SOLITONS, SCATTERING AND BLOW-UP FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH COMBINED POWER-TYPE NONLINEARITIES ON $\mathbb{R}^d \times \mathbb{T}$

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Abstract. We investigate the long time dynamics of the nonlinear Schrödinger equation (NLS) with combined powers on the waveguide manifold $\mathbb{R}^d \times \mathbb{T}$. Different from the previously studied NLS-models with single power on the waveguide manifolds, where the non-scale-invariance is mainly due to the mixed nature of the underlying domain, the non-scale-invariance of the present model is both geometrical and structural. By considering different combinations of the nonlinearities, we establish both qualitative and quantitative properties of the soliton, scattering and blow-up solutions. As one of the main novelties of the paper compared to the previous results for the NLS with single power, we particularly construct two different rescaled families of variational problems, which leads to an NLS with single power in different limiting profiles respectively, to establish the periodic dependence results.

Contents

1. Introduction

1.1. **Background and motivation.** In this paper, we consider the nonlinear Schrödinger equation (NLS)

$$(i\partial_t + \Delta_{x,y})u = \mu|u|^p u - |u|^q u \tag{1.1}$$

and its corresponding stationary equation

$$-\Delta_{x,y}u + \omega u = -\mu|u|^p u + |u|^q u \tag{1.2}$$

on the waveguide manifold $\mathbb{R}^d_x \times \mathbb{T}_y$ with $\mu \in \{-1, 1\}$, $p, q \in (\frac{4}{d}, \frac{4}{d-1})$ and $p < q$, where $\mathbb{T} = \mathbb{T}_y$ is the 2π-periodic torus. The NLS arises as a fundamental model in numerous physical scenarios such as the Bose-Einstein condensates and nonlinear optics. For detailed physical background on (1.1), see e.g. [38, 39, 25] and the references therein.

Because of its close connection to many mathematical areas, the NLS has also attracted much attention from the mathematical community in recent years. Different from the classical references, where people mainly focused on the NLS-models posed either on an infinite domain (such as the Euclidean space $\mathbb{R}^d$) or on a bounded domain with suitable boundary conditions (e.g. the periodic or Dirichlet boundary conditions), we study in this paper the NLS occupying the mixed semi-periodic domain $\mathbb{R}^d \times \mathbb{T}$, the so-called waveguide manifold. The main interest in studying such models on domains of mixed type is twofold: On the one hand, these models are of physical importance since they arise naturally in actual physical applications, where one or more confinements are applied in order to guide the Schrödinger wave to propagate in a designed way[1]. On the other hand, the domain $\mathbb{R}^d \times \mathbb{T}$, or more generally $\mathbb{R}^d \times \mathbb{T}^m$, can be seen as an interpolation between the infinite Euclidean space and a compact torus, thus it is

2020 *Mathematics Subject Classification.* 35Q55, 35R01, 37K06, 37L50.
Key words and phrases. nonlinear Schrödinger equation, waveguide manifolds, scattering, solitons, blow-up.

[1] This also explains where name "waveguide" comes from.

an interesting question whether properties of the Schrödinger wave on the sole domains could also be inherited to the waveguide manifolds.

Recently, there has been a number of papers devoting to the study for NLS on waveguide manifolds, see e.g. [43, 44, 42, 23, 22, 11, 12, 14, 48, 47, 46, 29, 31, 34, 21, 33, 32]. It is worth pointing out that only models with single nonlinearity on waveguide were studied in the previously mentioned papers. In many actual physical experiments, however, the NLS-models with combined powers were indeed used which serve as a correction for the theoretically preset models. As one of the most famous examples, the cubic-quintic NLS-model, which corrects the focusing cubic NLS where the predicted collapse phenomenon does not happen in actual experiments, plays a fundamental role in the study of nonlinear optics. Generalizing the cubic and quintic nonlinearities to nonlinearities of arbitrary order leads to the study of (1.1). For the readers' interest, we refer e.g. to [41, 13, 27, 10, 8, 37, 26, 7, 30, 35, 1] for some recent studies on the NLS with combined powers.

Back to our study, we restrict ourselves in this paper to the so-called *intercritical* (in other word, strictly larger than mass-critical and smaller than energy-critical) regime $p, q \in (\frac{4}{d}, \frac{4}{d-1})$, where the nonlinear effects between the particles could either become weaker and weaker as time goes by and scattering of a solution (i.e. a solution becomes asymptotically linear in the sense of (1.7)) might take place, or on the contrary become stronger and stronger which dominate the linear dispersive effects and consequently lead to a possible collapse of the particles. We consider the case where the nonlinearity of higher order q is focusing, and the lower power of order p can be tuned to be either focusing or defocusing. For simplicity, we consider the case where the prefactor of the q-power is -1 and the one of the p-power is chosen from the set $\{\pm 1\}$.

1.2. **Main results.** We now state the main results of this paper. As our first concern, we shall investigate the existence of the soliton solutions which solve the stationary NLS (1.2), since they might be the only observable quantities in actual physical experiments. From a mathematical point of view, the solitons can also be seen as a balance point between the linear and nonlinear effects and thus can be used to characterize a sharp threshold bifurcating the scattering and blow-up solutions, see Theorem 1.4 and Theorem 1.9 below for corresponding rigourous statements.

Mathematically, it is nowadays a standard routine to solve the stationary NLS (1.2) by appealing to suitable variational methods, namely, one formulates suitable variational problems and looks for optimizers of the corresponding energy functionals on either a Nehari mainifold or in a Sobolev space with given normalized mass. Using the Lagrange multiplier theorem one then immediately infers that those minimizers will also be solving (1.2). In the context of the waveguide setting, however, the standard methods can not be directly applied in a straightforward way due to the non-scale-invariance of the torus. For NLS-models with single focusing powers, this issue was solved by the second author in a series of papers [29, 31, 34, 33] by introducing the so-called *semivirial-vanishing geometry*. Following the same lines in the previous papers, we adapt the framework of the semivirial-vanishing geometry to the present setting, in order to derive the existence and periodic dependence results of the solitons solutions of the NLS (1.2) with combined powers.

Before introducing the main results concerning the soliton solutions, several variational quantities are defined in order. Firstly, the mass and energy are defined in the standard way as follows:

$$M(u) = \|u\|^2_{L^2(\mathbb{R}^d\times\mathbb{T})}, \quad E(u) = \frac{1}{2}\|\nabla_{x,y} u\|^2_{L^2(\mathbb{R}^d\times\mathbb{T})} + \frac{\mu}{p+2}\|u\|^{p+2}_{L^{p+2}(\mathbb{R}^d\times\mathbb{T})} - \frac{1}{q+2}\|u\|^{q+2}_{L^{q+2}(\mathbb{R}^d\times\mathbb{T})}.$$

For $\omega > 0$ define the action energy function $S_\omega(u)$ by

$$S_\omega(u) = E(u) + \frac{\omega}{2}M(u).$$

We will also make use of the so-called *semivirial* functional

$$Q(u) = \|\nabla_x u\|^2_{L^2(\mathbb{R}^d\times\mathbb{T})} + \frac{\mu p d}{2(p+2)}\|u\|^{p+2}_{L^{p+2}(\mathbb{R}^d\times\mathbb{T})} - \frac{qd}{2(q+2)}\|u\|^{q+2}_{L^{q+2}(\mathbb{R}^d\times\mathbb{T})}.$$

Due to their different geometric nature, we shall define two different variational problems for $\mu = \mp 1$ respectively. More precisely, we define the variational problems m_c and γ_ω by

$$m_c := \inf\{E(u) : M(u) = c, Q(u) = 0\},$$
$$\gamma_\omega := \inf\{S_\omega(u) : Q(u) = 0\}.$$

Our first main result is concerned with the existence of ground states for (1.2) and reads as follows:

Theorem 1.1 (Existence of ground states)**.** *Let $d \geq 1$ and $\frac{4}{d} < p < q < \frac{4}{d-1}$. Then the following statements hold true:*

(i) *Let $\mu = -1$. Then for any $c \in (0, \infty)$ the variational problem m_c possesses an optimizer u_c which also solves (1.2) with some $\omega = \omega_c > 0$.*

(ii) *Let $\mu = 1$. Then for any $\omega \in (0, \infty)$ the variational problem γ_ω possesses an optimizer u_ω which also solves (1.2) with the given ω.*

We shall follow the same lines in [29] to prove Theorem 1.1 (i), where we replace the Liouville's theorem applied in [29] by the one deduced in [28] that also works in the setting of combined powers. For Theorem 1.1 (ii), we point out that solving a variational problem involving the action energy functional is in general much easier in comparison to looking for normalized ground states (which is the case in Theorem 1.1 (i)) due to the presence of the frequency parameter ω. However, in the setting of waveguide manifold, it is *a priori* unclear whether the Lagrange multiplier equation is elliptic since the partial Laplacians $-\Delta_x$ and $-\partial_y^2$ might have different signs, and we note that the ellipticity of the total Laplacian is necessary for the proof of the Pohozaev's identity since certain elliptic regularity will be invoked in the proof. We then appeal to the mountain pass geometry on a suitable Nehari manifold in order to solve this problem. For details, we refer to Subsection 2.6 below.

Another interesting problem is the periodic dependence of the soliton solutions, which is formulated as follows: notice that by the boundedness of a torus we may assume that (1.1) and (1.2) are constant along the periodic direction. In this case the solitons, for example, will automatically reduce to the ones on $\mathbb{R}^d$. As a natural question, we may ask whether those solitons deduced in Theorem 1.1 coincide with the ones on $\mathbb{R}^d$ or not. We refer this kind of questions to as the *periodic dependence problems* of the solitons. Such problems were firstly studied by Terracini, Tzvetkov and Visciglia [42] where the NLS on a generalized product space with a single mass-subcritical[2] nonlinearity was studied. By combining the rescaled variational problems introduced in [42] and the framework of the semivirial-vanishing geometry, similar periodic dependence results for the focusing NLS with at least mass-critical nonlinearity have been recently established by the second author, see [29, 31, 33].

We now state the periodic dependence of the ground state solutions deduced in Theorem 1.1. To formulate the main result we shall still introduce some necessary notations. Let $\widehat{M}(u), \widehat{E}(u), \widehat{Q}(u), \widehat{S}_\omega(u)$ be the quantities defined by (1.18), (1.19) and (1.23). Define also the variational problems

$$\widehat{m}_c := \inf\{\widehat{E}(u) : u \in H^1(\mathbb{R}^d \times \mathbb{T}), \widehat{M}(u) = c, \widehat{Q}(u) = 0\},$$
$$\widehat{\gamma}_\omega := \inf\{\widehat{S}_\omega(u) : u \in H^1(\mathbb{R}^d \times \mathbb{T}), \widehat{Q}(u) = 0\}.$$

The second main result about the periodic dependence of the ground state solutions given in Theorem 1.1 is stated as follows:

Theorem 1.2 (Periodic dependence of the ground states)**.** *Let $d \geq 1$ and $\frac{4}{d} < p < q < \frac{4}{d-1}$.*

(i) *Let additionally $\mu = -1$. Then there exist $0 < c_* \leq c^* < \infty$ such that*
- *For all $c \in (0, c_*)$ we have $m_c < 2\pi \widehat{m}_{(2\pi)^{-1}c}$ and any minimizer u_c of m_c satisfies $\partial_y u_c \neq 0$.*
- *For all $c \in (c^*, \infty)$ we have $m_c = 2\pi \widehat{m}_{(2\pi)^{-1}c}$ and any minimizer u_c of m_c satisfies $\partial_y u_c = 0$.*

(ii) *Let additionally $\mu = 1$. Then there exist $0 < \omega_* \leq \omega^* < \infty$ such that*
- *For all $\omega \in (0, \omega_*)$ we have $\gamma_\omega = 2\pi \widehat{\gamma}_\omega$ and any minimizer u_ω of γ_ω satisfies $\partial_y u_\omega = 0$.*
- *For all $\omega \in (\omega^*, \infty)$ we have $\gamma_\omega < 2\pi \widehat{\gamma}_\omega$ and any minimizer u^ω of γ_ω satisfies $\partial_y u^\omega \neq 0$.*

In order to prove our results, we follow an approach similar to [42, 29, 31, 33], specifically we shall use suitable rescaled variational problems to prove Theorem 1.2. New difficulties however arise due to the presence of the combined powers. More precisely, unlike in [42] and [29, 31, 33], we are unable to establish the periodic dependence results by using only one single parameterized family of rescaled variational problems, since certain divergence happens when considering the limiting profile of the parameterized problems. We will design well-tailored rescaled variational problems, according to the orders of the nonlinearities, to overcome this problem.

Remark 1.3. We notice that due to the combined powers we are unable to compare the rescaled variational problem with the unparameterized constant variational problem as in [42, 29, 31, 33], thus it remains an interesting and also challenging open problem whether the thresholds in Theorem 1.2 (e.g. c_* and c^*) will coincide or not. △

[2]We note that in [42] the exponent of the nonlinearity is mass-subcritical w.r.t. the whole space dimension, while in our paper the mass-criticality condition is defined only in term of the Euclidean dimension, therefore the mass-critical threshold exponent in our paper is strictly larger than the one in [42].

Once the existence of ground states is proved, we have that $u(t) = e^{it}\tilde{u}$, where $\tilde{u}$ solves (1.2), is a global, non-scattering solution to the time-dependent equation (1.1). Similar to the single nonlinearity case, it is worth wondering which conditions on the initial datum ensure existence of solution for all times, or conversely lead to formation of singularity in finite time.

In next theorem, we establish the existence of blowing-up solutions for (1.1) and the blow-up rate of such solutions under suitable assumptions. In what follows, and in the rest of the paper, $T_{\max} > 0$ and $T_{\min} > 0$ are the forward and backward maximal time of existence, respectively, namely the maximal time of existence of the solution $u(t)$ is $I_{\max} = (-T_{\min}, T_{\max})$.

Theorem 1.4 (Existence of blow-up solutions). *Let $d \geq 2$, $\frac{4}{d} < p < q < \frac{4}{d-1}$ and $u_0 \in H^1(\mathbb{R}^d \times \mathbb{T})$ satisfy*

$$E(u_0) < m_{M(u_0)} \quad \text{and} \quad Q(u_0) < 0$$

provided $\mu = -1$, or

$$S_\omega(u_0) < \gamma_\omega \quad \text{and} \quad Q(u_0) < 0$$

provided $\mu = 1$ and $\omega > 0$. Assume moreover that the initial datum enjoys the following radiality assumption on the non-compact direction: $u_0(x, y) = u_0(|x|, y)$. Then the solution u to (1.1) *with $u(0) = u_0$ blows-up in finite time, i.e., $I_{\max}$ is bounded.*

Our strategy classically combines variational estimates and a localization argument in the virial estimates. Nonetheless, due the anisotropy of the underlying domain, the proof is not straightforward as in the classical Euclidean case, and we make use of a Fourier expansion in the compact direction to carefully estimate the contribution of the localized potential energy terms jointly with the decay of radial Sobolev functions.

Remark 1.5. To the best of our knowledge, this paper is the first one dealing with formation of singularity in finite time without the assumption of finite variance in the context of waveguide manifolds. Indeed, as in the classical case of a Euclidean domain, for any $d \geq 1$, provided $u_0 \in H^1(\mathbb{R}^d \times \mathbb{T}) \cap L^2(\mathbb{R}^d \times \mathbb{T}, |x|^2 dxdy)$, we have from (4.8) the usual identity $V''_{|x|^2}(t) = 8Q < 0$, so the Glassey's convexity argument gives a finite time blow-up result directly, see [29]. △

Remark 1.6. In the one-dimensional Euclidean component case (i.e. $d = 1$), we can not hope to remove the assumption of finite variance, as otherwise we can not control the remainder in a localized virial estimate. △

In light of the results in Theorem 1.4 and previous remarks, it is worth mentioning that if we remove both the symmetry assumption and the finiteness of the $L^2(\mathbb{R} \times \mathbb{T}, |x|^2 dxdy)$-norm of the initial datum, we can prove the so-called grow-up phenomenon, namely we can prove that if the solution is global forward in time. A rigorous and precise statement is given as follows.

Theorem 1.7 (Grow-up solutions). *Under the hypothesis of Theorem 1.4, without any restriction on the space dimension and without assuming any symmetry, we have the following dichotomy: either $u(t)$ blows-up in finite time, or the $T_{\max} = +\infty$ (and similarly for $T_{\min}$) and satisfies $\limsup_{t \to +\infty} \|\nabla_{x,y} u\|_{L^2(\mathbb{R}^d \times \mathbb{T})} = +\infty$.*

In order to show Theorem 1.7, we can invoke the results by [19] on intercritical NLS equations posed on $\mathbb{R}^d$, which rely on the uniform in time control of $Q(u(t))$ as in Lemma 4.1, virial estimates, and an almost finite speed of propagation. As the proof is very similar to the purely Euclidean setting, we sketch the main steps in Appendix B.

A natural question arising after the proof of the existence of blowing-up solutions would be whether one could describe the blowing-up solutions in a more quantitative way, or in other words whether the rate of the blow-up could be quantified. The last result concerning solutions with finite time singularity formation is the following theorem about an upper bound rate. The proof relies on the estimates established to prove the blow-up results of Theorem 1.4 and a well-known scheme by Merle, Raphaël, and Szeftel [36].

Theorem 1.8 (An upper bound of the blow-up rate). *Let the assumptions of Theorem 1.4 be retained. Then for the blow-up solution u to* (1.1) *with $u(0) = u_0$ given in Theorem 1.4, we have:*

(i) if $\mu = 1$ and $\omega > 0$, then

$$\int_t^{T_{\max}} (T_{\max} - \tau) \|\nabla_{x,y} u(\tau)\|^2_{L^2(\mathbb{R}^d \times \mathbb{T})} d\tau \leq C(T_{\max} - t)^{\frac{2q(d-1)}{(d-2)q+4}} \quad \text{for} \quad t \to T^-_{\max}. \tag{1.3}$$

(ii) if $\mu = -1$, then

$$\int_t^{T_{\max}} (T_{\max} - \tau)\|\nabla_{x,y} u(\tau)\|^2_{L^2(\mathbb{R}^d\times\mathbb{T})} d\tau \leq C(T_{\max} - t)^{\frac{2p(d-1)}{(d-2)p+4}} \quad \text{for} \quad t \to T^-_{\max}. \tag{1.4}$$

As a consequence, there exists a time sequence $t_n \to T^-_{\max}$ such that

$$\|\nabla_{x,y} u(t_n)\|_{L^2(\mathbb{R}^d\times\mathbb{T})} \leq C(T_{\max} - t_n)^{-\frac{4-p}{(d-2)q+4}} \tag{1.5}$$

provided $\mu = -1$, and

$$\|\nabla_{x,y} u(t_n)\|_{L^2(\mathbb{R}^d\times\mathbb{T})} \leq C(T_{\max} - t_n)^{-\frac{4-q}{(d-2)p+4}} \tag{1.6}$$

provided $\mu = 1$ and $\omega > 0$. A similar result holds for $-T_{\min}$.

At the end of the introductory section, we conclude the analysis concerning the dynamical properties of solutions to (1.1) by considering global solutions, and in particular scattering solutions. As written above, the standing wave $u(t) = e^{it}\tilde{u}$ is a global non-scattering solution, in the sense that it is not decaying in time. It is worth wondering under which conditions on the initial data, solutions to (1.1) are global, i.e., they exist for every time $t \in \mathbb{R}$, and moreover how they behave for large times. Specifically, we will establish conditions leading to global well-posedness of the Cauchy problem associated to (1.1), and we will also prove that a solution *scatters*, namely they behave as a linear Schrödinger wave for large time in the energy topology, see (1.7) below. The global well-posedness and scattering for small data is classical and is a consequence of perturbation arguments. The next theorem, by using the ground state solutions deduced in Theorem 1.1, ensures scattering for large data.

