STANDING WAVES FOR A SCHRÖDINGER SYSTEM WITH THREE WAVES INTERACTION

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ABSTRACT. We study standing waves for a system of nonlinear Schrödinger equations with three waves interaction arising as a model for the Raman amplification in a plasma. We consider the mass-critical and mass-supercritical regimes and we prove existence of ground states along with a synchronised mass collapse behavior. In addition, we show that the set of ground states is stable under the associated Cauchy flow. Furthermore, in the mass-supercritical setting we construct an excited state which corresponds to a strongly unstable standing wave. Moreover, a semi-trivial limiting behavior of the excited state is drawn accurately. Finally, by a refined control of the excited state's energy, we give sufficient conditions to prove global existence or blow-up of solutions to the corresponding Cauchy problem.

1. INTRODUCTION

In this paper, we consider a three-components system of nonlinear Schrödinger equations related to the Raman amplification in a plasma, as derived by Colin, Colin, and Ohta in [19], which reads as follows:

$$\begin{cases} i\partial_t \psi_1 = -\Delta\psi_1 - |\psi_1|^{p-2}\psi_1 - \alpha\psi_3\overline{\psi}_2, \\ i\partial_t \psi_2 = -\Delta\psi_2 - |\psi_2|^{p-2}\psi_2 - \alpha\psi_3\overline{\psi}_1, \\ i\partial_t \psi_3 = -\Delta\psi_3 - |\psi_3|^{p-2}\psi_3 - \alpha\psi_1\psi_2. \end{cases} \tag{1.1}$$

Here, $\psi_i = \psi_i(t,x)$ with $i = 1,2,3$, are complex-valued functions $\psi_i : \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{C}$, with $\overline{\psi}_i$ denoting the complex conjugate, the space dimension is $N \leq 3$, α is a positive real parameter, and the power nonlinearity p is in the range $2_* \leq p < 2^*$, where

$$\begin{cases} 2_* = 2 + \frac{4}{N}, \\ 2^* = \infty \text{ if } N \leq 2, \quad 2^* = \frac{2N}{N-2} \text{ if } N = 3. \end{cases}$$

Namely, we consider the mass-critical or mass-supercritical and energy subcritical power-type nonlinearities.

It is standard to see that the Cauchy problem associated to (1.1) is locally well-posed in the energy space, i.e., for a fixed initial datum

$$(\psi_{0,1}, \psi_{0,2}, \psi_{0,3})(x) := (\psi_1, \psi_2, \psi_3)(0,x) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N),$$

there exists a solution $(\psi_1, \psi_2, \psi_3) \in C([0, T_{\max}), H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N))$, where $T_{\max} > 0$ is the positive maximal time of existence (a similar notion can be given for negative times).

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See the monograph [11]. Moreover, the blow-up alternative holds true, in the sense that either $T_{\max} = \infty$ (the solution is global), or $T_{\max} < \infty$ and the homogenous Sobolev norm of the solution diverges as $t \to T_{\max}^-$. More precisely $\lim_{t \to T_{\max}^-} \left(\sum_{i=1}^3 \|\nabla \psi_i(t)\|_{L^2(\mathbb{R}^N)}^2 \right) = \infty$. In addition, the following quantities are conserved along the flow: the energy, defined by

$$E(t) = E\left(\vec{\psi}(t)\right) = \sum_{i=1}^3 \left(\frac{1}{2}\|\nabla\psi_i(t)\|_{L^2(\mathbb{R}^N)}^2 - \frac{1}{p}\|\psi_i(t)\|_{L^p(\mathbb{R}^N)}^p \right) - \alpha\mathrm{Re}\int_{\mathbb{R}^N} \left(\psi_1\psi_2\overline{\psi}_3\right)(t)dx,$$
(1.2)

and the mixed masses

$$Q_1(t) = Q_1\left(\vec{\psi}(t)\right) = \|\psi_1(t)\|_{L^2(\mathbb{R}^N)}^2 + \|\psi_3(t)\|_{L^2(\mathbb{R}^N)}^2$$
$$Q_2(t) = Q_2\left(\vec{\psi}(t)\right) = \|\psi_2(t)\|_{L^2(\mathbb{R}^N)}^2 + \|\psi_3(t)\|_{L^2(\mathbb{R}^N)}^2,$$
(1.3)

where we used the compact notation

$$\vec{\psi} = \vec{\psi}(t,x) = (\psi_1(t,x), \psi_2(t,x), \psi_3(t,x)) \in H^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{C}^3).$$

As usual, conservation means that the previous quantities are not dependent on time, or alternatively $E(t) = E(0)$, $Q_1(t) = Q_1(0)$, and $Q_2(t) = Q_2(0)$ for any time t in the maximal interval of existence $[0, T_{\max})$. The conservation laws can be showed by a standard regularization argument, see [11].

Furthermore, we note that (1.1) can be written as

$$\partial_t \vec{\psi}(t,x) = -iE'\left(\vec{\psi}(t,x)\right),$$

and that

$$E\left(e^{i\theta_1}u_1, e^{i\theta_2}u_2, e^{i(\theta_1+\theta_2)}u_3\right) = E(\vec{u}),$$

for any $(\theta_1, \theta_2) \in \mathbb{R}^2$, and any function $\vec{u} = (u_1, u_2, u_3) \in H^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{C}^3)$.

The main purpose of the paper, is to study existence and stability properties of standing waves solution to (1.1). Let us recall that standing wave for (1.1) is a solution of the form $(\psi_1(t,x), \psi_2(t,x), \psi_3(t,x))$ with $\psi_1(t,x) = e^{i\lambda_1 t}u_1(x)$, $\psi_2(t,x) = e^{i\lambda_2 t}u_2(x)$ and $\psi_3(t,x) = e^{i\lambda_3 t}u_3(x)$, where $\lambda_1, \lambda_2, \lambda_3$ are real numbers and $\vec{u} \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ satisfies the system of elliptic equations

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = |u_1|^{p-2}u_1 + \alpha u_3\overline{u}_2, \\ -\Delta u_2 + \lambda_2 u_2 = |u_2|^{p-2}u_2 + \alpha u_3\overline{u}_1, \\ -\Delta u_3 + \lambda_3 u_3 = |u_3|^{p-2}u_3 + \alpha u_1 u_2, \end{cases}$$
(1.4)

where $\lambda_3 = \lambda_1 + \lambda_2$.

