divisors of small genus on algebraic surfaces and projective embeddings*", Proceedings of the conference "Hirzebruch 65", Tel Aviv 1993, Contemp. Math., A.M.S. (1994), subseries 'Israel Mathematical Conference Proceedings' Vol. **9** (1996), 109–140.

[CFHR] F.Catanese, M.Franciosi, K.Hulek, M.Reid, "*Embeddings of Curves and Surfaces*", preprint of 'Dipartimento di Matematica dell' Università degli Studi di Pisa' n.1.163.708, (1996).

[G] M. Green "*Koszul cohomology and the geometrty of projective varieties*", J. Diff. Geom. **19** (1984), 125–171

[G-L] M. Green, R. Lazarsfeld "*On the projective normality of complete series on an algebraic curve*", Inv. Math. **83** (1986), 73–90

[Gr–Ha] A. Grothendieck (notes by R. Hartshorne), "*Local cohomology*", Springer, LNM **41** (1967).

[Ha] R. Hartshorne, "*Algebraic Geometry*", Springer (1977).

[Ia] A.Iarrobino "*Punctual Hilbert scheme*", Mem. A.M.S. **188**, 1977

[Ko] J. Kollar, “*Rational curves on algebraic varieties*”, Springer (1996)

[Me] A.L.Mayer “*Compactification of variety of moduli of curves*”, Lectures 2 ,3 Seminar on degeneration of algebraic varieties (mimeographed note), Inst. for Advanced Study, Princeton, N.J. (1969)

[Mu-1] D. Mumford ,”*Further comments on boundary points*”, (mimeographed note), AMS Summer school at Woods Hoole (1964)

[Mu-2] D. Mumford , ”*Varieties defined by quadratic equations*” Corso C.I.M.E. 1969, in ’Questions on algebraic varieties’, Rome, Cremonese (1970), 30–100

[Re] C.J. Rego “*The compactified Jacobian*”, Ann. scient. Ec. Norm. Sup. 4 serie, **13** (1980), 211–223

[R] M. Reid, “*Nonnormal del Pezzo surfaces*”, Publications of RIMS **30**:5 (1995), 695–727.

[St] N. St. Donat ”*Sur les equations definissant une courbe algebrique*” C.R. Acad. Sci. Paris, Ser A, **274** (1972), 324–327

[Se] J-P. Serre, “*Courbes algébriques et corps de classes*”, Hermann, Paris, 1959 (English translation, Springer, 1990).

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