Theorem 1.9 (Large data scattering)**.** *Let $d \geq 1$, $\frac{4}{d} < p < q < \frac{4}{d-1}$ and $u_0 \in H^1(\mathbb{R}^d \times \mathbb{T})$ satisfy*

$$E(u_0) < m_{M(u_0)} \quad \text{and} \quad Q(u_0) > 0$$

provided $\mu = -1$, or

$$S_\omega(u_0) < \gamma_\omega \quad \text{and} \quad Q(u_0) > 0$$

provided $\mu = 1$ and $\omega > 0$. Then the solution u to (1.1) with $u(0) = u_0$ is global and scattering in the sense that there exist $\phi^\pm \in H^1(\mathbb{R}^d \times \mathbb{T})$ such that

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\Delta_{x,y}}\phi^\pm\|_{H^1(\mathbb{R}^d\times\mathbb{T})} = 0. \tag{1.7}$$

In the case of the NLS with single power, the large data scattering result was proved in [29] and [34] in the cases $d < 5$ and $d \geq 5$, by using the *concentration compactness* and *interaction Morawetz-Dodson-Murphy* methods, respectively. The main reason for such a difference is that the nonlinearity becomes less regular, or more precisely its derivative is no longer Lipschitz, in higher dimensional spaces $d \geq 5$. We shall notice that one may still derive large data scattering results on higher dimensional Euclidean spaces $\mathbb{R}^d$ by appealing to suitable fractional calculus, see e.g. [45], but so far we don't know whether such fractional calculus is also available on product spaces. Alternatively, such difficulties were overcome by the second author [34] by appealing to the interaction Morawetz inequality recently developed by Dodson and Murphy [45, 18] which avoids the use of any fractional calculus. At this point, we notice that different from the variational analysis, the proof for Theorem 1.9 is essentially the same as in [29, 34]. For this reason, we shall present a sketch of the proof of Theorem 1.9 in the case $d \geq 5$ based on the interaction Morawetz inequality in Appendix A.

Remark 1.10. Notice that in the waveguide setting, even by using the interaction Morawetz inequality we are able to avoid the use of the fractional calculus in the Euclidean space, the fractional derivatives in the periodic direction is still needed. New ideas were then introduced in the recent paper [34] by the second author to overcome the additional technical difficulties. △

The rest of the paper is organized as follows: Sections 2 and 3 are devoted to the proofs of Theorem 1.1 and Theorem 1.2 respectively. In Section 4 we give the proofs for the blow-up results Theorem 1.4 and Theorem 1.8. In Appendix A we give a sketch for the proof of the large data scattering result Theorem 1.9 in the case $d \geq 5$ by using the modern method based on the interaction Morawetz inequality. Finally, a proof for the grow-up result Theorem 1.7 will be given in Appendix B.

Acknowledgment. Y. Luo was supported by the NSF grant of China (No. 12301301) and the NSF grant of Guangdong (No. 2024A1515010497). Z. Zhao was supported by the NSF grant of China (No. 12101046, 12271032) and the Beijing Institute of Technology Research Fund Program for Young Scholars.

1.3. **Notations and preliminaries.** For simplicity, we ignore in most cases the dependence of the function spaces on their underlying domains and hide this dependence in their indices. For example $L_x^2 = L^2(\mathbb{R}^d)$, $H^1_{x,y} = H^1(\mathbb{R}^d \times \mathbb{T})$ and so on. However, when the space is involved with time, we still display the underlying temporal interval such as $L_t^p L_x^q(I)$, $L_t^\infty L^2_{x,y}(\mathbb{R})$ etc. The norm $\|\cdot\|_p$ is defined by $\|\cdot\|_p := \|\cdot\|_{L^p_{x,y}}$.

Next, we define the variational quantities, such as mass and energy etc. that will be frequently used in the proof of the main results. For $u \in H^1_{x,y}$, define

$$E(u) := \frac{1}{2}\|\nabla_{x,y} u\|_2^2 + \frac{\mu}{p+2}\|u\|_{p+2}^{p+2} - \frac{1}{q+2}\|u\|_{q+2}^{q+2}, \tag{1.8}$$

$$M(u) := \|u\|_2^2, \quad S_\omega(u) := \frac{\omega}{2}M(u) + E(u), \tag{1.9}$$

$$Q(u) := \|\nabla_x u\|_2^2 + \frac{\mu p d}{2(p+2)}\|u\|_{p+2}^{p+2} - \frac{qd}{2(q+2)}\|u\|_{q+2}^{q+2}, \tag{1.10}$$

$$I(u) := \frac{1}{2}\|\partial_y u\|_2^2 + \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u\|_2^2 + \frac{1}{q+2}\Big(\frac{q}{p} - 1\Big)\|u\|_{q+2}^{q+2} = E(u) - \frac{2}{pd}Q(u). \tag{1.11}$$

For $\lambda \in [0,\infty]$ (as long as the quantities are well-defined), define

$$E_\lambda(u) := \frac{\lambda}{2}\|\partial_y u\|_2^2 + \frac{1}{2}\|\nabla_x u\|_2^2 + \frac{\mu\lambda^{\frac{p}{q}-1}}{p+2}\|u\|_{p+2}^{p+2} - \frac{1}{q+2}\|u\|_{q+2}^{q+2}, \tag{1.12}$$

$$E^\lambda(u) := \frac{\lambda}{2}\|\partial_y u\|_2^2 + \frac{1}{2}\|\nabla_x u\|_2^2 + \frac{\mu}{p+2}\|u\|_{p+2}^{p+2} - \frac{\lambda^{\frac{q}{p}-1}}{q+2}\|u\|_{q+2}^{q+2}, \tag{1.13}$$

$$S_{1,\lambda}(u) := \frac{1}{2}M(u) + E_\lambda(u), \quad S_1^\lambda(u) := \frac{1}{2}M(u) + E^\lambda(u), \tag{1.14}$$

$$Q_\lambda(u) := \|\nabla_x u\|_2^2 + \frac{\mu\lambda^{\frac{p}{q}-1}pd}{2(p+2)}\|u\|_{p+2}^{p+2} - \frac{qd}{2(q+2)}\|u\|_{q+2}^{q+2}, \tag{1.15}$$

$$Q^\lambda(u) := \|\nabla_x u\|_2^2 + \frac{\mu pd}{2(p+2)}\|u\|_{p+2}^{p+2} - \frac{\lambda^{\frac{q}{p}-1}qd}{2(q+2)}\|u\|_{q+2}^{q+2}, \tag{1.16}$$

$$I_\lambda(u) := \frac{\lambda}{2}\|\partial_y u\|_2^2 + \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u\|_2^2 + \frac{1}{q+2}\Big(\frac{q}{p} - 1\Big)\|u\|_{q+2}^{q+2}. \tag{1.17}$$

For $u \in H^1_x$, define

$$\widehat{E}(u) := \frac{1}{2}\|\nabla_x u\|_{L_x^2}^2 + \frac{\mu}{p+2}\|u\|_{L_x^{p+2}}^{p+2} - \frac{1}{q+2}\|u\|_{L_x^{q+2}}^{q+2} \tag{1.18}$$

$$\widehat{M}(u) := \|u\|_{L_x^2}^2, \quad \widehat{S}_\omega(u) := \widehat{E}(u) + \frac{\omega}{2}\widehat{M}(u), \tag{1.19}$$

$$\widehat{I}(u) := \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u\|_{L_x^2}^2 + \frac{1}{q+2}\Big(\frac{q}{p} - 1\Big)\|u\|_{L_x^{q+2}}^{q+2}, \tag{1.20}$$

$$\widehat{E}_\lambda(u) := \frac{1}{2}\|\nabla_x u\|_{L_x^2}^2 + \frac{\mu\lambda^{\frac{p}{q}-1}}{p+2}\|u\|_{L_x^{p+2}}^{p+2} - \frac{1}{q+2}\|u\|_{L_x^{q+2}}^{q+2}, \tag{1.21}$$

$$\widehat{E}^\lambda(u) := \frac{1}{2}\|\nabla_x u\|_{L_x^2}^2 + \frac{\mu}{p+2}\|u\|_{L_x^{p+2}}^{p+2} - \frac{\lambda^{\frac{q}{p}-1}}{q+2}\|u\|_{L_x^{q+2}}^{q+2}, \tag{1.22}$$

$$\widehat{S}_{1,\lambda}(u) := \frac{1}{2}\widehat{M}(u) + \widehat{E}_\lambda(u), \quad \widehat{S}_1^\lambda(u) := \frac{1}{2}\widehat{M}(u) + \widehat{E}^\lambda(u), \tag{1.23}$$

$$\widehat{Q}_\lambda(u) := \|\nabla_x u\|_{L_x^2}^2 + \frac{\mu\lambda^{\frac{p}{q}-1}pd}{2(p+2)}\|u\|_{L_x^{p+2}}^{p+2} - \frac{qd}{2(q+2)}\|u\|_{L_x^{q+2}}^{q+2}, \tag{1.24}$$

$$\widehat{Q}^\lambda(u) := \|\nabla_x u\|_{L_x^2}^2 + \frac{\mu pd}{2(p+2)}\|u\|_{L_x^{p+2}}^{p+2} - \frac{\lambda^{\frac{q}{p}-1}qd}{2(q+2)}\|u\|_{L_x^{q+2}}^{q+2}, \tag{1.25}$$

$$\widehat{I}^\lambda(u) := \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u\|_{L_x^2}^2 + \frac{\lambda^{\frac{q}{p}-1}}{q+2}\Big(\frac{q}{p} - 1\Big)\|u\|_{L_x^{q+2}}^{q+2}. \tag{1.26}$$

As convention, the number $\infty^{\frac{p}{q}-1}$ is defined as zero. We also define the sets

$$V(c) := \{u \in S(c) : Q(u) = 0\}, \tag{1.27}$$

(1.28) $$S(c) := \{u \in H^1_{x,y} : M(u) = c\}, \quad \widehat{S}(c) := \{u \in H^1_x : \widehat{M}(u) = c\}$$

(1.29) $$V_\lambda(c) := \{u \in S(c) : Q_\lambda(u) = 0\}, \quad V^\lambda(c) := \{u \in S(c) : Q^\lambda(u) = 0\},$$

(1.30) $$\widehat{V}_\lambda(c) := \{u \in \widehat{S}(c) : \widehat{Q}_\lambda(u) = 0\}, \quad \widehat{V}^\lambda(c) := \{u \in \widehat{S}(c) : \widehat{Q}^\lambda(u) = 0\}.$$

and the variational problems

(1.31) $$m_{c,\lambda} := \inf\{E_\lambda(u) : u \in V_\lambda(c)\}, \qquad m_c^\lambda := \inf\{E^\lambda(u) : u \in V^\lambda(c)\},$$

(1.32) $$\gamma_{1,\lambda} := \inf\{S_{1,\lambda}(u) : u \in H^1_{x,y},\ Q_\lambda(u) = 0\}, \quad \gamma_1^\lambda := \inf\{S_1^\lambda(u) : u \in H^1_{x,y},\ Q^\lambda(u) = 0\},$$

(1.33) $$\widehat{m}_{c,\lambda} := \inf\{\widehat{E}_\lambda(u) : u \in \widehat{V}_\lambda(c)\}, \qquad \widehat{m}_c^\lambda := \inf\{\widehat{E}^\lambda(u) : u \in \widehat{V}^\lambda(c)\},$$

(1.34) $$\widehat{\gamma}_{1,\lambda} := \inf\{\widehat{S}_{1,\lambda}(u) : u \in H^1_x,\ \widehat{Q}_\lambda(u) = 0\}, \quad \widehat{\gamma}_1^\lambda := \inf\{\widehat{S}_1^\lambda(u) : u \in H^1_x,\ \widehat{Q}^\lambda(u) = 0\}.$$

Finally, for a function $u \in H^1_{x,y}$, the scaling operator $u \mapsto u^t$ for $t \in (0,\infty)$ is defined by

(1.35) $$u^t(x,y) := t^{\frac{d}{2}} u(tx,y).$$

The following useful results for the variational problem $\widehat{m}_{c,0}$ are stated in order. For a proof, see e.g. [9, 24, 3, 2].

Lemma 1.11. *The following statements hold true:*

(i) *For any $c > 0$ the variational problem $\widehat{m}_{c,\infty}$ has an optimizer $P_c \in \widehat{S}(c)$. Moreover, P_c satisfies the standing wave equation*

(1.36) $$-\Delta_x P_c + \omega_c P_c = |P_c|^q P_c$$

with some $\omega_c > 0$.

(ii) *Any solution $P_c \in H^1_x$ of (1.36) with $\omega_c > 0$ is of class $W^{3,p}(\mathbb{R}^d)$ for all $p \in [2,\infty)$.*

(iii) *Any solution $P_c \in H^1_x$ of (1.36) with $\omega_c \geq 0$ satisfies $\widehat{Q}_\infty(P_c) = 0$.*

(iv) *The mapping $c \mapsto \widehat{m}_{c,0}$ is strictly monotone decreasing and continuous on $(0,\infty)$.*

2. Existence of normalized ground states: Proof of Theorem 1.1

2.1. **Some useful auxiliary lemmas.** Before proving Theorem 1.1, we collect firstly some useful auxiliary lemmas which will be used throughout the paper.

Lemma 2.1 (Concentration compactness, [42])**.** *Let $(u_n)_n$ be a bounded sequence in $H^1_{x,y}$. Assume also that there exists some $\alpha \in (0, \frac{4}{d-1})$ such that*

(2.1) $$\liminf_{n\to\infty} \|u_n\|_{\alpha+2} > 0.$$

Then there exist $(x_n)_n \subset \mathbb{R}^d$ and some $u \in H^1_{x,y} \setminus \{0\}$ such that up to a subsequence

(2.2) $$u_n(x + x_n, y) \rightharpoonup u(x,y) \quad \text{weakly in } H^1_{x,y}.$$

Lemma 2.2 (Scale-invariant Gagliardo-Nirenberg inequality on $\mathbb{R}^d \times \mathbb{T}$, [29])**.** *For $\alpha \in (\frac{4}{d}, \frac{4}{d-1})$ the following inequality holds for all $u \in H^1_{x,y}$:*

(2.3) $$\|u\|_{\alpha+2}^{\alpha+2} \lesssim \|\nabla_x u\|_2^{\frac{\alpha d}{2}} \|u\|_2^{\frac{4-\alpha(d-1)}{2}} (\|u\|_2^{\frac{\alpha}{2}} + \|\partial_y u\|_2^{\frac{\alpha}{2}})$$

As an immediate consequence of Lemma 2.2, we deduce the following useful properties of a minimizing sequence of the variational problem m_c.

Corollary 2.3. *For any $c \in (0,\infty)$ we have $m_c \in (0,\infty)$. Moreover, there exists a bounded sequence $(u_n)_n \subset V(c)$ (recall the definition of $V(c)$ in (1.27)) such that $m_c = E(u_n) + o_n(1)$ and $\liminf_{n\to\infty} \|u_n\|_{q+2} > 0$.*

Proof. That $m_c < \infty$ follows from $V(c) \neq \varnothing$, the latter being deduced from the scaling properties of the mapping $t \mapsto Q(u^t)$ given in Lemma 2.4 below. Next, let $(u_n)_n \subset V(c)$ be a minimizing sequence such that $E(u_n) = m_c + o_n(1)$. Then

$$\begin{aligned}\infty > m_c + o_n(1) = E(u_n) &= E(u_n) - \frac{2}{pd} Q(u_n) \\ &= \frac{1}{2}\|\partial_y u_n\|_2^2 + \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_n\|_2^2 + \Big(\frac{q}{p} - 1\Big)\frac{1}{q+2}\|u_n\|_{q+2}^{q+2}.\end{aligned}$$

Combining $q > p > 4/d$, which in turn implies $\min\{\frac{1}{2} - \frac{2}{pd}, \frac{q}{p} - 1\} > 0$, we infer that $(u_n)_n$ is a bounded sequence in $H^1_{x,y}$. Now by Lemma 2.2 and the fact that $p < q < 4/(d-1)$ we deduce

$$\begin{aligned}
\|\nabla_x u_n\|_2^2 &= \frac{pd}{2(p+2)}\|u_n\|_{p+2}^{p+2} + \frac{qd}{2(q+2)}\|u_n\|_{q+2}^{q+2} \\
&\lesssim \|\nabla_x u_n\|_2^{\frac{pd}{2}} \|u_n\|_2^{\frac{4-p(d-1)}{2}} (\|u_n\|_2^{\frac{p}{2}} + \|\partial_y u_n\|_2^{\frac{p}{2}}) \\
&\quad + \|\nabla_x u_n\|_2^{\frac{qd}{2}} \|u_n\|_2^{\frac{4-q(d-1)}{2}} (\|u_n\|_2^{\frac{q}{2}} + \|\partial_y u_n\|_2^{\frac{q}{2}}) \\
&\lesssim \|\nabla_x u_n\|_2^{\frac{pd}{2}} + \|\nabla_x u_n\|_2^{\frac{qd}{2}},
\end{aligned}$$

which combining $q > p > 4/d$ implies

$$\liminf_{n\to\infty}(\|u_n\|_{p+2}^{p+2} + \|u_n\|_{q+2}^{q+2}) \sim \liminf_{n\to\infty} \|\nabla_x u_n\|_2^2 > 0.$$

Using also interpolation we know that there exists some $\theta \in (0,1)$ such that

$$\begin{aligned}
1 &\lesssim \|u_n\|_{p+2}^{p+2} + \|u_n\|_{q+2}^{q+2} \lesssim \|u_n\|_{q+2}^{q+2} + \|u_n\|_2^{(p+2)(1-\theta)} \|u_n\|_{q+2}^{(p+2)\theta} \\
&\lesssim \|u_n\|_{q+2}^{(p+2)\theta}(1 + \|u_n\|_{q+2}^{(q+2)-(p+2)\theta}) \lesssim \|u_n\|_{q+2}^{(p+2)\theta}.
\end{aligned}$$

Thus

$$\liminf_{n\to\infty} \|u_n\|_{q+2} > 0. \tag{2.4}$$

Summing up, we obtain

$$\begin{aligned}
m_c &= \lim_{n\to\infty} \Big(\frac{1}{2}\|\partial_y u_n\|_2^2 + \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_n\|_2^2 + \Big(\frac{q}{p} - 1\Big)\frac{1}{q+2}\|u_n\|_{q+2}^{q+2}\Big) \\
&\gtrsim \liminf_{n\to\infty} \|\nabla_x u_n\|_{q+2}^{q+2} \gtrsim 1,
\end{aligned}$$

which completes the proof. □

2.2. **Dynamical properties of the mappings** $t \mapsto Q(u^t)$ **and** $c \mapsto m_c$**.** We state in this subsection the dynamical properties of the mappings $t \mapsto Q(u^t)$ and $c \mapsto m_c$, which will play a crucial role in the upcoming proofs.