Under certain conditions, the existence, uniqueness and multiplicity of solutions of (1.4) have been studied by many authors. We refer the reader to [14, 32, 37, 39] and the references therein. In particular, the authors in [19, 20] studied the orbital stability of scalar solutions (semi-trivial standing waves) for system (1.1) of the form $(e^{i\omega t}u, 0, 0)$, $(0, e^{i\omega t}u, 0)$, $(0, 0, e^{i\omega t}u)$, where $\omega > 0$ and $u \in H^1(\mathbb{R}^N, \mathbb{R})$ is the unique positive radial solution of

$$-\Delta u + \omega u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

In [17, 19], it is proved that when $2 < p < 2_*$, $(e^{i\omega t}u, 0, 0)$ and $(0, e^{i\omega t}u, 0)$ are orbitally stable for any $\alpha > 0$, while $(0, 0, e^{i\omega t}u)$ is orbitally stable if $0 < \alpha < \bar{\alpha}$ and it is orbitally unstable if

$\alpha > \bar{\alpha}$ for a suitable positive constant $\bar{\alpha} = \bar{\alpha}(N, p, \omega)$ (see also [33] for dimensions $N = 4, 5$). Solutions of the previous type with two trivial components are called scalar solutions.

In [1], it is instead proved the existence of stable standing waves (vector solutions) for the system (1.1) with $N = 1$, $2 < p < 6 = 2_*$ and $\alpha > 0$, by minimizing the energy $E(\vec{u})$ on the manifold

$$S(a_1, a_2) := \left\{ \vec{u} \in H^1(\mathbb{R}^N, \mathbb{C}^3) \text{ s.t. } \int_{\mathbb{R}^N} |u_1|^2 + |u_3|^2 dx = a_1^2, \quad \int_{\mathbb{R}^N} |u_2|^2 + |u_3|^2 dx = a_2^2 \right\},$$

$$(1.5)$$

where $a_1, a_2 > 0$. The results of [1] have been generalized in [28] to the higher dimensional case and to the model (1.1) with potentials (see also [36]). It is worth mentioning that in [37], the existence of non-scalar solutions were proved by minimizing the action function on the Nehari manifold, provided the coupling parameter α is large enough.

In this paper, illuminated by [27] and [38], we aim to consider standing waves and their stability for system (1.1) in the mass-critical or mass-supercritical regime and the energy subcritical one, namely we cover the range of non-linearities $2_* \leq p < 2^*$, where the corresponding energy functional $E(\vec{u})$ is not always bounded from below on $S(a_1, a_2)$. Note that the coupling terms are of mass-subcritical type and sign-indefinite, then we are dealing with a special mass-mixed case (i.e., the combination of mass-subcritical and mass-supercritical terms), which is more complicated.

Before introducing the main results, we recall some definition (see also [5]).

Definition 1.1. *We say that $\vec{u}_0$ is a ground state of (1.4) on $S(a_1, a_2)$ if*

$$dE|_{S(a_1,a_2)}(\vec{u}_0) = 0 \quad and \quad E(\vec{u}_0) = \inf \left\{ E(\vec{u}) \text{ s.t. } dE|_{S(a_1,a_2)}(u) = 0 \text{ and } \vec{u} \in S(a_1, a_2) \right\}.$$

We say that $\vec{v}_0$ is an excited state of (1.4) on $S(a_1, a_2)$ if

$$dE|_{S(a_1,a_2)}(\vec{v}_0) = 0 \quad and \quad E(\vec{v}_0) > \inf \left\{ E(\vec{u}) \text{ s.t. } dE|_{S(a_1,a_2)}(u) = 0 \text{ and } \vec{u} \in S(a_1, a_2) \right\}.$$

The set of ground states will be denoted by $\mathcal{G} = \mathcal{G}_{p,\alpha,N}$.

We emphasise, as in [1], that variational problems with the energy restricted on the manifold $S(a_1, a_2)$ is particularly appropriate for the study of the stability properties of the ground states, as both the energy and the partial mass functionals Q_1 and Q_2 are conserved along the flow generated by (1.1).

Definition 1.2. (i) We say that the set $\mathcal{G}$ is orbitally stable if $\mathcal{G} \neq \emptyset$ and for any $\varepsilon > 0$, there exists a $\delta > 0$ such that, provided that an initial datum $\vec{\psi}(0) = (\psi_1(0), \psi_2(0), \psi_3(0))$ for (1.1) satisfies

$$\inf_{\vec{u} \in \mathcal{G}} \|\vec{\psi}(0) - \vec{u}\|_{H^1(\mathbb{R}^N, \mathbb{C}^3)} < \delta,$$

then $\vec{\psi}$ is globally defined and

$$\inf_{\vec{u} \in \mathcal{G}} \|\vec{\psi}(t) - \vec{u}\|_{H^1(\mathbb{R}^N, \mathbb{C}^3)} < \varepsilon \qquad \forall t > 0,$$

where $\vec{\psi}(t)$ is the solution to (1.1) corresponding to the initial condition $\vec{\psi}(0)$.

(ii) A standing wave $(e^{i\lambda_1 t}u_1, e^{i\lambda_2 t}u_2, e^{i\lambda_3 t}u_3)$ is said to be strongly unstable if for any $\varepsilon > 0$ there exists $\vec{\psi}_0 \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ such that $\|\vec{u} - \vec{\psi}_0\|_{H^1(\mathbb{R}^N, \mathbb{C}^3)} < \varepsilon$, and $\vec{\psi}(t)$ blows-up in finite time, namely $T_{\max} < \infty$.

Throughout this article, we are not only interested in proving existence of standing waves and their stability properties, but also in proving suitable asymptotic results for different regime on the involved parameters α, a_1, and a_2. To this aim, before stating our first main result, we introduce another minimization problem:

$$m_0(a_1, a_2) := \inf_{\vec{u} \in S(a_1, a_2)} E_0(\vec{u}), \tag{1.6}$$

where

$$E_0(\vec{u}) := \frac{1}{2} \sum_{i=1}^{3} \|\nabla u_i\|_{L^2(\mathbb{R}^N)}^2 - \operatorname{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3 dx.$$

We can now state our main result regarding existence, stability, and mass-synchronised asymptotic of the ground states.