Lemma 2.4 (Property of the mapping $t \mapsto Q(u^t)$)**.** *Let $c > 0$ and $u \in S(c)$. Then the following statements hold true:*

(i) $\frac{\partial}{\partial t}E(u^t) = t^{-1}Q(u^t)$ *for all* $t > 0$.
(ii) *There exists some $t^* = t^*(u) > 0$ such that $u^{t^*} \in V(c)$.*
(iii) *We have $t^* < 1$ if and only if $Q(u) < 0$. Moreover, $t^* = 1$ if and only if $Q(u) = 0$.*
(iv) *Following inequalities hold:*

$$Q(u^t)\begin{cases} > 0, & t \in (0, t^*), \\ < 0, & t \in (t^*, \infty). \end{cases}$$

(v) $E(u^t) < E(u^{t^*})$ *for all $t > 0$ with $t \neq t^*$.*

Proof. (i) follows from direct calculation. Now define $y(t) := \frac{\partial}{\partial t}E(u^t)$. Then

$$\begin{aligned}
y(t) &= t\|\nabla_x u\|_2^2 - \frac{pd}{2(p+2)}t^{\frac{pd}{2}-1}\|u\|_{p+2}^{p+2} - \frac{qd}{2(q+2)}t^{\frac{qd}{2}-1}\|u\|_{q+2}^{q+2}, \\
y'(t) &= \|\nabla_x u\|_2^2 - \frac{2pd(pd-2)}{4(p+2)}t^{\frac{pd}{2}-2}\|u\|_{p+2}^{p+2} - \frac{2qd(qd-2)}{4(q+2)}t^{\frac{qd}{2}-2}\|u\|_{q+2}^{q+2}.
\end{aligned}$$

Using $q > p > 4/d$ we infer that $y'(0) = \|\nabla_x u\|_2^2 > 0$, $y'(t) \to -\infty$ as $t \to \infty$ and $y'(t)$ is strictly monotone decreasing on $(0,\infty)$. Thus there exists some $t_0 > 0$ such that $y'(t)$ is positive on $(0, t_0)$ and negative on (t_0, ∞). Consequently, we conclude that $y(t)$ has a zero at $t^* > t_0$, $y(t)$ is positive on $(0, t^*)$ and negative on (t^*, ∞). (ii) and (iv) now follow from the fact

$$y(t) = \frac{\partial E(u^t)}{\partial t} = \frac{Q(u^t)}{t}.$$

For (iii), assume first $Q(u) < 0$. Then

$$0 > Q(u) = \frac{Q(u^1)}{1} = y(1),$$

which is only possible as long as $t^* < 1$. Conversely, let $t^* < 1$. Then using the fact that $y(t)$ is monotone decreasing on (t^*, ∞) we obtain

$$Q(u) = y(1) < y(t^*) < 0.$$

This completes the proof of (iii). To see (v), integration by parts yields

$$E(u^{t^*}) = E(u^t) + \int_t^{t^*} y(s)\,ds.$$

Then (v) follows from the fact that $y(t)$ is positive on $(0, t^*)$ and $y(t)$ is negative on (t^*, ∞). □

Lemma 2.5 (Property of the mapping $c \mapsto m_c$). *The mapping $c \mapsto m_c$ is lower semicontinuous and monotone decreasing on* $(0, \infty)$.

Proof. Define the functions f and g by

$$f(a, b, c) := \max_{t>0}\{at^2 - bt^{\frac{pd}{2}} - ct^{\frac{qd}{2}}\} =: \max_{t>0} g(t, a, b, c).$$

We first infer the continuity of the function f on $(0, \infty)^3$. By direct calculation it is easy to verify that for given $a, b, c > 0$ there exists a unique $t_0 \in (0, \infty)$ such that $\partial_t g(t_0, a, b, c) = 0$ and $\partial_t^2 g(t_0, a, b, c) < 0$, hence $f(a, b, c) = g(t_0, a, b, c)$. Thus for given $(a_0, b_0, c_0) \in (0, \infty)^3$, using the implicit function theorem we may interpret the function f as

$$f(a, b, c) = g(h(a, b, c), a, b, c)$$

with some continuous function h for points (a, b, c) lying in a neighborhood of (a_0, b_0, c_0). This in turn proves the continuity of the function f.

We now show the monotonicity of $c \mapsto m_c$. It suffices to show that for any $0 < c_1 < c_2 < \infty$ and $\varepsilon > 0$ we have

$$m_{c_2} \le m_{c_1} + \varepsilon.$$

By the definition of m_{c_1} there exists some $u_1 \in V(c_1)$ such that

$$E(u_1) \le m_{c_1} + \frac{\varepsilon}{2}. \tag{2.5}$$

Let $\eta \in C_c^\infty(\mathbb{R}^d; [0, 1])$ be a cut-off function such that $\eta = 1$ for $|x| \le 1$ and $\eta = 0$ for $|x| \ge 2$. For $\delta > 0$, define

$$\tilde{u}_{1,\delta}(x, y) := \eta(\delta x) \cdot u_1(x, y).$$

Using dominated convergence theorem it is easy to verify that $\tilde{u}_{1,\delta} \to u_1$ in $H^1_{x,y}$ as $\delta \to 0$. Therefore,

$$\|\nabla_{x,y}\tilde{u}_{1,\delta}\|_2 \to \|\nabla_{x,y}u_1\|_2,$$
$$\|\tilde{u}_{1,\delta}\|_p \to \|u_1\|_p$$

for all $p \in [2, 2 + \frac{4}{d-1})$ as $\delta \to 0$. Combining the continuity of f we conclude that

$$\begin{aligned}
\max_{t>0} E(\tilde{u}^t_{1,\delta}) &= \max_{t>0}\Big\{\frac{t^2}{2}\|\nabla_x \tilde{u}_{1,\delta}\|_2^2 - \frac{t^{\frac{pd}{2}}}{p+2}\|\tilde{u}_{1,\delta}\|_{p+2}^{p+2} - \frac{t^{\frac{qd}{2}}}{q+2}\|\tilde{u}_{1,\delta}\|_{q+2}^{q+2}\Big\} + \frac{1}{2}\|\partial_y \tilde{u}_{1,\delta}\|_2^2 \\
&\le \max_{t>0}\Big\{\frac{t^2}{2}\|\nabla_x u_1\|_2^2 - \frac{t^{\frac{pd}{2}}}{p+2}\|u_1\|_{p+2}^{p+2} - \frac{t^{\frac{qd}{2}}}{q+2}\|u_1\|_{q+2}^{q+2}\Big\} + \frac{1}{2}\|\partial_y u_1\|_2^2 + \frac{\varepsilon}{4} \\
&= \max_{t>0} E(u_1^t) + \frac{\varepsilon}{4}
\end{aligned} \tag{2.6}$$

for sufficiently small $\delta > 0$. Now let $v \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} v \subset B(0, 4\delta^{-1} + 1)\backslash B(0, 4\delta^{-1})$ and define

$$v_0 := \frac{(c_2 - M(\tilde{u}_{1,\delta}))^{\frac{1}{2}}}{M(v)^{\frac{1}{2}}}\, v.$$

Notice that v_0 and $\tilde{u}_{1,\delta}$ have compact supports, which also implies $M(v_0) = c_2 - M(\tilde{u}_{1,\delta})$. Define

$$w_\lambda := \tilde{u}_{1,\delta} + v_0^\lambda$$

with some to be determined $\lambda > 0$. Then

$$\|w_\lambda\|_p^p = \|\tilde{u}_{1,\delta}\|_p^p + \|v_0^\lambda\|_p^p$$

for all $p \in [2, 2+\frac{4}{d-1})$. Particularly, $M(w_\lambda) = c_2$. Since v_0 is independent of $y \in \mathbb{T}$, we also infer that

$$\|\nabla_x w_\lambda\|_2 \to \|\nabla_x \tilde{u}_{1,\delta}\|_2,$$
$$\|\partial_y w_\lambda\|_2 = \|\partial_y \tilde{u}_{1,\delta}\|_2,$$
$$\|w_\lambda\|_p \to \|\tilde{u}_{1,\delta}\|_p$$

for all $p \in (2, 2+\frac{4}{d-1})$ as $\lambda \to 0$. Using the continuity of f once again we obtain

$$\max_{t>0} E(w_\lambda^t) \le \max_{t>0} E(\tilde{u}_{1,\delta}^t) + \frac{\varepsilon}{4}$$

for sufficiently small $\lambda > 0$. Finally, combing (2.5) and (2.6) we infer that

$$m_{c_2} \le \max_{t>0} E(w_\lambda^t) \le \max_{t>0} E(\tilde{u}_{1,\delta}^t) + \frac{\varepsilon}{4} \le \max_{t>0} E(u_1^t) + \frac{\varepsilon}{2} = E(u_1) + \frac{\varepsilon}{2} \le m_{c_1} + \varepsilon,$$

which implies the monotonicity of $c \mapsto m_c$ on $(0, \infty)$.

Next, we show the lower semicontinuity of the curve $c \mapsto m_c$. Since $c \mapsto m_c$ is non-increasing, it suffices to show that for any $c \in (0,\infty)$ and any sequence $c_n \downarrow c$ we have

$$m_c \le \lim_{n\to\infty} m_{c_n}.$$

Let $\varepsilon > 0$ be an arbitrary positive number. By the definition of m_{c_n} we can find some $u_n \in V(c_n)$ such that

$$E(u_n) \le m_{c_n} + \frac{\varepsilon}{2} \le m_c + \frac{\varepsilon}{2}. \tag{2.7}$$

We define $\tilde{u}_n = (c_n^{-1}c)^{\frac{1}{2}} \cdot u_n := \rho_n u_n$. Then $M(\tilde{u}_n) = c$ and $\rho_n \uparrow 1$. Since $u_n \in V(c_n)$, we obtain

$$\begin{aligned} m_c + \frac{\varepsilon}{2} &\ge m_{c_n} + \frac{\varepsilon}{2} \ge E(u_n) = E(u_n) - \frac{2}{pd} Q(u_n) \\ &= \frac{1}{2}\|\partial_y u_n\|_2^2 + \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_n\|_2^2 + \Big(\frac{q}{p} - 1\Big)\frac{1}{q+2}\|u_n\|_{q+2}^{q+2}. \end{aligned}$$

Thus $(u_n)_n$ is bounded in $H^1_{x,y}$ and up to a subsequence we infer that there exist $A, B, C \ge 0$ such that

$$\|\nabla_x u_n\|_2^2 = A + o_n(1), \quad \|\partial_y u_n\|_2^2 = B + o_n(1), \quad \|u_n\|_{\alpha+2}^{\alpha+2} = C_\alpha + o_n(1)$$

with $\alpha \in \{p, q\}$. Arguing as in the proof of Corollary 2.3, we may use the fact $Q(u_n) = 0$ and Lemma 2.2 to deduce $A, C_\alpha > 0$ and by previous arguments we know that f is continuous at the point (A, C_p, C_q). Using also the fact that $\rho_n \uparrow 1$ we conclude that

$$\begin{aligned} m_c &\le \max_{t>0} E(\tilde{u}_n^t) = \max_{t>0}\Big\{\frac{t^2\rho_n^2}{2}\|\nabla_x u_n\|_2^2 - \sum_{\alpha\in\{p,q\}} \frac{t^{\frac{\alpha d}{2}}\rho_n^{\alpha+2}}{\alpha+2}\|u_n\|_{\alpha+2}^{\alpha+2}\Big\} + \frac{\rho_n^2}{2}\|\partial_y u_n\|_2^2 \\ &\le \max_{t>0}\Big\{\frac{t^2 A}{2} - \sum_{\alpha\in\{p,q\}} \frac{t^{\frac{\alpha d}{2}} C_\alpha}{\alpha+2}\Big\} + \frac{1}{2}\|\partial_y u_n\|_2^2 + \frac{\varepsilon}{4} \\ &\le \max_{t>0}\Big\{\frac{t^2}{2}\|\nabla_x u_n\|_2^2 - \sum_{\alpha\in\{p,q\}} \frac{t^{\frac{\alpha d}{2}}}{\alpha+2}\|u_n\|_{\alpha+2}^{\alpha+2}\Big\} + \frac{1}{2}\|\partial_y u_n\|_2^2 + \frac{\varepsilon}{2} \\ &= \max_{t>0} E(u_n^t) + \frac{\varepsilon}{2} = E(u_n) + \frac{\varepsilon}{2} \le m_{c_n} + \varepsilon \end{aligned}$$

by choosing n sufficiently large. The continuity claim follows from the arbitrariness of ε. □

2.3. **Mountain pass geometry of $E(u)$ on $S(c)$.** As already mentioned in the introductory section, in the waveguide setting, it is *a priori* unclear whether the critical points of the variational problem m_c correspond to an elliptic problem, leading to possible failure of applying the Pohozaev's identity to infer that a critical point of m_c is also a solution of the stationary equation (1.2). We shall invoke a deformation argument in [3] to solve this issue.

We begin with defining the mountain pass geometry of $E(u)$ on $S(c)$

Definiton 2.6 (Mountain pass geometry of $E(u)$ on $S(c)$)**.** *We say that $E(u)$ has a mountain pass geometry on $S(c)$ at the level γ_c if there exists some $k > 0$ and $\varepsilon \in (0, m_c)$ such that*

$$\gamma_c := \inf_{g\in\Gamma(c)} \max_{t\in[0,1]} E(g(t)) > \max\{\sup_{g\in\Gamma(c)} E(g(0)), \sup_{g\in\Gamma(c)} E(g(1))\}, \tag{2.8}$$

where

$$\Gamma(c) := \{g \in C([0,1]; S(c)) : g(0) \in A_{k,\varepsilon}, E(g(1)) \leq 0\}$$

and

$$A_{k,\varepsilon} := \{u \in S(c) : \|\nabla_x u\|_2^2 \leq k, \|\partial_y u\|_2^2 \leq 2(m_c - \varepsilon)\}.$$

The following lemma establishes the fact that m_c characterizes the mountain pass level of $E(u)$ on $S(c)$.

Lemma 2.7. *There exist $k > 0$ and $\varepsilon \in (0, m_c)$ such that*

(i) *$m_c = \gamma_c$ holds.*
(ii) *$E(u)$ has a mountain pass geometry on $S(c)$ at the level m_c in the sense of Definition 2.6.*

Proof. We firstly prove that by choosing k sufficiently small we have $Q(u) > 0$ for all $u \in A_{k,\varepsilon}$, where k is independent of the choice of $\varepsilon \in (0, m_c)$. Indeed, by Lemma 2.2, the fact that $M(u) = c$, $\|\partial_y u\|_2^2 < 2m_c$ for $u \in A_{k,\varepsilon}$ and $q > p > 4/d$ we obtain

$$\begin{aligned} Q(u) &= \frac{1}{2}\|\nabla_x u\|_2^2 - \frac{pd}{2(p+2)}\|u\|_{p+2}^{p+2} - \frac{qd}{2(q+2)}\|u\|_{q+2}^{q+2} \\ &\geq \frac{1}{2}\|\nabla_x u\|_2^2 - C(\|\nabla_x u\|_2^{\frac{pd}{2}} + \|\nabla_x u\|_2^{\frac{qd}{2}}) > 0 \end{aligned}$$

as long as $\|\nabla_x u\|_2^2 \in (0, k)$ for some sufficiently small k. Next we construct the number ε. Arguing as in (2.4) we know that there exists some $\beta = \beta(c) > 0$ such that if $(u_n)_n \subset V(c)$ is a minimizing sequence for m_c, then

$$\liminf_{n\to\infty} \Big(\Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_n\|_2^2\Big) \geq \beta. \tag{2.9}$$

We may shrink β further such that $\beta < 4m_c$. Hence for any minimizing sequence $(u_n)_n$ we must have

$$\begin{aligned} m_c + \frac{\beta}{4} &\geq \frac{1}{2}\|\partial_y u_n\|_2^2 + \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_n\|_2^2 + \Big(\frac{q}{p} - 1\Big)\frac{1}{q+2}\|u_n\|_{q+2}^{q+2} \\ &> \frac{1}{2}\|\partial_y u_n\|_2^2 + \frac{\beta}{2} \end{aligned} \tag{2.10}$$

for all sufficiently large n. Now set $\varepsilon = \frac{\beta}{4}$. For $u \in A_{k,\varepsilon}$, using also Lemma 2.2 we infer that

$$\begin{aligned} E(u) &\leq \frac{1}{2}\|\partial_y u\|_2^2 + \frac{1}{2}\|\nabla_x u\|_2^2 + C(\|\nabla_x u\|_2^{\frac{pd}{2}} + \|\nabla_x u\|_2^{\frac{qd}{2}}) \\ &\leq m_c - \varepsilon + \Big(\frac{1}{2}k + Ck^{\frac{pd}{4}} + Ck^{\frac{qd}{4}}\Big). \end{aligned}$$

We now choose $k = k(\varepsilon)$ sufficiently small (without changing the fact that $Q(u) > 0$ for $u \in A_{k,\varepsilon}$) such that $\frac{1}{2}k + Ck^{\frac{\alpha d}{4}} < \frac{\varepsilon}{2}$. For this choice of k and ε we have $E(u) < m_c - \varepsilon/2 < m_c$ for all $u \in A_{k,\varepsilon}$ and by definition, (ii) follows immediately from (i).

It is left to show (i). Let $(u_n)_n$ be the given minimizing sequence satisfying (2.9). Let $u = u_n$ for some (to be determined) sufficiently large $n \in \mathbb{N}$. For any $\kappa \in (0, \beta/4)$ we can choose n sufficiently large such that $E(u) \leq m_c + \kappa$ and $(\frac{1}{2} - \frac{2}{pd})\|\nabla_x u\|_2^2 \geq \beta/2$. Then by (2.10) we know that $\|\partial_y u\|_2^2 \leq 2(m_c - \varepsilon)$ for all $\kappa \in (0, \beta/4)$. It is easy to check that $\|\partial_y(u^t)\|_2^2 = \|\partial_y u\|_2^2$ for all $t \in (0,\infty)$ and $\|\nabla_x(u^t)\|_2^2 = t^2\|\nabla_x u\|_2^2 \to 0$ as $t \to 0$. We then fix some $t_0 > 0$ sufficiently small such that $\|\nabla_x(u^{t_0})\|_2^2 < k$, which in turn implies that $(u^{t_0}) \in A_{k,\varepsilon}$. On the other hand,

$$E(u^t) = \frac{1}{2}\|\partial_y u\|_2^2 + \frac{t^2}{2}\|u\|_2^2 - \frac{t^{\frac{pd}{2}}}{p+2}\|u\|_{p+2}^{p+2} - \frac{t^{\frac{qd}{2}}}{q+2}\|u\|_{q+2}^{q+2} \to -\infty$$

as $t \to \infty$. We then fix some t_1 sufficiently large such that $E(u^{t_1}) < 0$. Now define $g \in C([0,1]; S(c))$ by

$$g(t) := u^{t_0 + (t_1 - t_0)t}. \tag{2.11}$$

Then $g \in \Gamma(c)$. By definition of γ_c and Lemma 2.4 we have

$$\gamma_c \leq \max_{t\in[0,1]} E(g(t)) = E(u) \leq m_c + \kappa.$$

Since κ can be chosen arbitrarily small, we conclude that $\gamma_c \leq m_c$. On the other hand, by our choice of k we already know that for any $g \in \Gamma(c)$ we have $Q(g(0)) > 0$. We now prove $Q(g(1)) < 0$ for any $g \in \Gamma(c)$. Assume the contrary that there exists some $g \in \Gamma(c)$ such that $Q(g(1)) \geq 0$. Then

$$\begin{aligned} 0 > E(g(1)) &\geq \frac{1}{2}\|\nabla_x g(1)\|_2^2 - \frac{1}{p+2}\|g(1)\|_{p+2}^{p+2} - \frac{1}{q+2}\|g(1)\|_{q+2}^{q+2} \\ &\geq \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x g(1)\|_2^2 + \Big(\frac{q}{p} - 1\Big)\frac{1}{q+2}\|g(1)\|_{p+2}^{p+2} \geq 0, \end{aligned}$$

a contradiction. Next, by continuity of g there exists some $t \in (0,1)$ such that $Q(g(t)) = 0$. Therefore

$$\max_{t \in [0,1]} E(g(t)) \geq m_c.$$

Taking infimum over $g \in \Gamma(c)$ we deduce $\gamma_c \geq m_c$, which completes the desired proof. □

Remark 2.8. By technical reason we will also shrink ε in the Definition 2.6 if necessary such that $\varepsilon \leq 1 - \frac{1}{2c_{p,q}}$, where

$$c_{p,q} := \frac{1}{2} + \frac{1}{2}\Big(d - \frac{4}{p}\Big)\Big(\frac{2q}{p} + \frac{4}{p} - d\Big)^{-1}.$$

The purpose of this choice of ε will become clear in the upcoming proof of Lemma 2.10. Notice also that $1 - \frac{1}{2c_{p,q}} \in (0,1)$ is equivalent to $c_{p,q} > \frac{1}{2}$, which by using $p > 4/d$ is also equivalent to $2q/p > d - 4/p$. However, this is always satisfied since $p < q < 4(d-1)^{-1}$. △

2.4. **Characterization of an optimizer as a standing wave equation.** We now prove that a minimizer of m_c is also a solution of the stationary equation (1.2). First we prove a different characterization of m_c that will be more useful in later analysis.