Theorem 1. *Let $N \leq 3$, $2_* \leq p < 2^*$, and $\alpha, a_1, a_2 > 0$. There exists a positive explicit constant $D = D(N, p, \alpha)$ such that if $\max\{a_1, a_2\} < D$, we have:*

(i) $\mathcal{G}$ *is nonempty, i.e., there exists a ground state of* (1.4) *on $S(a_1, a_2)$;*

(ii) *the set $\mathcal{G}$ is orbitally stable;*

(iii) *if $\vec{u} \in \mathcal{G}$, we fix $\alpha > 0$, and assume $a_2 = a_1 \to 0$, then we have*

$$\sup_{\vec{u} \in \mathcal{G}} \|\vec{u}(x) - \kappa \alpha^{-1} \vec{v}_0(\kappa^{\frac{1}{2}} x)\|_{H^1(\mathbb{R}^N, \mathbb{C}^3)} = o(1),$$

where $\vec{v}_0$ is a minimizer for $m_0 \left(\sqrt{2}\|w\|_{L^2(\mathbb{R}^N)}, \sqrt{2}\|w\|_{L^2(\mathbb{R}^N)} \right)$, $\kappa = \left(\frac{\alpha a_1}{\sqrt{2}\|w\|_{L^2(\mathbb{R}^N)}} \right)^{\frac{4}{4-N}}$ and w is the unique, real positive solution of $-\Delta w + w = w^2$;

(iv) *if $\vec{u} \in \mathcal{G}$ then $\sum_{i=1}^{3} \|\nabla u_i\|_{L^2(\mathbb{R}^N)}^2 \to 0$ as $\alpha \to 0$.*

We comment on the results given in Theorem 1 above.

Remark 1.3. To the best of the author's knowledge, this is the first result dealing with the existence and stability/instability results of standing waves for the Schrödinger system with three waves interaction in the mass-critical/supercritical nonlinearities. Moreover, it is worth mentioning that our result are not perturbative, indeed the constant D in the statement of Theorem 1 is given by

$$D := \left(\frac{3}{\alpha C^3(N, p)} \frac{p\gamma_p - 2}{2p\gamma_p - N} \right)^{\frac{N(p-2)-4}{4(p-3)}} \left(\frac{p(4-N)}{2(2p\gamma_p - N)C^p(N, p)} \right)^{\frac{4-N}{4(p-3)}}, \tag{1.7}$$

where $C(N, p)$ is the best constant in the following Gagliardo-Nirenberg inequality,

$$\|u\|_{L^p(\mathbb{R}^N)} \leq C(N, p)\|\nabla u\|_{L^2(\mathbb{R}^N)}^{\gamma_p}\|u\|_{L^2(\mathbb{R}^N)}^{1-\gamma_p}, \quad \forall u \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{1.8}$$

with

$$\gamma_p = \frac{N(p-2)}{2p}. \tag{1.9}$$

and $p \in [2, 2^*)$.

Remark 1.4. Theorem 1 shows that a ground state exists even if $E|_{S(a_1,a_2)}$ is unbounded from below, and, for a_1, a_2 small enough, the ground state is indeed a least action solution which reaches the infimum of the C^1 action functional $J(\vec{u}) = E(\vec{u}) + \sum_{i=1}^{3} \left(\frac{\lambda_i}{2} \|u_i\|_{L^2(\mathbb{R}^N)}^2 \right)$ among all nontrivial solutions to (1.4) (see [37, 39] for the existence of least action solutions), where λ_i $(i = 1, 2, 3)$ are the Lagrange multipliers corresponding the ground state.

Remark 1.5. The set $\mathcal{G}$, containing a priori complex-valued ground states, has the following structure:

$$\mathcal{G} = \left\{ (e^{i\theta_1} u_1, e^{i\theta_2} u_2, e^{i(\theta_1+\theta_2)} u_3) \text{ s.t. } \theta_1, \theta_2 \in \mathbb{R} \right\},$$

where $(u_1, u_2, u_3) \in S(a_1, a_2)$ is a positive, radial ground state of (1.4). See the proof of Theorem 1 later on.

Remark 1.6. The fact that $\mathcal{G}$ is orbitally stable indicates that the coupling term leads to the stabilization of ground state standing waves corresponding to (1.1). It is worth recalling that for the Schrödinger equation $i\partial_t \psi = -\Delta \psi - |\psi|^{p-2} \psi$, for p in the mass supercritical regime, the standing wave $\psi = e^{i\lambda t} u$ is strongly unstable, see [11], where $u \in H^1(\mathbb{R}^N)$ is the unique positive radial solution of $-\Delta u + \lambda u = |u|^{p-2} u$ for $\lambda > 0$.

Remark 1.7. In proving the existence of ground states, due to the indefinite sign of the three wave interaction term in the corresponding energy functional, we need to introduce additional constrain given by an inequality. This in turn makes appear further difficulties in proving the compactness of related minimizing sequences, and is different from constrained variational problems with a sign-definite type structure, see for example [3, 27, 34, 38, 40]. In order to get the synchronised mass collapse behavior of the ground state of (1.4) on $S(a_1, a_2)$ (point (iii) in Theorem 1), we prove the existence of ground states for the limit system

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = u_3 \overline{u}_2, \\ -\Delta u_2 + \lambda_2 u_2 = u_3 \overline{u}_1, \\ -\Delta u_3 + (\lambda_1 + \lambda_2) u_3 = u_1 u_2, \end{cases} \tag{1.10}$$

under the constraints

$$Q_1(\vec{u}) = a_1^2 \text{ and } Q_2(\vec{u}) = a_2^2. \tag{1.11}$$

If $\lambda_1 = \lambda_2$, the uniqueness of minimizer for $m_0(a_1, a_2)$ (see (1.6)) and ground state for (1.10) are proved in [32, 39].

Remark 1.8. For $N = 3$, by replacing the constraints in (1.11) by three independent prescribed mass constraints, namely we consider, for $\gamma, \mu, \nu > 0$,

$$\|u_1\|_{L^2(\mathbb{R}^3)}^2 = \gamma, \quad \|u_2\|_{L^2(\mathbb{R}^3)}^2 = \mu, \quad \|u_3\|_{L^2(\mathbb{R}^3)}^2 = \nu$$

and by slightly modifying the proof of Lemma 3.6, we also give a positive answer to the open problem proposed by Kurata and Osada in [28, Remark 4]. See the proof of Lemma 3.6 and subsequent comments.

We now give the results related to the existence and properties of excited states. In what follows, we consider mass-energy intracritical nonlinearities, namely $2_* < p < 2^*$.

Theorem 2. *Let $N \leq 3$, $2_* < p < 2^*$, $\max\{a_1, a_2\} < D$, and $a_1, a_2 > 0$. There exists $\alpha_0 = \alpha_0(a_1, a_2) > 0$ such that, for any $\alpha > \alpha_0$:*

(i) *there exists an excited state $\vec{v} = (v_1, v_2, v_3) \in S(a_1, a_2)$, with associated Lagrange multipliers $\lambda_1, \lambda_2 > 0$;*

(ii) *let $a_1 > 0$ and $a_2 \to 0^+$, then we have*

$$\left(\tilde{\kappa}^{-\frac{1}{p-2}} v_1(\tilde{\kappa}^{-\frac{1}{2}} x), v_2(x), v_3(x)\right) \to (w_p, 0, 0) \quad in \ H^1(\mathbb{R}^N, \mathbb{C}^3),$$

where $\tilde{\kappa} = \left(\dfrac{a_1^2}{\|w_p\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{p-2}{2-p\gamma_p}}$ and w_p is the unique positive solution of $-\Delta w + w = |w|^{p-2} w$.

Remark 1.9. Theorem 2 together with Theorem 1 yields the multiplicity of standing waves for problem (1.1). This indicates that the coupling term not only makes the ground states stable, but also enriches the solutions set. See the first paragraph of Subsection 1.1 for a description of what happens from a physical point of view.

Remark 1.10. The condition $\max\{a_1, a_2\} < D$ in Theorem 1 and Theorem 2 not only ensures that the corresponding energy functional E admits a convex-concave geometry, but also guarantees the existence of a natural constraint (the Pohozaev manifold, see later on in the paper), on which the critical points of E are indeed nontrivial solutions to the problem (1.4). $\alpha > \alpha_0$ is used for a better control of the energy level which excludes semi-trivial solutions. Point (ii) of Theorem 2 draws an accurate semi-trivial limiting behavior of the excited states as portion of the mass vanishes. The transition from mass-supercritical to mass-critical regime greatly changes the geometry of $E|_{S(a_1, a_2)}$, preventing the appearance of the excited state in the latter case. Moreover, if $p = 2_*$, similarly to the proof of Theorem 2, point (ii), the same semi-trivial limiting behavior of ground states obtained in Theorem 1 holds if and only if $a_1^2 = \|w_p\|^2_{L^2(\mathbb{R}^N)}$. It is worth mentioning that similar semi-trivial limits of ground states for mass-critical Schrödinger systems were obtained in [10, 22].

Based on the existence results on ground states and excited states, we can provide sufficient conditions for the global dynamics of solutions.

Firstly, with a control on the energy by means of the excited state obtained in Theorem 2, we show a global existence result. Define the Pohozaev functional P by

$$P(\vec{u}) := \sum_{i=1}^{3} \|\nabla u_i\|_2^2 - \gamma_p \sum_{i=1}^{3} \|u_i\|_p^p - \frac{N}{2} \alpha \mathrm{Re} \int_{\mathbb{R}^N} u_1 u_2 \overline{u}_3 \, dx, \tag{1.12}$$

then we have the following.

Theorem 3. *Under the assumptions of Theorem 2, let $\vec{\psi}$ be the solution of (1.1) with initial datum $\vec{\psi}_0 \in S(a_1, a_2)$ such that $P(\vec{\psi}_0) > 0$ and $E(\vec{\psi}_0) < E(\vec{v})$. Then, $\vec{\psi}$ exists globally in time.*

Secondly, we are able to prove that under certain sufficient condition on the initial datum, finite time blowing-up solutions exist.

Theorem 4. *Under the assumption of Theorem 2, let $\vec{\psi}$ be the solution of (1.1) with initial datum $\vec{\psi}_0 \in S(a_1, a_2)$, $P(\vec{\psi}_0) < 0$ and $E(\vec{\psi}_0) < E(\vec{v})$. If $|x|\vec{\psi}_0 \in L^2(\mathbb{R}^N, \mathbb{C}^3)$, the solution*

blows-up in finite time. The same conclusion holds true for $N = 2, 3$ if the solution is radial and $p \in (4, 6)$ for $N = 2$.

The previous Theorem implies the following instability result.

Corollary 1. *The standing wave constructed $\vec{\psi}(t, x) = \left(e^{i\lambda_1 t}v_1, e^{i\lambda_1 t}v_2, e^{i(\lambda_1+\lambda_2)t}v_3\right)$ with $\vec{v}$ as in Theorem 2 is strongly unstable.*

Remark 1.11. The set

$$\Lambda_0 := \{\vec{u} \in S(a_1, a_2) \text{ s.t. } P(\vec{u}) > 0 \text{ and } E(\vec{u}) < E(\vec{v})\}$$

is not empty and contains not only small initial data in the sense of the $L^2(\mathbb{R}^N)$-norm. Given $\gamma, \mu, \nu > 0$, in the same manner we can look for solutions $(u_1, u_2, u_3) \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ of (1.4) satisfying the conditions $\|u_1\|_2^2 = \gamma$, $\|u_2\|_2^2 = \mu$ and $\|u_3\|_2^2 = \nu$. Such solutions are of interest in physics and sometimes referred to as normalized solutions. In the present paper, we care more about solutions of (1.4) with prescribed partial sum of masses. This is not only because $Q_1(\vec{u})$ and $Q_2(\vec{u})$ are invariant with respect to the flow generated by (1.1) but also is suitable for studying dynamics of (1.1).

Remark 1.12. The last remark is on the fact that similar results as the ones described above, can be stated for $\alpha < 0$, provided we replace u_3 by $-u_3$ in (1.4).