Lemma 2.9. *For $c > 0$, define*

$$\tilde{m}_c := \inf\{I(u) : u \in S(c),\, Q(u) \leq 0\}, \tag{2.12}$$

where $I(u)$ is defined by (1.11)*. Then $m_c = \tilde{m}_c$.*

Proof. Let $(u_n)_n \subset S(c)$ be a minimizing sequence for the variational problem $\tilde{m}_c$, i.e.

$$I(u_n) = \tilde{m}_c + o_n(1), \quad Q(u_n) \leq 0 \quad \forall\, n \in \mathbb{N}. \tag{2.13}$$

By Lemma 2.4 we know that there exists some $t_n \in (0,1]$ such that $Q(u_n^{t_n})$ is equal to zero. Thus

$$m_c \leq E(u_n^{t_n}) = I(u_n^{t_n}) \leq I(u_n) = \tilde{m}_c + o_n(1).$$

Sending $n \to \infty$ we infer that $m_c \leq \tilde{m}_c$. On the other hand,

$$\tilde{m}_c \leq \inf\{I(u) : u \in V(c)\} = \inf\{E(u) : u \in V(c)\} = m_c,$$

which completes the proof. □

We shall still need to introduce the following preliminary concepts given in [5]. First recall that $S(c)$ is a submanifold of $H^1_{x,y}$ with codimension 1. Moreover, the tangent space $T_u S(c)$ for a point $u \in S(c)$ is given by

$$T_u S(c) = \{v \in H^1_{x,y} : \langle u, v\rangle_{L^2_{x,y}} = 0\}.$$

Denote the tangent bundle of $S(c)$ by $TS(c)$. Next, the energy functional $E|_{S(c)}$ restricted to $S(c)$ is a C^1-functional on $S(c)$ and for any $u \in S(c)$ and $v \in T_u S(c)$ we have

$$\langle E'|_{S(c)}(u), v\rangle = \langle E'(u), v\rangle.$$

We use $\|E'|_{S(c)}(u)\|_*$ to denote the dual norm of $E'|_{S(c)}(u)$ in the cotangent space $(T_u S(c))^*$, i.e.

$$\|E'|_{S(c)}(u)\|_* := \sup_{v \in T_u S(c),\, \|v\|_{H^1_{x,y}} \leq 1} |\langle E'|_{S(c)}(u), v\rangle|.$$

Let now

$$\tilde{S}(c) := \{u \in S(c) : E'|_{S(c)}(u) \neq 0\}.$$

According to [5, Lem. 4] there exists a locally Lipschitz pseudo gradient vector field $Y : \tilde{S}(c) \to TS(c)$ such that $Y(u) \in T_u S(c)$ and

$$\|Y(u)\|_{H^1_{x,y}} \leq 2\|E'|_{S(c)}(u)\|_* \quad \text{and} \quad \langle E'|_{S(c)}(u), Y(u)\rangle \geq \|E'|_{S(c)}(u)\|_*^2 \tag{2.14}$$

for $u \in \tilde{S}(c)$.

Having all the preliminaries we are ready to prove the claimed statement.

Lemma 2.10. *For any $c>0$ an optimizer u_c of m_c is a solution of* (1.2) *for some $\omega \in \mathbb{R}$.*

Proof. We borrow an idea from the proof of [3, Lem. 6.1] to show the claim. By Lagrange multiplier theorem we know that u_c solves (1.2) is equivalent to $E'|_{S(c)}(u)=0$. We hence assume the contrary $\|E'|_{S(c)}(u)\|_* \neq 0$, which implies that there exists some $\delta>0$ and $\mu>0$ such that

$$v \in B_{u_c}(3\delta) \Rightarrow \|E'|_{S(c)}(v)\|_* \geq \mu, \tag{2.15}$$

where $B_{u_c}(\delta) := \{v \in S(c) : \|u-v\|_{H^1_{x,y}} \leq \delta\}$. Let k and ε be given according to Lemma 2.7 and Remark 2.8. Define

$$\begin{aligned}\varepsilon_1 &:= \frac{1}{4}\Big(m_c - \max\{\sup_{g\in\Gamma(c)} E(g(0)), \sup_{g\in\Gamma(c)} E(g(1))\}\Big),\\ \varepsilon_2 &:= \min\{\varepsilon_1, m_c/4, \mu\delta/4\}.\end{aligned}$$

We now define the deformation mapping η as follows: Let the sets A, B and function $h : S(c) \to [0,\delta]$ be given by

$$\begin{aligned}A &:= S(c) \cap E^{-1}([m_c - 2\varepsilon_2, m_c + 2\varepsilon_2]),\\ B &:= B_{u_c}(2\delta) \cap E^{-1}([m_c - \varepsilon_2, m_c + \varepsilon_2]),\\ h(u) &:= \frac{\delta\, \mathrm{dist}(u, S(c)\setminus A)}{\mathrm{dist}(u, S(c)\setminus A) + \mathrm{dist}(u, B)}.\end{aligned}$$

Next, we define the pseudo gradient flow $W : S(c) \to H^1_{x,y}$ by

$$W(u) := \begin{cases} -h(u)\|Y(u)\|^{-1}_{H^1_{x,y}} Y(u), & \text{if } u \in \tilde{S}(c),\\ 0, & \text{if } u \in S(c)\setminus \tilde{S}(c).\end{cases}$$

One easily verifies that W is a locally Lipschitz function from $S(c)$ to $H^1_{x,y}$. Then by standard arguments (see for instance [5, Lem. 6]) there exists a mapping $\eta : \mathbb{R} \times S(c) \to S(c)$ such that $\eta(1,\cdot) \in C(S(c); S(c))$ and η solves the differential equation

$$\frac{d}{dt}\eta(t,u) = W(\eta(t,u)), \quad \eta(0,u) = u$$

for any $u \in S(c)$. We now claim that η satisfies the following properties:

(i) $\eta(1,v) = v$ if $v \in S(c) \setminus E^{-1}([m_c - 2\varepsilon_2, m_c + 2\varepsilon_2])$.
(ii) $\eta(1, E^{m_c+\varepsilon_2} \cap B_{u_c}(\delta)) \subset E^{m_c - \varepsilon}$.
(iii) $E(\eta(1,v)) \leq E(v)$ for all $v \in S(c)$.

Here, the symbol E^κ denotes the set $E^\kappa := \{v \in S(c) : E(v) \leq \kappa\}$. For (i), by definition we see that $h(v) = 0$, thus $\frac{d}{dt}\eta(t,v)|_{t=0} = W(v) = 0$ and $\eta(t,v) \equiv \eta(0,v) = v$. For (iii), using (2.14) and the non-negativity of h we obtain

$$\begin{aligned}E(\eta(1,v)) &= E(v) + \int_0^1 \frac{d}{ds} E(\eta(s,v))\,ds\\ &= E(v) - \int_{s\in[0,1],\eta(s,v)\in\tilde{S}(c)} \langle E'(\eta(s,v)), h(\eta(s,v))\|Y(\eta(s,v))\|^{-1}_{H^1_{x,y}} Y(\eta(s,v))\rangle\,ds\\ &\leq E(v) - \frac{1}{2}\int_{s\in[0,1],\eta(s,v)\in\tilde{S}(c)} h(\eta(s,v))\|E'|_{S(c)}(\eta(s,v))\|_*\,ds \leq E(v).\end{aligned}$$

It is left to prove (ii). We first show that for $v \in E^{m_c+\varepsilon_2} \cap B_{u_c}(\delta)$ one has $\eta(t,v) \in B_{u_c}(2\delta)$ for all $t \in [0,1]$. This follows from

$$\|\eta(t,v) - v\|_{H^1_{x,y}} = \|\int_0^t h(v)\|Y(v)\|^{-1}_{H^1_{x,y}} Y(v)\,ds\|_{H^1_{x,y}} \leq t h(v) \leq \delta.$$

By (2.15) this implies particularly that $\|E'|_{S(c)}(\eta(t,v))\|_* \geq \mu$. Consequently, using (2.14), $0 \leq h \leq \delta$ and $\varepsilon_2 \leq \mu\delta/4$ we obtain

$$\begin{aligned}E(\eta(1,v)) &\leq E(v) - \frac{1}{2}\int_{s\in[0,1],\eta(s,v)\in\tilde{S}(c)} h(\eta(s,v))\|E'|_{S(c)}(\eta(s,v))\|_*\,ds\\ &\leq m_c + \varepsilon_2 - \frac{\mu\delta}{2} \leq m_c - \varepsilon_2.\end{aligned}$$

Next, we recall the function g defined by (2.11) by setting $u = u_c$ therein. We claim that there exist $t_0 \ll 1$ and $t_1 \gg 1$ such that $g \in \Gamma(c)$. Indeed, from the proof of Lemma 2.7 it suffices to show $\|\partial_y u_c\|_2^2 \le 2(m_c - \varepsilon)$. Define the scaling operator T_λ by

$$T_\lambda u(x, y) := \lambda^{\frac{2}{p}} u(\lambda x, y). \tag{2.16}$$

Then

$$\begin{aligned}
\|T_\lambda(\nabla_x u)\|_2^2 &= \lambda^{2+\frac{4}{p}-d}\|\nabla_x u\|_2^2,\\
\|T_\lambda u\|_{p+2}^{p+2} &= \lambda^{2+\frac{4}{p}-d}\|u\|_{p+2}^{p+2}, \quad \|T_\lambda u\|_{q+2}^{q+2} = \lambda^{\frac{2q}{p}+\frac{4}{q}-d}\|u\|_{q+2}^{q+2},\\
\|T_\lambda(\partial_y u)\|_2^2 &= \lambda^{\frac{4}{p}-d}\|\partial_y u\|_2^2, \quad \|T_\lambda u\|_2^2 = \lambda^{\frac{4}{p}-d}\|u\|_2^2,\\
Q(T_\lambda u) &= \lambda^{2+\frac{4}{p}-d} Q(u) - \lambda^{2+\frac{4}{p}-d}\|u\|_{q+2}^{q+2}(\lambda^{2(q/p-1)} - 1).
\end{aligned}$$

It thus follows that if $Q(u) = 0$ and $\lambda \ge 1$, then $Q(T_\lambda u) \le 0$, where we also invoked the condition $p < q$. Combining Lemma 2.5, Lemma 2.9 and the fact that u_c is an optimizer of m_c we infer that $\frac{d}{d\lambda} I(T_\lambda u_c)|_{\lambda=1} \ge 0$, or equivalently

$$\begin{aligned}
\|\partial_y u_c\|_2^2 \le 2\Big(d - \frac{4}{p}\Big)^{-1}\Big(&\Big(2 + \frac{4}{p} - d\Big)\Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_c\|_2^2\\
&+ \Big(\frac{2q}{p} + \frac{4}{p} - d\Big)(q+2)^{-1}\Big(\frac{q}{p} - 1\Big)\|u_c\|_{q+2}^{q+2}\Big).
\end{aligned} \tag{2.17}$$

Using $p < q$, (2.17) also implies

$$\|\partial_y u_c\|_2^2 \le 2\Big(d - \frac{4}{p}\Big)^{-1}\Big(\frac{2q}{p} + \frac{4}{p} - d\Big)\Big(\Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_c\|_2^2 + (q+2)^{-1}\Big(\frac{q}{p} - 1\Big)\|u_c\|_{q+2}^{q+2}\Big). \tag{2.18}$$

Hence

$$\begin{aligned}
m_c = E(u_c) &= I(u_c)\\
&= \frac{1}{2}\|\partial_y u_c\|_2^2 + \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_c\|_2^2 + (q+2)^{-1}\Big(\frac{q}{p} - 1\Big)\|u_c\|_{q+2}^{q+2}\\
&\ge c_{p,q}\|\partial_y u_c\|_2^2,
\end{aligned}$$

where the number $c_{p,q}$ is given by Remark 2.8. Using the condition $\varepsilon \le 1 - \frac{1}{2c_{p,q}}$ from Remark 2.8 we obtain

$$\|\partial_y u_c\|_2^2 \le 2m_c\Big(1 - \Big(1 - \frac{1}{2c_{p,q}}\Big)\Big) \le 2m_c(1 - \varepsilon),$$

as desired. By (i) and Lemma 2.7 we know that $\eta(1, g(t)) \in \Gamma(c)$. We shall finally prove

$$\max_{t \in [0,1]} E(\eta(1, g(t))) < m_c,$$

which contradicts the characterization $m_c = \gamma_c$ deduced from Lemma 2.7 and closes the desired proof. Notice by definition of g and Lemma 2.4 we have $E(g(t)) \le E(u_c) = m_c$ for all $t \in [0, 1]$. Thus only the following scenarios can happen:

(a) $g(t) \in S(c) \setminus B_{u_c}(\delta)$. By (iii) and Lemma 2.4 (v) we have

$$E(\eta(1, g(t))) \le E(g(t)) < E(u_c) = m_c.$$

(b) $g(t) \in E^{m_c - \varepsilon_2}$. By (iii) we have

$$E(\eta(1, g(t))) \le E(g(t)) \le m_c - \varepsilon_2 < m_c.$$

(c) $g(t) \in E^{-1}([m_c - \varepsilon_2, m_c + \varepsilon_2]) \cap B_{u_c}(\delta)$. By (ii) we have

$$E(\eta(1, g(t))) \le m_c - \varepsilon_2 < m_c.$$

This completes the desired proof. □

2.5. **Proof of Theorem 1.1 (i).** Having all the preliminaries we are in a position to prove Theorem 1.1 (i).

Proof of Theorem 1.1 (i). We split our proof into three steps.

Step 1: Existence of a non-negative optimizer of m_c. By Lemma 2.9 we consider equivalently the variational problem $\tilde{m}_c$. Let $(u_n)_n \subset S(c)$ be a minimizing sequence of $\tilde{m}_c$ satisfying (2.13). By diamagnetic inequality we know that the variational problem $\tilde{m}_c$ is stable under the mapping $u \mapsto |u|$, thus we may w.l.o.g. assume that $u_n \geq 0$. By Corollary 2.3 and Lemma 2.9 we know that $\tilde{m}_c < \infty$, hence

$$\infty > \tilde{m}_c \gtrsim I(u_n) = \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_n\|_2^2 + \frac{1}{2}\|\partial_y u_n\|_2^2 + \frac{1}{q+2}\Big(\frac{q}{p} - 1\Big)\|u_n\|_{q+2}^{q+2}.$$

Combining $Q(u_n) \leq 0$ and $(u_n) \subset S(c)$ we conclude that $(u_n)_n$ is a bounded sequence in $H^1_{x,y}$. Now using (2.4) and Lemma 2.1 we may find some $H^1_{x,y} \setminus \{0\} \ni u \geq 0$ such that $u_n \rightharpoonup u$ weakly in $H^1_{x,y}$. By weakly lower semicontinuity of norms we deduce

$$M(u) =: c_1 \in (0, c], \quad I(u) \leq \tilde{m}_c. \tag{2.19}$$

We next show $Q(u) \leq 0$. Assume the contrary $Q(u) > 0$. By Brezis-Lieb lemma, $Q(u_n) \leq 0$ and the fact that $L^2_{x,y}$ is a Hilbert space we infer that

$$\begin{aligned} M(u_n - u) &= c - c_1 + o_n(1), \\ Q(u_n - u) &\leq -Q(u) + o_n(1). \end{aligned}$$

Therefore, for all sufficiently large n we know that $M(u_n - u) \in (0, c)$ and $Q(u_n - u) < 0$. By Lemma 2.4 we know that there exists some $t_n \in (0, 1)$ such that $Q((u_n - u)^{t_n}) = 0$. Consequently, Lemma 2.5, Brezis-Lieb lemma and Lemma 2.9 yield

$$\tilde{m}_c \leq I((u_n - u)^{t_n}) < I(u_n - u) = I(u_n) - I(u) + o_n(1) = \tilde{m}_c - I(u) + o_n(1).$$

Sending $n \to \infty$ and using the non-negativity of $I(u)$ we obtain $I(u) = 0$. This in turn implies $u = 0$, which is a contradiction and thus $Q(u) \leq 0$. If $Q(u) < 0$, then again by Lemma 2.4 we find some $s \in (0, 1)$ such that $Q(u^s) = 0$. But then using Lemma 2.5, Lemma 2.9 and the fact $c_1 \leq c$

$$\tilde{m}_{c_1} \leq I(u^s) < I(u) \leq \tilde{m}_c \leq \tilde{m}_{c_1},$$

a contradiction. We conclude therefore $Q(u) = 0$

Thus u is a minimizer of m_{c_1}. From Lemma 2.10 we know that u is a solution of (1.2) and it remains to show that the corresponding ω in (1.2) is positive and $M(u) = c$.

Step 2: Positivity of ω. First we prove that ω is non-negative. Testing (1.2) with u and followed by eliminating $\|\nabla_x u\|_2^2$ using $Q(u) = 0$ we obtain

$$\begin{aligned} \|\partial_y u\|_2^2 + \omega M(u) = 2\Big(d - \frac{4}{p}\Big)^{-1}\Big(\Big(2 + \frac{4}{p} - d\Big)\Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_c\|_2^2 \\ + \Big(\frac{2q}{p} + \frac{4}{p} - d\Big)(q+2)^{-1}\Big(\frac{q}{p} - 1\Big)\|u_c\|_{q+2}^{q+2}\Big). \end{aligned} \tag{2.20}$$

Combining (2.17) we infer that $\omega M(u) \geq 0$. Since $u \neq 0$ we conclude $\omega \geq 0$. We next show that $\omega = 0$ leads to a contradiction, which completes the proof of Step 2. Assume therefore that u satisfies the equation

$$-\Delta_{x,y} u = u^{p+1} + u^{q+1}. \tag{2.21}$$

First consider the case $d \geq 2$. By the Brezis-Kato estimate [6] (see also [40, Lem. B.3]) and the local L^p-elliptic regularity (see for instance [40, Lem. B.2]) we know that $u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^{d+1})$ for all $p \in [1, \infty)$. Hence by Sobolev embedding we also know that u and ∇u are of class $L^\infty_{\mathrm{loc}}(\mathbb{R}^{d+1})$. Taking ∂_j to (2.21) with $j \in \{1, \ldots, d+1\}$ we obtain

$$-\Delta_{x,y}\partial_j u = (p+1)u^p \partial_j u + (q+1)u^q \partial_j u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^{d+1}).$$

Hence by applying the local L^p-elliptic regularity again we deduce $u \in W^{3,p}_{\mathrm{loc}}(\mathbb{R}^{d+1})$ for all $p \in [1, \infty)$. Consequently, by Sobolev embedding we infer that $u \in C^2(\mathbb{R}^{d+1})$. But then by [28, Thm. 1.3] we know that any nonnegative C^2-solution of (2.21) must be zero, a contradiction.