1.1. **Physical background and motivations.** The study of the model as described by equations in (1.1) has a physical motivation, as the system (1.1) is a simplified model of a quasilinear Zakharov system related to the Raman amplification in a plasma. Roughly speaking, the Raman amplification is an instability phenomenon taking place when an incident laser field propagates into a plasma (see [25] and the introduction in [37]). As explained in [37], the laser field, entering a plasma, is backscattered by a Raman type process and the interaction of the two waves generates an electronic plasma wave. Then the three waves together produce a change in the ions' density which in turn affects the waves. This picture is described by three Schrödinger equations coupled with a wave equation (a Zakharov type system), as follows:

$$\begin{cases} \left(i\left(\partial_t + v_C\partial_y\right) + \alpha_1\partial_y^2 + \alpha_2\Delta_\perp\right)A_C = \frac{b^2}{2}nA_C - \gamma(\nabla \cdot E)A_R e^{-i\theta}, \\ \left(i\left(\partial_t + v_R\partial_y\right) + \beta_1\partial_y^2 + \beta_2\Delta_\perp\right)A_R = \frac{bc}{2}nA_R - \gamma\left(\nabla \cdot \overline{E}\right)A_C e^{i\theta}, \\ \left(i\partial_t + \delta_1\Delta\right)E = \frac{b}{2}nE + \gamma\nabla\left(\overline{A}_R A_C e^{i\theta}\right), \\ \left(\partial_t^2 - v_s^2\Delta\right)n = a\Delta\left(|E|^2 + b|A_C|^2 + c|A_R|^2\right), \end{cases} \quad (1.13)$$

where $\theta = k_1 y - k_1^2\delta_1 t$, $t \in \mathbb{R}$, $y \in \mathbb{R}$, and $\Delta_\perp = \partial_x^2 + \partial_z^2$. In this system, A_C denotes the envelope of the incident laser field, A_R is the backscattered Raman field, E is the electronic-plasma wave and n is the variation of ions' density. We refer to [17, 18] for a precise description of the physical coefficients appearing in the equations above.

After proving the local well-posedness of (1.13), in order to study the solitary waves towards an analysis of the global dynamics, the authors of [19] needed to introduce some modifications on (1.13), eventually leading to the system (1.1) studied in this paper. For the reader's convenience and sake of clarity, we report here the few steps as in [19] to derive the desired three NLS system.

In (1.13), by writing $E = Fe^{i\theta}$, by considering a trivial density of ions, i.e., $n = 0$, and by neglecting the ∇ terms, the longitudinal dispersion terms ∂_y^2, and the transverse ones $\Delta_\perp$, one reduces to the simplified system

$$\begin{cases} (i\partial_t + \alpha_2 \Delta_\perp) A_C = -\gamma i k_1 F A_R, \\ (i\partial_t + \beta_2 \Delta_\perp) A_R = \gamma i k_1 \overline{F} A_C, \\ (i\partial_t + \delta_1 \Delta) F = i k_1 \gamma \overline{A}_R A_C. \end{cases} \tag{1.14}$$

In order to model nonlinear effects, the other nonlinear terms as appearing in (1.1), were added in [19], hence by a simple change of variables, and the introduction of the power-type nonlinear terms, one gets

$$\begin{cases} i\partial_t v_1 = -\Delta v_1 - |v_1|^{p-2} v_1 - \alpha v_3 \overline{v}_2, \\ i\partial_t v_2 = -\Delta v_2 - |v_2|^{p-2} v_2 - \alpha v_3 \overline{v}_1, \\ i\partial_t v_3 = -\Delta v_3 - |v_3|^{p-2} v_3 - \alpha v_1 v_2, \end{cases}$$

which is (1.1).

1.2. **Notations.** In the paper, we use the following notations. $x \in \mathbb{R}^N$, $N \leq 3$, $t \in \mathbb{R}$, $L^p = L^p(\mathbb{R}^N)$ with norm $\|f\|_{L^p(\mathbb{R}^N)} = \|f\|_p$, $H^1(\mathbb{R}^N)$ is the usual Sobolev space, with $H^1(\mathbb{R}^N, \mathbb{C}^3)$ or $H^1(\mathbb{R}^N, \mathbb{R}^3)$ for vector valued functions, or $H^1(\mathbb{R}^N, \mathbb{R})$ and $H^1(\mathbb{R}^N, \mathbb{C})$ for scalar functions. $H^{-1}(\mathbb{R}^N)$ denote the dual space of $H^1(\mathbb{R}^N)$. $\int_{\mathbb{R}^N} f dx$ is denoted simply by $\int f$. Re and Im are for the real and imaginary part of a complex number, and $\overline{z}$ stands for the complex conjugate of z.

2. Preliminaries

In this section, we give some preliminaries useful for the rest of the paper.

Lemma 2.1. *Let $N \leq 3$, $2_* \leq p < 2^*$, and $(u_1, u_2, u_3) \in H^1(\mathbb{R}^N, \mathbb{C}^3)$ be a solution to (1.4). Then the following identity holds true:*

$$\sum_{i=1}^{3} \int |\nabla u_i|^2 = \gamma_p \sum_{i=1}^{3} \int |u_i|^p + \frac{N}{2} \alpha \mathrm{Re} \int u_1 u_2 \overline{u}_3. \tag{2.1}$$

Proof. Multiplying both sides of the equation in (1.4) by $\overline{u}_i$ ($i = 1, 2, 3$), integrating over $\mathbb{R}^N$ and taking the real part, adding these three equalities together, we have

$$\sum_{i=1}^{3} \|\nabla u_i\|_2^2 + \sum_{i=1}^{3} \lambda_i \|u_i\|_2^2 = \sum_{i=1}^{3} \|u_i\|_p^p + 3\alpha \mathrm{Re} \int u_1 u_2 \overline{u}_3. \tag{2.2}$$

Multiplying both sides of the first equation by $x \cdot \nabla \overline{u}_1$, integrating over $\mathbb{R}^N$ and taking the real part, we get

$$-\mathrm{Re} \int \Delta u_1 x \cdot \nabla \overline{u}_1 + \lambda_1 \mathrm{Re} \int u_1 x \cdot \nabla \overline{u}_1 = \mathrm{Re} \int |u_1|^{p-2} u_1 x \cdot \nabla \overline{u}_1 + \mathrm{Re} \int u_3 \overline{u}_2 x \cdot \nabla \overline{u}_1.$$

By direct calculations, we obtain

$$-\mathrm{Re} \int \Delta u_1 x \cdot \nabla \overline{u}_1 = \frac{N-2}{2} \|\nabla u_1\|_2^2, \qquad \mathrm{Re} \int u_1 x \cdot \nabla \overline{u}_1 = \frac{N}{2} \|u_1\|_2^2,$$

$$\text{Re} \int |u_1|^{p-2} u_1 x \cdot \nabla \overline{u}_1 = \frac{N}{p} \|u_1\|_p^p,$$

and

$$\text{Re} \int u_3 \overline{u}_2 x \cdot \nabla \overline{u}_1 = -N \text{Re} \int u_1 u_2 \overline{u}_3 - \text{Re} \int u_1 u_2 x \cdot \nabla \overline{u}_3 - \text{Re} \int u_1 \overline{u}_3 x \cdot \nabla \overline{u}_2.$$