Next we consider the case $d = 1$. For $n \in \mathbb{N}$ let $\phi_n \in C_c^\infty(\mathbb{R}; [0, 1])$ be a radially symmetric decreasing function such that $\phi_n(t) \equiv 1$ on $|t| \leq n$, $\phi_n(t) \equiv 0$ for $|t| \geq n+1$ and $\sup_{n \in \mathbb{N}} \|\phi_n\|_{L^\infty_t} \lesssim 1$. Define also

$$D_n := \{x \in \mathbb{R} : |x| \in (n, n+1)\}.$$

Since $u \neq 0$ is non-negative, by monotone convergence theorem we know that

$$\int_{\mathbb{R} \times \mathbb{T}} (u^{p+1} + u^{q+1})\phi_n \, dxdy > 0$$

for all $n \gg 1$. On the other hand, using the fact that $\operatorname{supp} \nabla_x \phi_n \subset D_n$ and Hölder we see that

$$\int_{\mathbb{R}\times\mathbb{T}} \nabla_x u \cdot \nabla_x \phi_n \, dxdy \leq \|\nabla_x u\|_{L^2(D_n\times\mathbb{T})} \|\nabla_x \phi_n\|_{L^2(D_n\times\mathbb{T})} \lesssim \|\nabla_x u\|_{L^2(D_n\times\mathbb{T})} ((n+1)-n)^{\frac{1}{2}} \lesssim \|\nabla_x u\|_{L^2(D_n\times\mathbb{T})}. \tag{2.22}$$

But then $\infty > \|\nabla_x u\|_2^2 \geq \sum_{n\geq 1} \|\nabla_x u\|^2_{L^2(D_n\times\mathbb{T})}$ yields $\|\nabla_x u\|_{L^2(D_n\times\mathbb{T})} = o_n(1)$. By the fact that ϕ_n is independent of y we also know $\int_{\mathbb{R}\times\mathbb{T}} (-\partial_y^2 u)\phi_n \, dxdy = 0$. Summing up, by testing (1.2) with ϕ_n and rearranging terms we obtain

$$0 = \int_{\mathbb{R}\times\mathbb{T}} (u^{p+1} + u^{q+1})\phi_n \, dxdy + o_n(1) > 0$$

for n sufficiently large, a contradiction. This completes the proof of Step 2.

Step 3: $M(u) = c$ and conclusion. Finally, we prove $M(u) = c$. Assume therefore $c_1 < c$. By Lemma 2.5 and (2.19) we know that m_{c_1} is a local minimizer of the mapping $c \mapsto m_c$, which in turn implies that the inequality in (2.17) is in fact an equality. Now using (2.17) (as an equality) and (2.20) we infer that $\omega M(u) = 0$, which is a contradiction since $\omega > 0$ and $u \neq 0$. We thus conclude $M(u) = c$. That u is positive follows immediately from the strong maximum principle. This completes the desired proof. □

2.6. **Proof of Theorem 1.1 (ii).** In this subsection we give the proof of Theorem 1.1 (ii). Again, due to the waveguide setting we are unable to appeal to the Pohozaev's identity to infer that an optimizer of γ_ω is also solving (1.2). As in the case of Theorem 1.1 (i), we shall use the mountain pass geometry of $S_\omega(u)$ on $H^1_{x,y}$ to solve this problem.

Definiton 2.11 (Mountain pass geometry of $S_\omega(u)$ on $H^1_{x,y}$)**.** *We say that $S_\omega(u)$ has a mountain pass geometry on $H^1_{x,y}$ at the level ζ_ω if there exists some $k > 0$ and $\varepsilon \in 2\gamma_\omega$ such that*

$$\zeta_\omega := \inf_{g\in\Lambda(\omega)} \max_{t\in[0,1]} S_\omega(g(t)) > \max\{\sup_{g\in\Lambda(\omega)} S_\omega(g(0)), \sup_{g\in\Lambda(\omega)} S_\omega(g(1))\}, \tag{2.23}$$

where

$$\Lambda(\omega) := \{g \in C([0,1]; H^1_{x,y}) : g(0) \in B_{k,\varepsilon}, S_\omega(g(1)) \leq -1\}$$

and

$$B_{k,\varepsilon} := \{u \in H^1_{x,y} : k/2 \leq \|\nabla_x u\|_2^2 \leq k, \, \|\partial_y u\|_2^2 + \omega M(u) \leq 2(\gamma_\omega - \varepsilon)\}.$$

Lemma 2.12. *The following statements hold:*

- (i) *There exists some $0 < k_0 \ll 1$ such that for any $\varepsilon \leq \gamma_\omega/2$ and $0 < k \leq k_0$ we have $Q(u) > 0$ for all $u \in B_{k,\varepsilon}$.*
- (ii) *If $S_\omega(u) \leq -1$, then $Q(u) < 0$.*

Proof. Using Lemma 2.2 we have for $u \in B_{k,\varepsilon}$

$$Q(u) \geq \|\nabla_x u\|_2^2 - \frac{qd}{2(q+2)} \|u\|_{q+2}^{q+2} \geq \|\nabla_x u\|_2^2 - C_\omega \|\nabla_x u\|_2^{\frac{qd}{2}}.$$

Since $q > 4/d$, we can find some $k_0 \ll 1$ such that for any $k \in (0, k_0]$ the function $z \mapsto z^2 - C_\omega z^{\frac{qd}{4}}$ is positive on $[\frac{k}{2}, k]$, and the proof of (i) is complete. For (ii), by direct computation one infers that

$$S_\omega(u) - \frac{2}{pd} Q(u) = \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u\|_2^2 + \frac{1}{2}(\|\partial_y u\|_2^2 + \omega M(u)) + \Big(\frac{q}{p} - 1\Big)\frac{1}{q+2}\|u\|_{q+2}^{q+2} \geq 0,$$

from which we conclude that $Q(u) \lesssim S_\omega \leq -1$ and the proof of (ii) is complete. □

Lemma 2.13. *There exist $k > 0$ and $\varepsilon \in (0, \gamma_\omega)$ such that*

- (i) *$\gamma_\omega = \zeta_\omega$ holds.*
- (ii) *$S_\omega(u)$ has a mountain pass geometry on $H^1_{x,y}$ at the level γ_ω in the sense of Definition 2.11*

Proof. The proof is similar to the one of Lemma 2.7 and we omit the details here. □

Notice that from Lemma 2.7 we know that by choosing $k \ll 1$, every curve g from $\Lambda(\omega)$ must go through the sphere $\{Q(u) = 0\}$. Thus using the mountain pass geometry in the context of [20, Thm. 4.1] (setting $F = \{Q(u) = 0\}$ therein) we obtain the existence of a Palais-Smale sequence for the variational problem γ_ω, stated as follows.

Lemma 2.14 (Existence of a Palais-Smale sequence)**.** *There exists a sequence* $(u_n)_n \subset H^1_{x,y}$ *such that*

$$S_\omega(u) = \gamma_\omega + o_n(1), \quad \mathrm{dist}_{H^1_{x,y}}(u_n, \{Q = 0\}) = o_n(1), \quad \|S'_\omega(u_n)\|_{H^{-1}_{x,y}} = o_n(1).$$

Mimicking the proof of Corollary 2.3 we are able to show the following useful properties of the constructed Palais-Smale sequence.

Lemma 2.15. *Let* $(u_n)_n$ *be the Palais-Smale sequence constructed in Lemma 2.14. Then* $(u_n)_n$ *is bounded in* $H^1_{x,y}$ *and satisfies* $\liminf_{n\to\infty} \|u_n\|_{q+2} > 0$.

We are now ready to give the complete proof of Theorem 1.1 (ii).

Proof of Theorem 1.1 (ii). Let $(u_n)_n$ be the Palais-Smale sequence given by Lemma 2.14. Using also Lemma 2.15 and Lemma 2.1 we know that u_n converges to some $u \neq 0$ weakly in $H^1_{x,y}$. Using $\|S'_\omega(u_n)\|_{H^1_{x,y}} = o_n(1)$ from Lemma 2.14 we know that u solves (1.2). It remains to show that u is an optimizer of γ_ω.

Define

$$\tilde{\gamma}_\omega := \inf\{\tilde{I}(u) : u \in H^1_{x,y} \setminus \{0\},\, Q(u) \leq 0\},$$

where $\tilde{I}(u)$ is given by $\tilde{I}(u) := S_\omega(u) - \frac{2}{pd}Q(u)$. Following the same arguments in Lemma 2.9 one easily verifies $\gamma_\omega = \tilde{\gamma}_\omega$. Using $\mathrm{dist}_{H^1_{x,y}}(u_n, \{Q = 0\}) = o_n(1)$ given in Lemma 2.14 we deduce that

$$\tilde{I}(u_n) = \gamma_\omega + o_n(1), \quad Q(u_n) = o_n(1).$$

By the weakly lower semicontinuity of norms we infer that $\tilde{I}(u) \leq \gamma_\omega$. We still need to show $Q(u) = 0$. Assume first that $Q(u) < 0$. By Lemma 2.4 there exists some $t \in (0,1)$ such that $Q(u^t) = 0$. Then

$$\gamma_\omega \leq \tilde{I}(u^t) < \tilde{I}(u) \leq \gamma_\omega,$$

a contradiction. Suppose now $Q(u) > 0$. Then using the Brezis-Lieb inequality similarly as in the proof of Theorem 1.1 (i) we know that $Q(u_n - u) < 0$ for all $n \gg 1$. But then

$$\gamma_\omega \leq \tilde{I}(u_n - u) = \tilde{I}(u_n) - \tilde{I}(u) + o_n(1) = \gamma_\omega - \tilde{I}(u) + o_n(1),$$

from which we conclude that $\tilde{I}(u) = 0$ and consequently $u = 0$, a contradiction. This completes the proof. □

3. Periodic dependence of the ground states: Proof of Theorem 1.2

In this section, we give the proof for Theorem 1.2. The following lemma will be playing a crucial role throughout the whole section.

Lemma 3.1. *Let* $m_{1,\lambda}, m_1^\lambda, \widehat{m}_{1,\lambda}, \widehat{m}_1^\lambda$ *be the quantities defined through* (1.31) *and* (1.33). *Then there exist some* $0 < \lambda_*, \lambda^* < \infty$ *such that*

- *For all* $\lambda \in (0, \lambda_*)$ *we have* $m_1^\lambda < 2\pi\widehat{m}^\lambda_{(2\pi)^{-1}}$ *and any minimizer* u^λ *of* m_1^λ *satisfies* $\partial_y u^\lambda \neq 0$.
- *For all* $\lambda \in (\lambda^*, \infty)$ *we have* $m_{1,\lambda} = 2\pi\widehat{m}_{(2\pi)^{-1},\lambda}$ *and any minimizer* u_λ *of* $m_{1,\lambda}$ *satisfies* $\partial_y u_\lambda = 0$.

Moreover, we shall mostly concentrate on the proof of Theorem 1.2 in the variational context where the mass is normalized. The proof for the variational problem, where the frequency ω is fixed, is quite similar to the former and thus will be sketched out at the end of the proof of Theorem 1.2 by taking suitable modification into account.

3.1. Proof of Lemma 3.1.

We firstly characterize $m_{1,\lambda}$ in the limit $\lambda \to \infty$.

Lemma 3.2. *We have*

$$\lim_{\lambda\to\infty} m_{1,\lambda} = 2\pi\widehat{m}_{(2\pi)^{-1},\infty}. \tag{3.1}$$

Additionally, let $u_\lambda \in V_\lambda(1)$ *be a positive optimizer of* $m_{1,\lambda}$ *which also satisfies*

$$-\Delta_x u_\lambda - \lambda\partial_y^2 u_\lambda + \beta_\lambda u_\lambda = \lambda^{\frac{p}{q}-1}|u_\lambda|^p u_\lambda + |u_\lambda|^q u_\lambda \quad \text{on } \mathbb{R}^d \times \mathbb{T} \tag{3.2}$$

for some $\beta_\lambda > 0$ *(whose existence is guaranteed by Theorem 1.1). Then*

$$\lim_{\lambda\to\infty} \lambda\|\partial_y u_\lambda\|_2^2 = 0. \tag{3.3}$$

Proof. It suffices to restrict ourselves to the case $\lambda > 1$. By assuming that a candidate in $V_\lambda(1)$ is independent of y we already conclude

$$m_{1,\lambda} \leq 2\pi \widehat{m}_{(2\pi)^{-1},\lambda} \leq 2\pi \widehat{m}_{(2\pi)^{-1},\infty} < \infty. \tag{3.4}$$

To see the second inequality in (3.4), we may simply take $(v_n)_n \subset \widehat{Q}_\infty((2\pi)^{-1})$ that approaches $\widehat{m}_{(2\pi)^{-1},\infty}$. This particularly yields $\widehat{Q}_\lambda(v_n) < 0$ for all $n \in \mathbb{N}$. By a similar argument as the one given in the proof of Lemma 2.9 it follows

$$\widehat{m}_{(2\pi)^{-1},\lambda} \leq \widehat{I}(v_n) = \widehat{m}_{(2\pi)^{-1},\infty} + o_n(1)$$

The claim follows by sending $n \to \infty$.

Next we prove

$$\lim_{\lambda\to\infty} \|\partial_y u_\lambda\|_2^2 = 0. \tag{3.5}$$

Suppose that (3.5) does not hold. Then we must have

$$\lim_{\lambda\to\infty} \lambda\|\partial_y u_\lambda\|_2^2 = \infty.$$

Since $Q_\lambda(u_\lambda) = 0$ and $q > p > 4/d$,

$$\begin{aligned} m_{1,\lambda} &= E_\lambda(u_\lambda) - \frac{2}{pd} Q_\lambda(u_\lambda) = I_\lambda(u_\lambda) \\ &= \frac{\lambda}{2}\|\partial_y u_\lambda\|_2^2 + \Big(\frac{1}{2} - \frac{2}{pd}\Big)\|\nabla_x u_\lambda\|_2^2 + \frac{1}{q+2}\Big(\frac{q}{p} - 1\Big)\|u_\lambda\|_{q+2}^{q+2} \\ &\geq \frac{\lambda}{2}\|\partial_y u_\lambda\|_2^2 \to \infty \end{aligned} \tag{3.6}$$

as $\lambda \to \infty$, which contradicts (3.4) and in turn proves (3.5). Using (3.4) and (3.6) we infer that

$$\|\nabla_x u_\lambda\|_2^2 \leq C m_{1,\lambda} \leq 2\pi C \widehat{m}_{(2\pi)^{-1},\infty} < \infty, \tag{3.7}$$

where C is some positive constant independent of λ. Therefore $(u_\lambda)_\lambda$ is a bounded sequence in $H^1_{x,y}$, whose weak limit is denoted by u. Arguing similarly as in the proof of Corollary 2.3 and using the fact that $\lambda^{\frac{p}{q}-1} \to 0$ as $\lambda \to \infty$ we infer that

$$\liminf_{\lambda\to\infty} \|u_\lambda\|_{q+2}^{q+2} = \liminf_{\lambda\to\infty} (\lambda^{\frac{p}{q}-1}\|u_\lambda\|_{p+2}^{p+2} + \|u_\lambda\|_{q+2}^{q+2}) \sim \liminf_{\lambda\to\infty} \|\nabla_x u_\lambda\|_2^2 > 0.$$

Hence by Lemma 2.1 we may also assume that $u \neq 0$. Using (3.5) we know that u is independent of y and thus $u \in H^1_x$. Moreover, applying the weakly lower semicontinuity of norms we know that $\widehat{M}(u) \in (0, (2\pi)^{-1}]$. On the other hand, using $Q(u_\lambda) = 0$, $M(u_\lambda) = 1$, $\lambda^{\frac{p}{q}-1} < 1$ for $\lambda > 1$ and Hölder we obtain from (3.2) that

$$\beta_\lambda \lesssim 1 + \|u_\lambda\|_{q+2}^{q+2}.$$

Thus $(\beta_\lambda)_\lambda$ is a bounded sequence in $(0, \infty)$, whose limit is denoted by β. We now test (3.2) with $\phi \in C_c^\infty(\mathbb{R}^d)$ and integrate both sides over $\mathbb{R}^d \times \mathbb{T}$. Notice particularly that the term $\int_{\mathbb{R}^d\times\mathbb{T}} \partial_y^2 u_\lambda \phi \, dxdy = 0$ for any $\lambda > 0$ since ϕ is independent of y. Using the weak convergence of u_λ to u in $H^1_{x,y}$ and $\lim_{\lambda\to\infty} \lambda^{\frac{p}{q}-1} = 0$, by sending $\lambda \to \infty$ we obtain

$$-\Delta_x u + \beta u = |u|^q u \quad \text{in } \mathbb{R}^d. \tag{3.8}$$

In particular, by Lemma 1.11 we know that $\widehat{Q}_\infty(u) = 0$. Combining the weakly lower semicontinuity of norms and (3.4) we deduce

$$2\pi \widehat{E}_\infty(u) = 2\pi \widehat{I}(u) \leq \liminf_{\lambda\to\infty} I_\lambda(u_\lambda) = \liminf_{\lambda\to\infty} E_\lambda(u_\lambda) = \liminf_{\lambda\to\infty} m_{1,\lambda} \leq 2\pi \widehat{m}_{(2\pi)^{-1},\infty}.$$

However, by Lemma 1.11 the mapping $c \mapsto \widehat{m}_{c,\infty}$ is strictly monotone decreasing on $(0, \infty)$, from which we conclude that $\widehat{M}(u) = (2\pi)^{-1}$ and u is an optimizer of $\widehat{m}_{(2\pi)^{-1},\infty}$. Finally, using the weakly lower semicontinuity of norms we conclude

$$
\begin{aligned}
m_{1,\lambda} = I_\lambda &= \frac{\lambda}{2}\|\partial_y u_\lambda\|_2^2 + \Big(\frac{1}{2}-\frac{2}{pd}\Big)\|\nabla_x u_\lambda\|_2^2 + \frac{1}{q+2}\Big(\frac{q}{p}-1\Big)\|u_\lambda\|_{q+2}^{q+2} \\
&\geq \Big(\frac{1}{2}-\frac{2}{pd}\Big)\|\nabla_x u_\lambda\|_2^2 + \frac{1}{q+2}\Big(\frac{q}{p}-1\Big)\|u_\lambda\|_{q+2}^{q+2} \\
&\geq 2\pi\Big(\Big(\frac{1}{2}-\frac{2}{pd}\Big)\|\nabla_x u\|_{L_x^2}^2 + \frac{1}{q+2}\Big(\frac{q}{p}-1\Big)\|u\|_{L_x^{q+2}}^{q+2}\Big) + o_\lambda(1) \\
&= 2\pi\widehat{E}_\infty(u) + o_\lambda(1) \geq 2\pi\widehat{m}_{(2\pi)^{-1},\infty} + o_\lambda(1).
\end{aligned} \tag{3.9}
$$

Letting $\lambda \to \infty$ and taking (3.4) into account we conclude (3.1). Finally, (3.3) follows directly from the computation in (3.9) without neglecting $\lambda\|u_\lambda\|_2^2$ therein. This completes the desired proof. □

Lemma 3.3. *Let u_λ and u be the functions given in the proof of Lemma 3.2. Then $u_\lambda \to u$ strongly in $H^1_{x,y}$.*

Proof. This simply follows from the observation that in the proof of Lemma 3.2, all the inequalities involving the weakly lower semicontinuity of norms are in fact equalities. Hence $\|u_\lambda\|_{H^1_{x,y}} \to \|u\|_{H^1_{x,y}} = 2\pi\|u\|_{H^1_x}$ as $\lambda \to \infty$, which in turn implies the strong convergence of u_λ to u in $H^1_{x,y}$. □

The following lemma shares the same proof of [42, Lem. 3.5]. The only difference is that in our setting we have an additional nonlinear term $\lambda^{\frac{p}{q}-1}|u_\lambda|^p u_\lambda$. This is however harmless since we are pushing λ to infinity and consequently $\lim_{\lambda\to\infty}\lambda^{\frac{p}{q}-1} = 0$. We thus omit the proof of the following lemma and refer the details to [42, Lem. 3.5].