Therefore, we get

$$-\frac{N-2}{2} \|\nabla u_1\|_2^2 - \frac{N}{2} \lambda_1 \|u_1\|_2^2 = -\frac{N}{p} \|u_1\|_p^p - N\alpha \text{Re} \int u_1 u_2 \overline{u}_3 - \alpha \text{Re} \int u_1 u_2 x \cdot \nabla \overline{u}_3 \\ - \alpha \text{Re} \int u_1 \overline{u}_3 x \cdot \nabla \overline{u}_2. \tag{2.3}$$

Similarly, by multiplying both sides of the second and third equation of (1.4) with $x \cdot \nabla \overline{u}_2$ and $x \cdot \nabla \overline{u}_3$, respectively, integrating over $\mathbb{R}^N$ and taking the real part, we have

$$-\frac{N-2}{2} \|\nabla u_2\|_2^2 - \frac{N}{2} \lambda_2 \|u_2\|_2^2 = -\frac{N}{p} \|u_2\|_p^p + \alpha \text{Re} \int u_3 \overline{u}_1 x \cdot \nabla \overline{u}_2 \tag{2.4}$$

and

$$-\frac{N-2}{2} \|\nabla u_3\|_2^2 - \frac{N}{2} \lambda_3 \|u_3\|_2^2 = -\frac{N}{p} \|u_3\|_p^p + \alpha \text{Re} \int u_1 u_2 x \cdot \nabla \overline{u}_3. \tag{2.5}$$

By adding (2.3)-(2.5), we obtain

$$\frac{N-2}{2} \sum_{i=1}^{3} \|\nabla u_i\|_2^2 + \frac{N}{2} \sum_{i=1}^{3} \lambda_i \|u_i\|_2^2 = \frac{N}{p} \sum_{i=1}^{3} \|u_i\|_p^p + \alpha N \text{Re} \int u_1 u_2 \overline{u}_3. \tag{2.6}$$

Combining (2.2) and (2.6), it gives that

$$\sum_{i=1}^{3} \|\nabla u_i\|_2^2 = \frac{N(p-2)}{2p} \sum_{i=1}^{3} \|u_i\|_p^p + \frac{N}{2} \alpha \text{Re} \int u_1 u_2 \overline{u}_3.$$

The proof is complete. □

We now introduce the L^2-norm-preserving dilation operator

$$s \star \vec{u}(x) := \left(s^{\frac{N}{2}} u_1(sx), s^{\frac{N}{2}} u_2(sx), s^{\frac{N}{2}} u_3(sx) \right)$$

with $s > 0$. As $\lim_{s \to \infty} E(s \star \vec{u}) = -\infty$, we see that $\inf_{\vec{u} \in S(a_1, a_2)} E(\vec{u}) = -\infty$ for $2_* < p < 2^*$. Furthermore, we introduce (see [3]) the Pohozaev set

$$\mathcal{P}_{a_1, a_2} := \left\{ \vec{u} \in S(a_1, a_2) : P(\vec{u}) := \sum_{i=1}^{3} \|\nabla u_i\|_2^2 - \gamma_p \sum_{i=1}^{3} \|u_i\|_p^p - \frac{N}{2} \alpha \text{Re} \int u_1 u_2 \overline{u}_3 = 0 \right\}, \tag{2.7}$$

where γ_p is given in (1.9).

The Pohozaev set $\mathcal{P}_{a_1, a_2}$ is related to the fiber maps

$$\Psi_{\vec{u}}(s) = E(s \star \vec{u}) = \frac{s^2}{2} \sum_{i=1}^{3} \int |\nabla u_i|^2 - \frac{s^{p\gamma_p}}{p} \sum_{i=1}^{3} \int |u_i|^p - s^{\frac{N}{2}} \alpha \text{Re} \int u_1 u_2 \overline{u}_3. \tag{2.8}$$

Indeed, we have $s\Psi'_{\vec{u}}(s) = P(s \star \vec{u})$. Note that $\mathcal{P}_{a_1,a_2}$ can be divided into the disjoint union $\mathcal{P}_{a_1,a_2} = \mathcal{P}^+_{a_1,a_2} \cup \mathcal{P}^0_{a_1,a_2} \cup \mathcal{P}^-_{a_1,a_2}$, where

$$
\begin{aligned}
\mathcal{P}^+_{a_1,a_2} &:= \left\{ \vec{u} \in \mathcal{P}_{a_1,a_2} \text{ s.t. } \Psi''_{\vec{u}}(1) > 0 \right\}, \\
\mathcal{P}^0_{a_1,a_2} &:= \left\{ \vec{u} \in \mathcal{P}_{a_1,a_2} \text{ s.t. } \Psi''_{\vec{u}}(1) = 0 \right\}, \\
\mathcal{P}^-_{a_1,a_2} &:= \left\{ \vec{u} \in \mathcal{P}_{a_1,a_2} \text{ s.t. } \Psi''_{\vec{u}}(1) < 0 \right\}.
\end{aligned}
\tag{2.9}
$$

We first study the case $2_* < p < 2^*$, namely the mass-energy intracritical case. To show that the energy functional $E|_{S(a_1,a_2)}$ has a concave-convex geometry (i.e., a structure with a local minimum and a global maximum, where the local minimum is strictly less than zero and the global maximum is strictly greater than zero; see Lemma 2.2 below), we introduce the following constraint:

$$
\mathcal{M} := \left\{ (u_1, u_2, u_3) \in H^1(\mathbb{R}^N, \mathbb{C}^3) \text{ s.t. } \operatorname{Re} \int u_1 u_2 \overline{u}_3 > 0 \right\}.
\tag{2.10}
$$