Lemma 3.4 ([42])**.** *There exists some λ_0 such that $\partial_y u_\lambda = 0$ for all $\lambda > \lambda_0$.*

After having all the preliminaries, we are now able to give the proof of Lemma 3.1.

Proof of Lemma 3.1. Define

$$\lambda^* := \inf\{\tilde{\lambda} \in (0,\infty) : m_{1,\lambda} = 2\pi\widehat{m}_{(2\pi)^{-1},\lambda}\,\forall\,\lambda \geq \tilde{\lambda}\}.$$

From Lemma 3.4 we already know that $\lambda^* \in (0,\infty)$ and the second part of Lemma 3.1 holds for the defined number λ^*. It is left to construct the positive number $\lambda_* \in (0,\infty)$ as required in Lemma 3.1.

We first show for any $c \in (0,\infty)$ it holds $\lim_{\lambda\to 0}\widehat{m}_c^\lambda = \widehat{m}_c^0$. Denote by U^λ a minimizer of $\widehat{m}_c^\lambda$ whose existence can be deduced by using a similar proof as the one of Theorem 1.1. In particular, $\widehat{Q}^\lambda(U^0) \leq \widehat{Q}^0(U^0) = 0$, which combining Lemma 2.9 implies

$$0 \leq \sup_{\lambda\in(0,1]}\widehat{m}_c^\lambda \leq \sup_{\lambda\in(0,1]}\widehat{I}^\lambda(U^0) \in (0,\infty).$$

Hence $(\widehat{m}_c^\lambda)_{\lambda\in(0,1)}$ is a bounded sequence. Using the arguments given in the proof of Corollary 2.3 it follows that $(U^\lambda)_{\lambda\in(0,1)}$ is a bounded sequence in H^1_x with $\liminf_{\lambda\to 0}\|U^\lambda\|_{L_x^{p+2}} > 0$. We may now use the completely same arguments given in the proof of Theorem 1.1 to show that (up to a subsequence) U^λ converges to some $\tilde{U}^0$ strongly in H^1_x with $\tilde{U}^0$ being a minimizer of $\widehat{m}_c^0$. This in turn implies the desired claim.

To proceed, we next construct an auxiliary function ρ as follows: Let $a \in (0,\pi)$ such that $a > \pi - 3\pi\Big(\frac{3}{p+3}\Big)^{\frac{2}{p}}$. This is always possible for a sufficiently close to π. Then we define ρ by

$$
\rho(y) = \begin{cases}
0, & y \in [0,a]\cup[2\pi-a,2\pi], \\
(\pi-a)^{-1}\Big(\frac{p+3}{3}\Big)^{\frac{1}{p}}(y-a), & y \in [a,\pi], \\
(\pi-a)^{-1}\Big(\frac{p+3}{3}\Big)^{\frac{1}{p}}(2\pi-a-y), & y \in [\pi,2\pi-a].
\end{cases}
$$

By direct computation and $q > p$ one easily verifies that $\rho \in H^1_y$ and

$$\|\rho\|_{L_y^2}^2 = \|\rho\|_{L_y^{p+2}}^{p+2} < \min\{2\pi, \|\rho\|_{L_y^{q+2}}^{q+2}\}. \tag{3.10}$$

Now let $P^\lambda \in H^1_x$ be an optimizer of $\widehat{m}^\lambda_{\|\rho\|_{L_y^2}^{-2}}$. Using arguments as in the proof of Corollary 2.3 one easily verifies that $(P^\lambda)_{\lambda\in[0,1)}$ is a bounded sequence in H^1_x. By Lemma 1.11, the mapping $c \mapsto \widehat{m}_c^0$ is strictly decreasing on $(0,\infty)$. Hence $\|\rho\|_{L_y^2}^{-2} > (2\pi)^{-1}$ implies $\delta := \widehat{m}^0_{(2\pi)^{-1}} - \widehat{m}^0_{\|\rho\|_{L_y^2}^{-2}} > 0$. Next, define

$\psi^\lambda(x,y) := \rho(y)P^\lambda(x)$. Then $(\psi^\lambda)_{\lambda\in(0,1]}$ is a bounded sequence in $H^1_{x,y}$ with $M(\psi^\lambda) = \|\rho\|^2_{L^2_y}\widehat{M}(P^\lambda) = 1$. For given λ let $t^\lambda \in (0,\infty)$ be given such that $Q^\lambda((\psi^\lambda)^{t^\lambda}) = 0$, where $(\psi^\lambda)^{t^\lambda}$ is defined by (1.35). Using (3.10) it follows $Q^0(\psi^0) = 0$. This, in conjunction with standard continuity arguments, implies $t^\lambda \to 1$ as $\lambda \to 0$. Using also $\lim_{\lambda\to 0}\widehat{m}^\lambda_c = \widehat{m}^0_c$ for any $c \in (0,\infty)$ we conclude that

$$
\begin{aligned}
m^\lambda_1 &\leq E^\lambda((\psi^\lambda)^{t^\lambda}) \leq \|\rho\|^2_{L^2_y}\widehat{E}^\lambda((P^\lambda)^{t^\lambda}) + o_\lambda(1)\\
&= \|\rho\|^2_{L^2_y}\widehat{m}^\lambda_{\|\rho\|^{-2}_{L^2_y}} + o_\lambda(1) = \|\rho\|^2_{L^2_y}\widehat{m}^0_{\|\rho\|^{-2}_{L^2_y}} + o_\lambda(1)\\
&\leq 2\pi(\widehat{m}^0_{(2\pi)^{-1}} - \delta) + o_\lambda(1) = 2\pi\widehat{m}^\lambda_{(2\pi)^{-1}} - 2\pi\delta + o_\lambda(1)
\end{aligned}
$$

as $\lambda \to 0$. This implies $m^\lambda_1 < 2\pi\widehat{m}^\lambda_{(2\pi)^{-1}}$ for all sufficiently small λ.

Finally, we borrow an idea from [16] to show that any minimizer of $m_{1,\lambda}$ for $\lambda > \lambda_*$ must be y-independent. Assume the contrary that an optimizer u_λ of $m_{1,\lambda}$ satisfies $\|\partial_y u_\lambda\|^2_2 \neq 0$. Since $\lambda > \lambda_*$, there exists some κ strictly lying between λ_* and λ. Then

$$
2\pi\widehat{m}_{(2\pi)^{-1},\kappa} = m_{1,\kappa} \leq E_\kappa(u_\lambda) = E_\lambda(u_\lambda) + \frac{\kappa-\lambda}{2}\|\partial_y u_\lambda\|^2_2 < E_\lambda(u_\lambda) = m_{1,\lambda} = 2\pi\widehat{m}_{(2\pi)^{-1},\lambda}.
$$

Nevertheless, using the characterization (2.12) for $\widehat{m}_{c,\lambda}$ one easily deduces that $\widehat{m}_{(2\pi)^{-1},\kappa} \geq \widehat{m}_{(2\pi)^{-1},\lambda}$ and we hence obtain a contradiction. This completes the desired proof. □

3.2. **Proof of Theorem 1.2.** We are in a final position to prove Theorem 1.2.

Proof of Theorem 1.2. For $c > 0$ and $\alpha \in \{p,q\}$ let $\kappa_{c,\alpha} := c^{\frac{1}{d-\frac{4}{\alpha}}}$. Define also

$$
T_{\lambda,\alpha}u(x,y) := \lambda^{\frac{2}{\alpha}}u(\lambda x, y).
$$

Then $u \mapsto T_{\kappa_{c,\alpha},\alpha}u$ defines a bijection between $V(c)$ and $V_{\kappa^2_{c,q}}(1)$ and $V^{\kappa^2_{c,p}}(1)$ respectively. By using simple scaling arguments one also infers that

$$
m_c = c^{\frac{d-4/q-2}{d-4/q}} m_{1,\kappa^2_{c,q}} = c^{\frac{d-4/p-2}{d-4/p}} m_1^{\kappa^2_{c,p}}.
$$

By same arguments we also deduce that $\widehat{m}_{(2\pi)^{-1}c} = c^{\frac{d-4/q-2}{d-4/q}}\widehat{m}_{(2\pi)^{-1},\kappa^2_{c,q}} = c^{\frac{d-4/p-2}{d-4/p}}\widehat{m}^{\kappa^2_{c,p}}_{(2\pi)^{-1}}$ for $c > 0$. Notice also that the mapping $c \mapsto \kappa_{c,\alpha}$ is strictly monotone increasing on $(0,\infty)$. Thus by Lemma 3.1 there exists some $c_*, c^* \in (0,\infty)$ such that

- For all $c \in (0, c_*)$ we have

$$
m_c = c^{\frac{d-4/p-2}{d-4/p}} m_1^{\kappa^2_{c,p}} < c^{\frac{d-4/p-2}{d-4/p}} 2\pi\widehat{m}^{\kappa^2_{c,p}}_{(2\pi)^{-1}} = 2\pi\widehat{m}_{(2\pi)^{-1}c}.
$$

- For all $c \in (c^*, \infty)$ we have

$$
m_c = c^{\frac{d-4/q-2}{d-4/q}} m_{1,\kappa^2_{c,q}} = c^{\frac{d-4/q-2}{d-4/q}} 2\pi\widehat{m}_{(2\pi)^{-1},\kappa^2_{c,q}} = 2\pi\widehat{m}_{(2\pi)^{-1}c}.
$$

By the definitions of c_* and c^* it is also clear that $c_* \leq c^*$. This completes the proof of the first part of Theorem 1.2.

It remains to prove the second part of Theorem 1.2. In fact, the proof is very similar to the one of the first part, we hence only give the key steps of the proof without establishing the full details.

Mimicking the proof of Lemma 3.1, we aim to prove the following claim: Let $\gamma_{1,\lambda}, \gamma^\lambda_1, \widehat{\gamma}_{1,\lambda}, \widehat{\gamma}^\lambda_1$ be the quantities defined through (1.32) and (1.34). Then there exist some $0 < \lambda_*, \lambda^* < \infty$ such that

(i) For all $\lambda \in (0,\lambda_*)$ we have $\gamma^\lambda_1 < 2\pi\widehat{\gamma}^\lambda_1$ and any minimizer u^λ of γ^λ_1 satisfies $\partial_y u^\lambda \neq 0$.
(ii) For all $\lambda \in (\lambda^*,\infty)$ we have $\gamma_{1,\lambda} = 2\pi\widehat{\gamma}_{1,\lambda}$ and any minimizer u_λ of $\gamma_{1,\lambda}$ satisfies $\partial_y u_\lambda = 0$.

The proof of (i) follows from a straightforward modification of the proof of Theorem 1.2. However, the proof of (ii) is completely different. The main reason here is that the variational problem $\widehat{\gamma}^0_1$ corresponds to a defocusing problem which admits no minimizers, hence we are unable to use a profile P^0 in the limiting case as in the proof of Lemma 3.1.

We use a different argument to overcome this issue: Let ρ be the function defined in the proof of Lemma 3.1. For $\lambda > 0$ also let P^λ be an optimizer of $\widehat{\gamma}^\lambda_1$. In this case, we also define $\psi^\lambda := \rho P^\lambda$. Using

(3.10) it follows $Q^\lambda(P^\lambda) < 2\pi \widehat{Q}(P^\lambda) = 0$. Hence arguing as in the proof of Lemma 2.9 (by also noticing that $\mu = 1$) and using (3.10) we obtain

$$\begin{aligned}\gamma_1^\lambda &\le S_1^\lambda(\psi^\lambda) - \frac{2}{qd}Q^\lambda(\psi^\lambda) = \frac{\lambda}{2}\|\partial_y \rho\|_{L_y^2}^2 \|P^\lambda\|_{L_x^2}^2 + \|\rho\|_{L_y^2}^2 (\widehat{S}_1^\lambda(P^\lambda) - \frac{2}{qd}\widehat{Q}^\lambda(P^\lambda)) \\ &\le \lambda \|\partial_y\rho\|_{L_y^2}^2 \widehat{\gamma}_1^\lambda + (2\pi - \tilde{\delta})\widehat{\gamma}_1^\lambda = 2\pi\widehat{\gamma}_1^\lambda - (\tilde{\delta} - \lambda\|\partial_y\rho\|_{L_y^2}^2)\widehat{\gamma}_1^\lambda < 2\pi\widehat{\gamma}_1^\lambda\end{aligned}$$

for $\lambda < \tilde{\delta}\|\partial_y\rho\|_{L_y^2}^{-2}$, where $\tilde{\delta} := 2\pi - \|\partial_y\rho\|_{L_y^2}^2 \in (0, 2\pi)$. This completes the proof of the claim.

Next, one easily verifies that $u \mapsto T_{\sqrt{\omega^{-1}},\alpha} u$ defines a bijection between the sets $\{u \in H_{x,y}^1 : Q(u) = 0\}$ and $\{u \in H_{x,y}^1 : Q_{\omega^{-1}}(u) = 0\}$ ($\alpha = q$) respectively $\{u \in H_{x,y}^1 : Q^{\omega^{-1}}(u) = 0\}$ ($\alpha = p$). Moreover, we have

$$\begin{aligned}\gamma_\omega &= \omega^{-\frac{d-4/q-2}{2}}\gamma_{1,\omega^{-1}} = \omega^{-\frac{d-4/p-2}{2}}\gamma_1^{\omega^{-1}}, \\ \widehat{\gamma}_\omega &= \omega^{-\frac{d-4/q-2}{2}}\widehat{\gamma}_{1,\omega^{-1}} = \omega^{-\frac{d-4/p-2}{2}}\widehat{\gamma}_1^{\omega^{-1}}.\end{aligned}$$

The desired claim then follows by using the same scaling arguments given previously and by noticing that the mapping $\omega \mapsto \omega^{-\frac{1}{2}}$ is strictly monotone decreasing on $(0, \infty)$. This completes the desired proof. ☐

4. Qualitative and quantitative blow-up results: Proof of Theorem 1.4 and 1.8

This section is devoted to proving the blow-up results Theorem 1.4 and 1.8. We start with the proof of the qualitative one.

4.1. **Existence of blow-up solutions: Proof of Theorem 1.4.**

We start with some a preliminary lemma.

Lemma 4.1. *Suppose that the initial datum u_0 satisfies*

$$E(u_0) < m_{\|u_0\|_2^2} \quad \text{and} \quad Q(u_0) < 0$$

when $\mu = -1$, or

$$S_\omega(u_0) < \gamma_\omega \quad \text{and} \quad Q(u_0) < 0$$

when $\mu = 1$. Then there exists a strictly positive δ such that $Q(u(t)) \le -\delta$ for any $t \in (-T_{\min}, T_{\max})$. More precisely, there exists a constant $\delta' > 0$, independent of t, such that

$$Q(u(t)) \le -\delta' \|\nabla_{x,y} u(t)\|_2^2 \tag{4.1}$$

for any $t \in (-T_{\min}, T_{\max})$.

Proof. Let us consider the case $\mu = -1$. Firstly, we suppose by the absurd that $Q(u(t)) > 0$ for some time $t \in (-T_{\min}, T_{\max})$. Then by the continuity in time of the function $Q(u(t))$, there exists $\tilde{t}$ such that $Q(u(\tilde{t})) = 0$. By definition of the functional m_c and the conservation of the mass, we have therefore that $m_{\|u_0\|_2^2} \le E(u(\tilde{t})) = E(u_0)$, which is a contradiction with respect to the hypothesis.

For the control away from zero, by means of Lemma 2.4 we infer the existence of $\tilde{\lambda} \in (0, 1)$ such that $Q(u_0^{\tilde{\lambda}}) = 0$ and

$$\frac{d}{d\lambda}(E(u_0^\lambda))(\lambda) \ge \frac{d}{d\lambda}(E(u_0^\lambda))(1) = Q(u_0)$$

for $\lambda \in (\tilde{\lambda}, 1)$. Hence,

$$\begin{aligned}E(u_0) &= E(u_0^{\tilde{\lambda}}) + \int_{\tilde{\lambda}}^1 \frac{d}{d\lambda}(E(u_0^\lambda))(\lambda)d\lambda \ge E(u_0^{\tilde{\lambda}}) + (1 - \tilde{\lambda})\frac{d}{d\lambda}(E(u_0^\lambda))(1) \\ &= E(u_0^{\tilde{\lambda}}) + (1 - \tilde{\lambda})Q(u_0) > m_{\|u_0\|_2^2} + Q(u_0),\end{aligned} \tag{4.2}$$

which in turn implies the bound of the Lemma with $\delta = E(u_0) - m_{\|u_0\|_2^2}$. By the energy conservation, for any other t belonging to the maximal time of existence, it suffices to repeat the argument above.

Similarly, in the focusing-defocusing case $\mu = 1$, if by the absurd we have the existence of $\tilde{t}$ such that $Q(u(\tilde{t})) = 0$, then we conclude, by conservation of mass and energy, that $S_\omega(u_0) = S_\omega(u(\tilde{t})) \ge \gamma_\omega$, which contradicts the hypothesis. Moreover, recalling that the scaling u^λ (see (1.35)) leaves invariant the mass, we have an estimate analogous to (4.2). Specifically,

$$
\begin{aligned}
S_\omega(u_0) &= E(u_0) + \frac{\omega}{2}M(u_0) = E(u_0^{\tilde\lambda}) + \int_{\tilde\lambda}^1 \frac{d}{d\lambda}(E(u_0^\lambda))(\lambda)d\lambda + \frac{\omega}{2}M(u_0) \\
&\geq E(u_0^{\tilde\lambda}) + (1-\tilde\lambda)\frac{d}{d\lambda}(E(u_0^\lambda))(1) + \frac{\omega}{2}M(u_0) = E(u_0^{\tilde\lambda}) + \frac{\omega}{2}M(u_0^{\tilde\lambda}) + (1-\tilde\lambda)Q(u_0) \\
&= S_\omega(u_0^{\tilde\lambda}) + (1-\tilde\lambda)Q(u_0) > \gamma_\omega + Q(u_0),
\end{aligned} \tag{4.3}
$$

then the claim follows by (4.3) and $\delta = S_\omega(u_0) - \gamma_\omega$. By the energy conservation, for any other t belonging to the maximal time of existence, it suffices to repeat the argument above.

In order to prove the refined control (4.1), recall the following identities for $\mu = 1$ and $\mu = -1$, respectively,

$$
\begin{aligned}
Q(u) &= \|\nabla_x u\|_2^2 + \frac{dp}{2(p+2)}\|u\|_{p+2}^{p+2} - \frac{dq}{2(q+2)}\|u\|_{q+2}^{q+2} \\
&= \frac{dq}{2}E(u) + \left(1 - \frac{dq}{4}\right)\|\nabla_{x,y}u\|_2^2 - \|\partial_y u\|_2^2 + \frac{d}{2}\frac{p-q}{p+2}\|u\|_{p+2}^{p+2}
\end{aligned} \tag{4.4}
$$

and

$$
\begin{aligned}
Q(u) &= \|\nabla_x u\|_2^2 - \frac{dp}{2(p+2)}\|u\|_{p+2}^{p+2} - \frac{dq}{2(q+2)}\|u\|_{q+2}^{q+2} \\
&= \frac{dp}{2}E(u) + \left(1 - \frac{dp}{4}\right)\|\nabla_{x,y}u\|_2^2 - \|\partial_y u\|_2^2 + \frac{d}{2}\frac{p-q}{q+2}\|u\|_{q+2}^{q+2}.
\end{aligned} \tag{4.5}
$$

Consider first the case $\mu = -1$. By (4.5) and the fact $q > p > \frac{4}{d}$, it is straightforward to see that

$$\left(\frac{dp}{4} - 1\right)\|\nabla_{x,y}u\|_2^2 \leq \frac{dp}{2}E(u) - Q(u),$$

hence, by means of Lemma 4.1, for any $\varepsilon > 0$

$$Q(u) + \varepsilon\left(\frac{dp}{4} - 1\right)\|\nabla_{x,y}u\|_2^2 \leq \varepsilon\frac{dp}{2}E(u) + (1-\varepsilon)Q(u) \leq \varepsilon\frac{dp}{2}E(u) + (1-\varepsilon)\delta.$$

The conclusion follows by taking ε small enough, recalling the conservation of the energy.

As for the defocusing-focusing case, namely $\mu = 1$, we repeat the same argument, by using (4.4), Lemma 4.1, and again $q > p > \frac{4}{d}$. ☐

We are now ready to give the proof of the blow-up results in Theorem 1.4.

Proof of Theorem 1.4. Given a smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$, we introduce the virial function

$$V_\phi(t) := \int_{\mathbb{R}^d \times \mathbb{T}} \phi(x)|u(t,x,y)|^2 dxdy. \tag{4.6}$$

The following identities are nowadays classical (see e.g., [9]), and when no confusion may arise, we omit the domain $\mathbb{R}^d \times \mathbb{T}$ along with the space variables (x, y) to lighten the notation:

$$V_\phi'(t) = 2\,\mathrm{Im}\int \nabla_x\phi \cdot \nabla_x u(t)\overline{u}(t)dxdy \tag{4.7}$$

and

$$
\begin{aligned}
V_\phi''(t) = &-\int \Delta_x^2\phi|u(t)|^2dxdy + 4\sum_{j,k}\mathrm{Re}\int \partial^2_{x_jx_k}\phi\,\partial_{x_j}u(t)\partial_{x_k}\overline{u}(t)dxdy \\
&+ \frac{2\mu p}{p+2}\int \Delta_x\phi|u(t)|^{p+2}dxdy - \frac{2q}{q+2}\int \Delta_x\phi|u(t)|^{q+2}dxdy.
\end{aligned} \tag{4.8}
$$

Suppose now that the initial datum u_0 is radial with respect to the Euclidean variable, i.e., $u_0 = u_0(x,y) = u_0(|x|,y)$. Then the radial symmetry of the solution remains for any time in the maximal lifespan. Let $\theta : [0,\infty) \to [0,2]$ a smooth function satisfying

$$\theta(r) = \begin{cases} 2 & \text{if} \quad 0 \leq r \leq 1, \\ 0 & \text{if} \quad r \geq 2. \end{cases} \tag{4.9}$$

We define the function $\Theta : [0,\infty) \to [0,\infty)$ by

$$\Theta(r) := \int_0^r \int_0^s \theta(\tau) d\tau ds.$$

For $\varrho > 0$, we define the radial function $\phi_\varrho : \mathbb{R}^d \to \mathbb{R}$ by

$$\phi_\varrho(x) = \phi_\varrho(r) := \varrho^2 \Theta(r/\varrho), \quad r = |x|. \tag{4.10}$$

Straightforward calculations yield to

$$V''_{\phi_\varrho}(t) \le 8Q(u(t)) - \frac{2\mu p}{p+2} \int |k(x)||u(t)|^{p+2} dxdy + \frac{2q}{q+2} \int |k(x)||u(t)|^{q+2} dxdy + C\varrho^{-2} \tag{4.11}$$

where $k = k(x) = k(|x|)$ is a non-positive radial function supported outside a ball of radius ϱ, centered at the origin of $\mathbb{R}^d$.

At this point we observe that if $\mu = 1$ (defocusing-focusing case), the lower order term can be simply estimated by zero, hence we may focus on the higher order term. Let us denote by

$$\mathfrak{m}(u)(x) = \frac{1}{2\pi} \int_{\mathbb{T}} u(x,y) dy$$

the mean of u with respect to the variable on $\mathbb{T}$, the compact component of the product manifold. By the estimate $|a+b|^c \lesssim |a|^c + |b|^c$ for $c \ge 1$, and the triangular inequality we estimate the two terms $\||k|^{1/(q+2)} \mathfrak{m}(u)\|_{q+2}^{q+2}$ and $\||k|^{1/(q+2)}(u - \mathfrak{m}(u))\|_{q+2}^{q+2}$. As $\mathfrak{m}(u)$ is independent of y, by the Strauss embedding (see e.g. [15]), the Minkowski's inequality and the Jensen's inequality, we get

$$\begin{aligned} \int |k(x)||\mathfrak{m}u|^{q+2} dxdy &\lesssim \int |k(x)||\mathfrak{m}(u)(x)|^{q+2} dx \\ &\lesssim \varrho^{-\frac{(d-1)q}{2}} \|\nabla_x \mathfrak{m}(u)\|_{L^2_x}^{q/2} \|\mathfrak{m}(u)\|_{L^2_x}^{q/2} \|\mathfrak{m}(u)\|_{L^2_x}^2 \\ &\lesssim \varrho^{-\frac{(d-1)q}{2}} \|\nabla_{x,y} u\|_2^{q/2} \|u\|_2^{q/2+2}. \end{aligned}$$

The term $\||k|^{1/(q+2)}(u - \mathfrak{m}(u))\|_{q+2}^{q+2}$ is estimated as follows: first recall the Sobolev embedding

$$\|u - \mathfrak{m}(u)\|_{L_y^{q+2}} \lesssim \|u\|_{\dot{H}_y^{\frac{q}{2(q+2)}}},$$

see e.g. [4]. Then writing also $u - \mathfrak{m}(u)$ in its Fourier expansion along the y-direction we obtain

$$\begin{aligned} \||k|^{1/(q+2)}(u - \mathfrak{m}(u))\|_{q+2}^{q+2} &= \int |k(x)||u - \mathfrak{m}(u)|^{q+2} dydx \\ &\lesssim \int \left(|k(x)| \left(\sum_j |j|^{q/(q+2)} |u_j(x)|^2 \right)^{\frac{q+2}{2}} \right) dx \\ &= \left\| |k|^{1/(q+2)} \||j|^{q/(2q+4)} |u_j|\|_{l_j^2} \right\|_{L_x^{q+2}}^{q+2}. \end{aligned}$$

As $2 + q \ge 2$, we continue by using the Minkowski's inequality,

$$\||k|^{1/(q+2)}(u - \mathfrak{m}(u))\|_{q+2}^{q+2} \le \left(\sum_j |j|^{q/(q+2)} \||k|^{1/(q+2)} |u_j|\|_{L_x^{q+2}}^2 \right)^{\frac{q+2}{2}}$$

and by employing again the Strauss embedding theorem as before, we obtain

$$\||k|^{1/(q+2)}(u - \mathfrak{m}(u))\|_{q+2}^{q+2} \lesssim \varrho^{-\frac{(d-1)q}{2}} \left(\sum_j |j|^{q/(q+2)} \|\nabla_x u_j\|_{L_x^2}^{q/(q+2)} \|u_j\|_{L_x^2}^{(q+4)/(q+2)} \right)^{\frac{q+2}{2}}.$$

By means of the Hölder's inequality with exponents $\eta = \frac{2q+4}{q}$ and $\gamma = \frac{2q+4}{q+4}$, in conjunction with the conservation of the mass, we end-up with

$$\begin{aligned} \||k|^{1/(q+2)}(u - \mathfrak{m}(u))\|_{q+2}^{q+2} &\lesssim \varrho^{-\frac{(d-1)q}{2}} \left(\left(\sum_j |j|^2 \|\nabla_x u_j\|_{L_x^2}^2 \right)^{\frac{q}{2(q+2)}} \left(\sum_j \|u_j\|_{L_x^2}^2 \right)^{\frac{q+4}{2(q+2)}} \right)^{\frac{q+2}{2}} \\ &\lesssim \varrho^{-\frac{(d-1)q}{2}} \|\nabla_{x,y} u\|_2^{q/2} \|u\|_2^{\frac{q+4}{2}} \lesssim \varrho^{-\frac{(d-1)q}{2}} \|\nabla_{x,y} u\|_2^{q/2}. \end{aligned}$$

Then the following virial estimate is established:

$$V''_{\phi_\varrho}(t) \le 8Q(u(t)) + C\varrho^{-2} + C\varrho^{-\frac{(d-1)q}{2}} \|\nabla_{x,y} u\|_2^{q/2}, \quad \forall t \in I_{\max}. \tag{4.12}$$

The desired blow-up claim follows then from (4.12), (4.1), and a convexity arguments.

In the focusing-focusing case, i.e. $\mu = -1$, we are nevertheless unable to simply estimate the contribution from the lower-order term by using its non-negativity property. Alternatively, one may verbatim repeat the estimate as for the higher order term. By doing so, we get in turn

$$\begin{aligned} V''_{\phi_\varrho}(t) &\le 8Q(u(t)) + C\varrho^{-2} + C\varrho^{-\frac{(d-1)p}{2}} \|\nabla_{x,y} u\|_2^{p/2} + C\varrho^{-\frac{(d-1)q}{2}} \|\nabla_{x,y} u\|_2^{q/2} \\ &\le 8Q(u(t)) + C\varrho^{-2} + C\varrho^{-\frac{(d-1)p}{2}} \|\nabla_{x,y} u\|_2^{p/2}, \quad \forall t \in I_{\max}. \end{aligned} \tag{4.13}$$

Note that $C\varrho^{-\frac{(d-1)q}{2}} \|\nabla_{x,y} u\|_2^2$ is absorbed in the contribution in p for $\varrho \gg 1$. In both cases we can conclude with the desired blow up results by taking ϱ sufficiently large. This completes the proof of Theorem 1.4. □

4.2. **Proof of Theorem 1.8.** In order to prove the blow-up rate results, we are inspired by the scheme introduced by Merle, Raphaël, and Szeftel in the context of critical equations, see [36].

First consider the defocusing-focusing case $\mu = 1$. By means of (4.4) in conjunction with (4.12), we get

$$\begin{aligned} V''_{\phi_\varrho}(t) &\le 8Q(u(t)) + C\varrho^{-2} + C\varrho^{-\frac{(d-1)q}{2}} \|\nabla_{x,y} u\|_2^{q/2} \\ &\le 4dqE(u(t)) + (8-2dq)\|\nabla_{x,y} u(t)\|_2^2 + C\varrho^{-2} + C\varrho^{-\frac{(d-1)q}{2}} \|\nabla_{x,y} u(t)\|_2^{q/2}, \quad \forall t \in I_{\max}, \end{aligned}$$

where ϕ_ϱ is defined in (4.10). By Young's inequality, we have for any $\varepsilon > 0$ and any $t \in I_{\max}$:

$$V''_{\phi_\varrho}(t) \le 4dqE(u(t)) + (8-2dq)\|\nabla_{x,y} u(t)\|_2^2 + C\varrho^{-2} + \varepsilon\|\nabla_{x,y} u(t)\|_2^2 + C\varepsilon^{-\frac{q}{4-q}} \varrho^{-\frac{2(d-1)q}{4-q}}.$$

We select now ε to be equal to $-(4-dq)$. Hence the above inequality reduces to

$$V''_{\phi_\varrho}(t) \le 4dqE(u(t)) + (4-dq)\|\nabla_{x,y} u(t)\|_2^2 + C\varrho^{-2} + C\varrho^{-\frac{2(d-1)q}{4-q}}.$$

Note that $1 < \frac{(d-1)q}{4-q}$ is always satisfied as we are working in the mass supercritical case. Therefore by the conservation of energy and $1 < \frac{(d-1)q}{4-q}$, provided $\varrho > 0$ is taken sufficiently small, we get

$$(dq-4)\|\nabla_{x,y} u(t)\|_2^2 + V''_{\phi_\varrho}(t) \le C\varrho^{-\frac{2(d-1)q}{4-q}}. \tag{4.14}$$

Now, consider times $0 < t_0 < t < T_{\max}$. Integrating (4.14) twice on (t_0, t) gives

$$\begin{aligned} (dq-4)\int_{t_0}^t \int_{t_0}^s \|\nabla_{x,y} u(\tau)\|_2^2 d\tau ds + V_{\phi_\varrho}(t) &\le C\varrho^{-\frac{2(d-1)q}{4-q}}(t-t_0)^2 + (t-t_0)V'_{\phi_\varrho}(t_0) + V_{\phi_\varrho}(t_0) \\ &\le \varrho^{-\frac{2(d-1)q}{4-q}}(t-t_0)^2 + C\varrho(t-t_0)\|\nabla u(t_0)\|_2 \\ &\quad + C\varrho^2, \end{aligned}$$

where we have used the conservation of mass and the estimates below:

$$\begin{aligned} &V_{\phi_\varrho}(t_0) \le C\varrho^2 \|u(t_0)\|_2^2 \le C\varrho^2, \\ &V'_{\phi_\varrho}(t_0) \le C\varrho \|\nabla_{x,y} u(t_0)\|_2 \|u(t_0)\|_2 \le C\varrho \|\nabla_{x,y} u(t_0)\|_2. \end{aligned}$$

Fubini's Theorem then implies

$$\int_{t_0}^t \int_{t_0}^s \|\nabla_{x,y} u(\tau)\|_2^2 d\tau ds = \int_{t_0}^t \left(\int_\tau^t ds\right) \|\nabla_{x,y} u(\tau)\|_2^2 d\tau = \int_{t_0}^t (t-\tau)\|\nabla_{x,y} u(\tau)\|_2^2 d\tau.$$

Recall that V_{ϕ_ϱ} is non-negative. Then we get

$$\int_{t_0}^t (t-\tau)\|\nabla_{x,y} u(\tau)\|_2^2 d\tau \le \varrho^{-\frac{2(d-1)q}{4-q}}(t-t_0)^2 + C\varrho(t-t_0)\|\nabla_{x,y} u(t_0)\|_2 + C\varrho^2.$$

In the limit $t \to T_{\max}$, by means of the Young's inequality we obtain

$$\begin{aligned}\int_{t_0}^{T_{\max}} (T_{\max} - \tau)\|\nabla_{x,y} u(\tau)\|_2^2 d\tau &\leq \varrho^{-\frac{2(d-1)q}{4-q}} (T_{\max} - t_0)^2 + C\varrho(T_{\max} - t_0)\|\nabla_{x,y} u(t_0)\|_2 + C\varrho^2 \\ &\leq \varrho^{-\frac{2(d-1)q}{4-q}} (T_{\max} - t_0)^2 + (T_{\max} - t_0)^2\|\nabla_{x,y} u(t_0)\|_2^2 + C\varrho^2.\end{aligned}$$

Optimizing in ϱ by choosing $\varrho^{-\frac{2(d-1)q}{4-q}} (T_{\max} - t_0)^2 = \varrho^2$, or equivalently $\varrho = (T_{\max} - t_0)^{\frac{4-q}{(d-2)q+4}}$, we deduce

$$\int_{t_0}^{T_{\max}} (T_{\max} - \tau)\|\nabla_{x,y} u(\tau)\|_2^2 d\tau \leq C(T_{\max} - t_0)^{\frac{8-2q}{(d-2)q+4}} + (T_{\max} - t_0)^2\|\nabla_{x,y} u(t_0)\|_2^2, \tag{4.15}$$

for any $0 < t_0 < T_{\max}$. By introducing the function

$$g(t) := \int_{t}^{T_{\max}} (T_{\max} - \tau)\|\nabla_{x,y} u(\tau)\|_2^2 d\tau, \tag{4.16}$$

from (4.15) and the Fundamental Theorem of Calculus we get

$$g(t) \leq C(T_{\max} - t)^{\frac{8-2q}{(d-2)q+4}} - (T_{\max} - t)g'(t), \quad \forall\, 0 < t < T_{\max}$$

which can be straightforwardly rewritten as

$$\frac{d}{dt}\left(\frac{g(t)}{T_{\max} - t}\right) = \frac{1}{(T_{\max} - t)^2}(g(t) + (T_{\max} - t)g'(t)) \leq C(T_{\max} - t)^{\frac{8-2q}{(d-2)q+4} - 2}.$$

Integrating over the interval $(0, t)$ the above inequality gives

$$\frac{g(t)}{T_{\max} - t} \leq \frac{g(0)}{T_{\max}} + \frac{C}{(T_{\max} - t)^{\frac{4-dq}{(d-2)q+4}}} - \frac{C}{(T_{\max})^{\frac{4-dq}{(d-2)q+4}}}$$

which in turn implies that

$$\frac{g(t)}{T_{\max} - t} \leq \frac{C}{(T_{\max} - t)^{\frac{4-dq}{(d-2)q+4}}} \quad \text{as} \quad t \to T_{\max}^-.$$

Therefore, we have

$$\int_{t}^{T_{\max}} (T_{\max} - \tau)\|\nabla_{x,y} u(\tau)\|_2^2 d\tau \leq C(T_{\max} - t)^{\frac{2q(d-1)}{(d-2)q+4}} \quad \text{as} \quad t \to T_{\max}^-.$$

The above estimate can be reformulated as

$$\frac{1}{T_{\max} - t}\int_{t}^{T_{\max}} (T_{\max} - \tau)\|\nabla_{x,y} u(\tau)\|_2^2 d\tau \leq \frac{C}{(T_{\max} - t)^{\frac{4-dq}{(d-2)q+4}}}. \tag{4.17}$$

At this point we consider a sequence $T_n \to T_{\max}^-$, and we note that for any n the function g introduced in (4.16) is a continuous function on $[T_n, T_{\max}]$ and differentiable on the interior points $(T_n, T_{\max})$. Hence, the mean value theorem gives the existence of a time $t_n \in (T_n, T_{\max})$ such that the left-hand side of (4.17) satisfies

$$\frac{1}{T_{\max} - T_n}\int_{T_n}^{T_{\max}} (T_{\max} - \tau)\|\nabla_{x,y} u(\tau)\|_2^2 d\tau = (T_{\max} - t_n)\|\nabla_{x,y} u(t_n)\|_2^2.$$

Using (4.17), we have

$$\|\nabla u(t_n)\|_2 \leq \frac{C}{(T_{\max} - t_n)^{\frac{4-q}{(d-2)q+4}}}$$

This concludes the proof for the blow-up rate in the case $\mu = 1$.

As for the focusing-focusing case ($\mu = -1$), using (4.5) instead of (4.4), we see that nothing changes with respect to the defocusing-focusing case, except for the coefficient of the homogeneous Sobolev norm term. In this case, we may simply repeat the same arguments given previously by taking the range of p into account instead of considering the higher order exponent q, we omit the repeating details. This completes the desired proof.

Appendix A. Large data scattering: Proof of Theorem 1.9

In this section, we focus on the large data scattering result for (1.1), i.e. proving Theorem 1.9. As already pointed out in the introductory section and also in the recent papers [29, 34] by the second author, the proof of Theorem 1.9 is quite different in the cases $d < 5$ and $d \geq 5$, mainly due to the fact that the nonlinearity becomes less regular in high-dimensional spaces.