In the spirit of Soave [38] and Wei and Wu [40], for $\vec{u} \in \mathcal{M}$, we see that the presence of the mass subcritical terms $\operatorname{Re} \int u_1 u_2 \overline{u}_3$ induces a convex-concave geometry of $E|_{S(a_1,a_2)}$ if $\alpha > 0$ and $a_1, a_2 > 0$ are small. For $\vec{u} \in S(a_1, a_2)$, we have $\|u_1\|_2 \le a_1$, $\|u_2\|_2 \le a_2$ and $\|u_3\|_2 \le \min\{a_1, a_2\}$. By Gagliardo-Nirenberg inequality and Young inequality, we have

$$
\begin{aligned}
\frac{1}{p} \sum_{i=1}^3 \|u_i\|_p^p &\le \frac{1}{p} C^p(N,p) \left(\sum_{i=1}^2 a_i^{p(1-\gamma_p)} \|\nabla u_i\|_2^{p\gamma_p} + \max\left\{ a_1^{p(1-\gamma_p)}, a_2^{p(1-\gamma_p)} \right\} \|\nabla u_3\|_2^{p\gamma_p} \right) \\
&\le A_1 \left(\sum_{i=1}^3 \|\nabla u_i\|_2^2 \right)^{\frac{p\gamma_p}{2}},
\end{aligned}
\tag{2.11}
$$

where $A_1 := \frac{C^p(N,p)}{p} \left(\max\{a_1, a_2\} \right)^{p(1-\gamma_p)}$. Similarly, we have

$$
\begin{aligned}
\left| \alpha \operatorname{Re} \int u_1 u_2 \overline{u}_3 \right| &\le \alpha \int |u_1||u_2||u_3| \\
&\le \frac{\alpha}{3} C^3(N,p) \left(\max\{a_1, a_2\} \right)^{3-3\gamma_3} \left(\sum_{i=1}^3 \|\nabla u_i\|_2^2 \right)^{\frac{3\gamma_3}{2}} \\
&= A_2 \left(\sum_{i=1}^3 \|\nabla u_i\|_2^2 \right)^{\frac{N}{4}},
\end{aligned}
\tag{2.12}
$$

where $A_2 := \frac{\alpha}{3} C^3(N, p) \left(\max\{a_1, a_2\} \right)^{\frac{6-N}{2}}$. Then, combining (2.11) and (2.12) with the dentition of the energy, we get

$$
\begin{aligned}
E(\vec{u}) &= \sum_{i=1}^{3} \left(\frac{1}{2} \|\nabla u_i\|_2^2 - \frac{1}{p} \|u_i\|_p^p \right) - \alpha \mathrm{Re} \int u_1 u_2 \overline{u}_3 \\
&\geq \frac{1}{2} \sum_{i=1}^{3} \|\nabla u_i\|_2^2 - A_1 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{\frac{p\gamma_p}{2}} - A_2 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{\frac{N}{4}} \\
&= h \left(\left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{\frac{1}{2}} \right),
\end{aligned}
\tag{2.13}
$$

where

$$
h(\rho) = \frac{\rho^2}{2} - A_1 \rho^{p\gamma_p} - A_2 \rho^{\frac{N}{2}}.
\tag{2.14}
$$

The next Lemma below shows that the functional E has a concave-convex structure on $S(a_1, a_2)$.

Lemma 2.2. *Let $N \leq 3$, $2_* < p < 2^*$, and $\alpha, a_1, a_2 > 0$. Let D be as in (1.7) and h as in (2.14).*

(i) If $\max\{a_1, a_2\} < D$, then $h(\rho)$ has a local minimum at negative level and a global maximum at positive level. Moreover, there exist $R_0 = R_0(a_1, a_2)$, $R_1 = R_1(a_1, a_2)$, and ρ^* such that, $R_0 < \max\{a_1, a_2\} D^{-1} \rho^* < \rho^* < R_1$, and

$$
h(R_0) = h(R_1) = 0, \quad h(\rho) > 0 \iff \rho \in (R_0, R_1).
$$

(ii) If $\max\{a_1, a_2\} = D$, then $h(\rho)$ has a local minimum at negative level and a global maximum at level zero. Moreover, we have

$$
h(\rho^*) = 0 \quad and \quad h(\rho) < 0 \iff \rho \in (0, \rho^*) \cup (\rho^*, +\infty).
$$

Proof. (i) We first prove that h has exactly two critical points. Indeed,

$$
h'(\rho) = 0 \iff \hat{h}(\rho) = \frac{N A_2}{2}, \quad \text{with} \quad \hat{h}(\rho) = \rho^{2 - \frac{N}{2}} - p\gamma_p A_1 \rho^{p\gamma_p - \frac{N}{2}}.
$$

We have that $\hat{h}(\rho)$ is increasing on $[0, \bar{\rho})$ and decreasing on $(\bar{\rho}, +\infty)$, with the point ρ being $\bar{\rho} = \left(\frac{4-N}{p\gamma_p(2p\gamma_p - N)A_1} \right)^{\frac{1}{p\gamma_p - 2}}$. Since $2 < p\gamma_p$, we get

$$
\max_{\rho \geq 0} \hat{h}(\rho) = \hat{h}(\bar{\rho}) = \frac{2p\gamma_p - 4}{2p\gamma_p - N} \left(\frac{4-N}{p\gamma_p(2p\gamma_p - N)A_1} \right)^{\frac{4-N}{2p\gamma_p - 4}} > \frac{N A_2}{2}
$$

if and only if

$$
\max\{a_1, a_2\} < D_0 := \left(\frac{3}{\alpha C^3(N,p)} \frac{2(2p\gamma_p - 4)}{N(2p\gamma_p - N)} \right)^{\frac{N(p-2)-4}{4(p-3)}} \left(\frac{4-N}{\gamma_p(2p\gamma_p - N)C^p(N,p)} \right)^{\frac{4-N}{4(p-3)}}.
$$

As $\lim_{s \to 0^+} \hat{h}(s) = 0^+$ and $\lim_{s \to +\infty} \hat{h}(s) = -\infty$, we see that h has exactly two critical points if $\max\{a_1, a_2\} < D_0$.