From an analytical point of view, we may consider (1.1) as a perturbed version of the NLS with a single nonlinearity by an intercritical perturbation, thus the proof of Theorem 1.9 is essentially the same compared to the ones given in [29, 34]. For the sake of completeness, we follow the same lines in [34] to present a sketch of the proof of Theorem 1.9 in the case $d \geq 5$, by making use of the modern tool *interaction Morawetz-Dodson-Murphy (IMDM) inequality* which has been recently developed in [18]. We shall also omit the almost identical proofs in most cases and refer to [34] for details. Instead, we focus on explaining the ideas for proving the main scattering result.

Finally, we also note that the proof of Theorem 1.9 in the case $d \leq 4$ can be similarly deduced as in the paper [29] via the standard *concentration compactness* method. To keep the paper as concise and short as possible, we omit the latter details.

A.1. **Scattering Criterion.** First notice that since (1.1) possesses an energy-subcritical nature and we work with a problem in the energy space, a solution of (1.1) can always be extended beyond its lifespan as long as its H^1-norm remains bounded. In particular, a solution of (1.1) will be a global solution when certain variational assumption is satisfied (see Section A.2 below). The main issue here is that to guarantee a global solution of (1.1) is also scattering, further control of the solution in infinite time becomes necessary. Such motivation leads to the useful scattering criterion Lemma A.2. To formulate Lemma A.2, some notion of the exotic Strichartz estimates will also be introduced.

Lemma A.1 (Exotic Strichartz estimates on $\mathbb{R}^d \times \mathbb{T}$, [34])**.** *For any $\alpha \in (\frac{4}{d}, \frac{4}{d-1})$ there exist $\mathbf{a}, \mathbf{r}, \mathbf{b}, \mathbf{s} \in (2, \infty)$ such that*

$$(\alpha+1)\mathbf{s}' = \mathbf{r}, \quad (\alpha+1)\mathbf{b}' = \mathbf{a}, \quad \alpha/\mathbf{r} < \min\{1, \frac{2}{d}\}, \quad \frac{2}{\mathbf{a}} + \frac{d}{\mathbf{r}} = \frac{2}{\alpha}.$$

Moreover, for any $\gamma \in \mathbb{R}$ we have the following exotic Strichartz estimate:

$$\| \int_{t_0}^{t} e^{i(t-s)\Delta_{x,y}} F(s)\, ds \|_{L_t^{\mathbf{a}} L_x^{\mathbf{r}} H_y^{\gamma}(I)} \lesssim \|F\|_{L_t^{\mathbf{b}'} L_x^{\mathbf{s}'} H_y^{\gamma}(I)}. \tag{A.1}$$

When $d \geq 5$, we can additionally assume that there exists some $0 < \beta \ll 1$ such that $\mathbf{r}$ can be chosen as an arbitrary number from $(\frac{\alpha(\alpha+1)d}{\alpha+2}, \frac{\alpha(\alpha+1)d}{\alpha+2} + \beta)$.

Lemma A.2 (Scattering criterion, [34])**.** *Let u be a global solution of* (1.1) *and assume that*

$$\|u\|_{L_t^\infty H_{x,y}^1(\mathbb{R})} \leq A.$$

Then for any $\sigma > 0$ there exist $\varepsilon = \varepsilon(\sigma, A)$ sufficiently small and $T_0 = T_0(\sigma, \varepsilon, A)$ sufficiently large such that if for all $a \in \mathbb{R}$ there exists $T \in (a, a+T_0)$ such that $[T - \varepsilon^{-\sigma}, T] \subset (a, a+T_0)$ and

$$\|u\|_{L_t^{\mathbf{a}_q} L_x^{\mathbf{r}_q} H_y^s(T-\varepsilon^{-\sigma}, T)} \lesssim \varepsilon^{\mu} \tag{A.2}$$

for some $\mu > 0$, where $(\mathbf{a}_q, \mathbf{r}_q)$ is the exotic-admissible pair in Lemma A.1 corresponding to the exponent q, then u scatters forward in time.

Proof. This follows immediately from the proof of [34, Lem. 6] by dealing the combined nonlinearities separately and using interpolation to bound the estimates for the nonlinearity of order p by the ones of the larger order q. □

The scattering criterion will be applied in conjunction with the following useful local control result. The latter can be similarly deduced by using the proof of [34, Lem. 5], where again we only need to cheat the both nonlinearities separately and use suitable interpolation inequalities.

Lemma A.3 (Local control of a solution, [34])**.** *Let u be a global solution of* (1.1) *with $\|u\|_{L_t^\infty H_{x,y}^1(\mathbb{R})} < \infty$ and let $s \in (\frac{1}{2}, 1 - s_q)$, where $s_q = \frac{d}{2} - \frac{2}{q}$. Then for any L_x^2-admissible pair (ℓ_1, ℓ_2) (namely (ℓ_1, ℓ_2) satisfies $\frac{2}{\ell_1} + \frac{d}{\ell_2} = \frac{d}{2}$) with $(\ell_1, \ell_2, d) \neq (2, \infty, 2)$ we have*

$$\|u\|_{L_t^{\ell_1} W_x^{1-s, \ell_2} H_y^s(I)} \lesssim \langle I \rangle^{\frac{1}{\ell_1}}. \tag{A.3}$$

A.2. **Variational analysis and the IMDM-estimates.** As mentioned in Subsection A.1, we will need to establish the uniform H^1-boundedness for the solution of (1.1) by appealing to suitable variational arguments that can be deduced nowadays in a quite standard way by using the functional inequalities such as the Gagliardo-Nirenberg or Sobolev inequalities. In the context of the waveguide setting, suitable scale-invariant (w.r.t. the x-direction) replacement of such functional inequalities becomes necessary, see e.g. Lemma 2.2. Moreover, to fit the non-local nature of the Morawetz inequalities of interaction type, additional spatial translation shall also be taken into account in the functional inequalities, see [18, Lem. 2.1]. All the consideration leads to the following coercivity result. For a proof, see e.g. [34, 1].

Lemma A.4 (Coercivity property, [34, 1])**.** *Let u be a solution of* (1.1) *with $u(0) \in \mathcal{A}$, where*

$$\begin{cases} \mathcal{A} := \{u \in S(c) : E(u) < m_c,\, Q(u) > 0\}, & \textit{when } \mu = -1, \\ \mathcal{A} := \{u \in H^1_{x,y} \setminus \{0\} : S_\omega(u) < \gamma_\omega,\, Q(u) > 0\}, & \textit{when } \mu = 1. \end{cases}$$

Then u is global and $u(t) \in \mathcal{A}$ for all $t \in \mathbb{R}$. Moreover, there exist $0 < \delta \ll 1$ and $R_0 \gg 1$ such that for all $R \geq R_0$, $z \in \mathbb{R}^d$, $t \in \mathbb{R}$ we have

$$Q(\chi_R(\cdot - z)u^\xi(t)) \geq \delta \|\nabla_x(\chi_R(\cdot - z))u^\xi(t)\|_2^2, \tag{A.4}$$

where $u^\xi(t,x,y) := e^{ix\cdot\xi}u(t,x,y)$ and

$$\xi = \xi(t,z,R) = \begin{cases} -\frac{\int \mathrm{Im}(\chi_R^2(x-z)\bar{u}(t,x,y)\nabla_x u(t,x,y))\,dxdy}{\int \chi_R^2(x-z)|u(t,x,y)|^2\,dxdy}, & \textit{if } \int \chi_R^2(x-z)|u(t,x,y)|^2\,dxdy \neq 0, \\ 0, & \textit{if } \int \chi_R^2(x-z)|u(t,x,y)|^2\,dxdy = 0. \end{cases}$$

Having established the previous coercivity property of the solution u of (1.1), we may immediately deduce the following IMDM-inequality by using the interaction Morawetz potential initiated by Dodson and Murphy [17, 18] which plays a crucial role in the rest of the paper. In the waveguide setting, we shall simply apply the interaction Morawetz inequality along the x-direction and integrate the quantities over the y-direction without taking other operations. This can also be seen from the observation that the NLS only possesses dispersive effects in the infinite Euclidean space.

Lemma A.5 (IMDM-inequality, [18, 34])**.** *Let u be a global solution of* (1.1) *satisfying the assumptions in Lemma A.4. Then for any $\varepsilon > 0$ there exist $T_0 = T_0(\varepsilon) \gg 1$, $J = J(\varepsilon) \gg 1$, $R_0 = R_0(\varepsilon, u_0) \gg 1$ and $\eta = \eta(\varepsilon) \ll 1$ such that for any $a \in \mathbb{R}$ we have*

$$\begin{aligned} &\frac{1}{JT_0}\int_a^{a+T_0}\int_{R_0}^{R_0e^J}\frac{1}{R^d}\int_{(\mathbb{R}^d_{x_a}\times\mathbb{T}_{y_a})\times(\mathbb{R}^d_{x_b}\times\mathbb{T}_{y_b})\times\mathbb{R}^d_z} \\ &\qquad |\chi_R(x_b - z)u(t,x_b,y_b)|^2|\nabla_x(\chi_R(x_a - z)u^\xi(t,x_a,y_a))|^2 d(x_a,y_a)d(x_b,y_b)dz\frac{dR}{R}dt \lesssim \varepsilon. \end{aligned} \tag{A.5}$$

A.3. **Conclusion.** In this final subsection, we explain briefly how we are able to prove Theorem 1.9 in the case $d \geq 5$ by using the lemmas stated in previous subsections.

Notice first that by the scattering criterion (Lemma A.2) we will need to prove

$$\|u\|_{L_t^{\mathbf{a}} L_x^{\mathbf{r}} H_y^s(T-\varepsilon^{-\sigma},T)} \lesssim \varepsilon^\mu$$

for some $\mu > 0$, where $(\mathbf{a},\mathbf{r}) = (\mathbf{a}_q,\mathbf{r}_q)$ is the exotic-admissible pair in Lemma A.1 corresponding to the exponent q. To deduce this, we firstly apply the pigeonhole principle to reduce the (both spatially and temporally) averaged inequality (A.5) to the localized form

$$\int_{t_0-\varepsilon^{-\sigma}}^{t_0}\sum_{w\in\mathbb{Z}^d}\|\chi_{R_1}(\cdot - \frac{R_1}{4}(w+\theta_0))u(t)\|^2_{L^2_{x,y}}\|\nabla_x(\chi_{R_1}(\cdot - \frac{R_1}{4}(w+\theta_0))u^\xi(t))\|^2_{L^2_{x,y}}dt \lesssim \varepsilon^{1-\sigma}, \tag{A.6}$$

where $\theta_0 : \mathbb{Z}^d \to [0,1]^d$ is a suitable function whose existence is guaranteed by the mean value theorem. Using the modified Gagliardo-Nirenberg inequality on $\mathbb{R}^d$ (see [18, Lem. 2.1])

$$\|u\|^2_{L_x^{\frac{2d}{d-1}}} \lesssim \|u\|_{L_x^2}\|\nabla_x u^\xi\|_{L_x^2} \tag{A.7}$$

and Hölder and Minkowski inequalities we obtain

$$\int_{t_0-\varepsilon^{-\sigma}}^{t_0}\sum_{w\in\mathbb{Z}^d}\|\chi_{R_1}(\cdot - \frac{R_1}{4}(w+\theta_0))u(t)\|^4_{L_x^{\frac{2d}{d-1}}L_y^2}dt \lesssim \varepsilon^{1-\sigma}. \tag{A.8}$$

Another application of the Hölder, Cauchy-Schwarz inequalities and the Sobolev embedding $H_x^1 \hookrightarrow L_x^{\frac{2d}{d-2}}$ yields

$$\sum_{w\in\mathbb{Z}^d} \|\chi_{R_1}(\cdot - \frac{R_1}{4}(w+\theta_0))u(t)\|^2_{L_x^{\frac{2d}{d-1}} L_y^2} \lesssim \|u(t)\|^2_{H^1_{x,y}} + O(R_1^{-2}\eta^{-2})\|u(t)\|^2_{L_x^2} \lesssim 1$$

by choosing $R_1 \gg 1$, hence

$$\int_{t_0-\varepsilon^{-\sigma}}^{t_0} \sum_{w\in\mathbb{Z}^d} \|\chi_{R_1}(\cdot - \frac{R_1}{4}(w+\theta_0))u(t)\|^2_{L_x^{\frac{2d}{d-1}} L_y^2}\,dt \lesssim \varepsilon^{-\sigma}. \tag{A.9}$$

Interpolating (A.8) and (A.9) we obtain

$$\begin{aligned}\|u\|^{\frac{2d}{d-1}}_{L_{t,x}^{\frac{2d}{d-1}} L_y^2(t_0-\varepsilon^{-\sigma},t_0)} &\lesssim \Big(\int_{t_0-\varepsilon^{-\sigma}}^{t_0} \sum_{w\in\mathbb{Z}^d} \|\chi_{R_1}(\cdot - \frac{R_1}{4}(w+\theta_0))u(t)\|^4_{L_x^{\frac{2d}{d-1}} L_y^2}\,dt\Big)^{\frac{1}{d-1}} \\ &\quad\times \Big(\int_{t_0-\varepsilon^{-\sigma}}^{t_0} \sum_{w\in\mathbb{Z}^d} \|\chi_{R_1}(\cdot - \frac{R_1}{4}(w+\theta_0))u(t)\|^2_{L_x^{\frac{2d}{d-1}} L_y^2}\,dt\Big)^{\frac{d-2}{d-1}} \lesssim \varepsilon^{\frac{1}{d-1}-\sigma}.\end{aligned} \tag{A.10}$$

Now the desired claim follows from (A.10), Lemma A.3 (setting $|I| = t_0 - (t_0 - \varepsilon^{-\sigma}) = \varepsilon^{-\sigma}$ therein) and suitable interpolation (noticing that $(\frac{2d}{d-1}, \frac{2d}{d-1})$ is not yet an L_x^2-admissible pair, hence Lemma A.3 is not directly applicable).

This essentially explains the idea for proving Theorem 1.9 in the case $d \geq 5$ by making use of the IMDM-estimates. For full details, we refer to [34].

Appendix B. The grow-up result: Proof of Theorem 1.7

In this section we give the proof of the grow-up result Theorem 1.7. Consider (4.6) with a cut-off function $\vartheta : \mathbb{R}^+ \mapsto [0,1]$ with $0 \leq \vartheta' \leq 4$ and

$$\vartheta(|x|) = \begin{cases} 0 & \text{if} \quad |x| \leq 1, \\ 1 & \text{if} \quad |x| \geq 2. \end{cases} \tag{B.1}$$

Consider $\vartheta_\varrho = \vartheta(|x|/\varrho)$. Recall the definition of $V_{\vartheta_\varrho}(t)$ as in (4.6). If we suppose that the solution is global and $\sup_{\mathbb{R}^+} \|\nabla_{x,y} u(t)\|_2$ is finite, then by the conservation of mass, there exists a constant $C > 0$ such that

$$V_{\vartheta_\rho}(t) = V_{\vartheta_\varrho}(0) + \int_0^t V'_{\vartheta_\varrho}(s)\,ds \leq o_\rho(1) + Ct\varrho^{-1},$$

where $V_{\vartheta_\varrho}(0) = o_\varrho(1)$ as $\varrho \to +\infty$ by means of the the dominated convergence theorem. Clearly

$$\int_{\{|x|\geq\varrho\}\times\mathbb{T}} |u(x,y,t)|^2\,dxdy \leq V_{\vartheta_\varrho}(t),$$

so we have that for any $\tilde{\delta} > 0$

$$\int_{\{|x|\geq\varrho\}\times\mathbb{T}} |u(x,y,t)|^2 dxdy \leq o_\varrho(1) + \tilde{\delta}, \quad \text{for} \quad t \leq \tilde{T} := C^{-1}\varrho\tilde{\delta}. \tag{B.2}$$

Going back to (4.6) and (4.8) with the localization function ϕ_ρ as defined in (4.10), and by recalling that for a radial function

$$\partial_{x_j} = \frac{x_j}{r}\partial_r, \quad \partial^2_{x_j x_k} = \left(\frac{\delta_{x_j x_k}}{r} - \frac{x_j x_k}{r^3}\right)\partial_r + \frac{x_j x_k}{r^2}\partial_r^2,$$

we have

$$\begin{aligned} V''_{\phi_\varrho}(t) &= 8Q(u(t)) - \int \Delta_x^2 \phi_\varrho |u(t)|^2 dxdy \\ &\quad + 4\int \left(\frac{\phi'_\varrho}{r} - 2\right)|\nabla_x u|^2 dxdy + 4\int\left(\frac{\phi''_\varrho}{r^2} - \frac{\phi'_\varrho}{r^3}\right)|x\cdot\nabla_x u|^2 dxdy \\ &\quad + \frac{2\mu p}{p+2}\int\left(\phi''_\varrho - (d-1)\frac{\phi_\varrho}{r} - 2d\right)|u(t)|^{p+2}dxdy \\ &\quad - \frac{2q}{q+2}\int\left(\phi''_\varrho - (d-1)\frac{\phi_\varrho}{r} - 2d\right)|u(t)|^{q+2}dxdy. \end{aligned} \tag{B.3}$$

By the support properties of the localization function and by interpolation it follows that

$$V''_{\phi_\varrho}(t) \le 8Q(u(t)) + C_p\|u\|^{\eta_p(p+2)}_{L^2(\{|x|\ge\varrho\}\times\mathbb{T})} + C_q\|u\|^{\eta_q(q+2)}_{L^2(\{|x|\ge\varrho\}\times\mathbb{T})} \tag{B.4}$$

where $\eta_p, \eta_q \in (0,1)$. Thus, by combining Lemma 4.1, (B.2), (B.4) we obtain

$$V''_{\phi_\varrho}(t) \le -8\delta + 2\max\{C_p, C_q\}\left(o_\varrho(1) + \tilde{\delta}^{\min\{\eta_p(p+2),\eta_q(q+2)\}}\right) \quad \text{for} \quad t \le \tilde{T}.$$

A choice $\tilde{\delta} \ll 1$ implies that for ϱ large enough

$$V''_{\phi_\varrho}(t) \le -4\delta < 0. \tag{B.5}$$

As $\phi \le |x|^2$, one notes that

$$\begin{aligned} V_{\phi_\varrho}(0) &\le \int_{\{|x|\le\sqrt{\varrho}\}\times\mathbb{T}} |x|^2|u(x,y,0)|^2 dxdy + \int_{\{\sqrt{\varrho}\le|x|\le 2\varrho\}\times\mathbb{T}} |x|^2|u(x,y,0)|^2 dxdy \\ &\le \varrho M(u_0) + 4\varrho^2 o_\varrho(1) = Co_\varrho(1)\varrho^2. \end{aligned} \tag{B.6}$$

Similarly,

$$V'_{\phi_\varrho}(0) \le Co_\varrho(1)\varrho. \tag{B.7}$$

It easily follows, by integrating twice in time over the interval $[0,\tilde{T}]$ the inequality (B.5) and by using (B.6), (B.7), along with the definition of $\tilde{T} \sim \varrho$, we conclude with

$$V_{\phi_\varrho}(\tilde{T}) = V_{\phi_\varrho}(0) + \tilde{T}V'_{\phi_\varrho}(0) + \int_0^{\tilde{T}}\int_0^s V''_{\phi_\varrho}(s)dsdt \le C(o_\varrho(1)-\delta)\varrho^2 \le -\frac{C\delta}{2}\varrho^2,$$

which is a contradiction with respect to the non-negativity of the function $V_{\phi_\varrho}(t)$.

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