Notice that

$$h(\rho) > 0 \iff \tilde{h}(\rho) > A_2 \quad \text{with} \quad \tilde{h}(\rho) = \frac{1}{2}\rho^{2-\frac{N}{2}} - A_1\rho^{p\gamma_p - \frac{N}{2}}.$$

It is not difficult to check that $\tilde{h}(\rho)$ is increasing on $[0, \rho_0)$ and decreasing on $(\rho_0, +\infty)$, where $\rho_0 = \left(\frac{4-N}{2(2p\gamma_p - N)A_1}\right)^{\frac{1}{p\gamma_p - 2}}$. We have

$$\max_{\rho \geq 0} \tilde{h}(\rho) = \tilde{h}(\rho_0) = \frac{p\gamma_p - 2}{2p\gamma_p - N}\left(\frac{4-N}{2(2p\gamma_p - N)A_1}\right)^{\frac{4-N}{2p\gamma_p - 4}} > A_2$$

provided

$$\max\{a_1, a_2\} < D := \left(\frac{3}{\alpha C^3(N,p)}\frac{p\gamma_p - 2}{2p\gamma_p - N}\right)^{\frac{N(p-2)-4}{4(p-3)}}\left(\frac{p(4-N)}{2(2p\gamma_p - N)C^p(N,p)}\right)^{\frac{4-N}{4(p-3)}}.$$

We have $h(\rho) > 0$ on an open interval (R_0, R_1) if and only if $\max\{a_1, a_2\} < D$. We claim that $D < D_0$. To this purpose, we only need to prove that $\left(\frac{4}{N}\right)^{N(p-3)}\left(\frac{1}{p-2}\right)^{4-N} > 1$ holds. As in [38, Lemma 5.2], by letting $z = \frac{4}{N}, y = p - 2$, we have

$$\left(\frac{4}{N}\right)^{N(p-3)}\left(\frac{1}{p-2}\right)^{4-N} > 1 \iff z^{y-1} > y^{z-1}.$$

Since $\frac{\log z}{z-1}$ is a monotone decreasing function for $z > 0$, we have $D < D_0$.
If $\max\{a_1, a_2\} < D$, combining $\lim_{s \to 0^+} h(s) = 0^-$ and $\lim_{s \to +\infty} h(s) = -\infty$, we see that h has a local minimum point at negative level in $(0, R_0)$ and a global maximum point at positive level in (R_0, R_1). Define

$$\rho^* := \left(\frac{p(4-N)}{2(2p\gamma_p - N)C^p(N,p)}\right)^{\frac{1}{p\gamma_p - 2}} D^{-\frac{p(1-\gamma_p)}{p\gamma_p - 2}}, \tag{2.15}$$

then $\rho^* < \rho_0$. By direct calculation, we have

$$\tilde{h}(\rho^*) > \frac{1}{2}(\rho^*)^{2-\frac{N}{2}} - \frac{C(N,p)}{p}D^{p(1-\gamma_p)}(\rho^*)^{p\gamma_p - \frac{N}{2}} = \frac{\alpha}{3}C^3(N,p)D^{\frac{6-N}{2}} > A_2,$$

then $h(\rho^*) > 0$ and $\rho^* > R_0$. Note that ρ^* is independent of a_1, a_2. In addition, it holds that

$$\tilde{h}\left(\frac{\max\{a_1, a_2\}}{D}\rho^*\right) = \frac{1}{2}\left(\frac{\max\{a_1, a_2\}}{D}\right)^{\frac{4-N}{2}}(\rho^*)^{2-\frac{N}{2}} - A_1\left(\frac{\max\{a_1, a_2\}}{D}\right)^{\frac{N(p-3)}{2}}(\rho^*)^{p\gamma_p - \frac{N}{2}}$$

$$> \left(\frac{\max\{a_1, a_2\}}{D}\right)^{\frac{4-N}{2}}\frac{\alpha}{3}C^3(N,p)D^{\frac{6-N}{2}} > A_2.$$

(ii) Similarly to the proof of (i), we have

$$R_0 = \bar{\rho} = \rho_0 = \rho^* = R_1, \quad \tilde{h}(\rho_0) = A_2, \quad \hat{h}(\bar{\rho}) > \frac{N}{2}A_2.$$

$\square$

Next, we study the structure of the manifold

$$\bar{\mathcal{P}}_{a_1,a_2} := \mathcal{P}_{a_1,a_2} \cap \mathcal{M}. \tag{2.16}$$

We will observe that a critical point for the functional E on $\bar{\mathcal{P}}_{a_1,a_2}$ is a critical point for the functional E on $S(a_1,a_2)$. Hence, $\bar{\mathcal{P}}_{a_1,a_2}$ is a natural constraint.

Lemma 2.3. *Let $N \leq 3$, $2_* < p < 2^*$, and $\alpha, a_1, a_2 > 0$. If $\max\{a_1, a_2\} \leq D$, then $\mathcal{P}^0_{a_1,a_2} = \emptyset$, and the set $\bar{\mathcal{P}}_{a_1,a_2}$ is a C^1-submanifold of codimension 1 in $S(a_1,a_2)$.*

Proof. We adopt the Soave's argument from [38]. It is sufficient to prove that $\mathcal{P}^0_{a_1,a_2}$ is empty. Indeed, if $\mathcal{P}^0_{a_1,a_2} = \emptyset$, we show that $\bar{\mathcal{P}}_{a_1,a_2}$ is a C^1-submanifold of codimension 1 in $S(a_1,a_2)$. Assume by contradiction that there exists a $\vec{u} \in \mathcal{P}^0_{a_1,a_2}$ such that $P(\vec{u}) = 0$, thus

$$\Psi''_{\vec{u}}(0) = \sum_{i=1}^{3} \int \left(2|\nabla u_i|^2 - p\gamma_p^2 |u_i|^p \right) - \frac{N^2}{4} \alpha \mathrm{Re} \int u_1 u_2 \overline{u}_3 = 0.$$

Let

$$f(y) := y\Psi'_{\vec{u}}(0) - \Psi''_{\vec{u}}(0)$$

$$= (y-2) \sum_{i=i}^{3} \int |\nabla u_i|^2 - (y - p\gamma_p)\gamma_p \sum_{i=i}^{3} \int |u_i|^p - \left(y - \frac{N}{2} \right) \frac{N}{2} \alpha \mathrm{Re} \int u_1 u_2 \overline{u}_3,$$

and observe that $f(y) = 0$, $\forall y \in \mathbb{R}$. Therefore, it follows from $f\left(\frac{N}{2}\right) = 0$ that

$$\left(2 - \frac{N}{2} \right) \sum_{i=2}^{3} \|\nabla u_i\|_2^2 = \gamma_p \left(p\gamma_p - \frac{N}{2} \right) \sum_{i=2}^{3} \|u_i\|_p^p. \tag{2.17}$$

By (2.11) and (2.17), we have

$$\left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{\frac{1}{2}} \geq \left(\frac{4-N}{\gamma_p(2p\gamma_p - N)C^p(N,p)} \right)^{\frac{1}{p\gamma_p-2}} (\max\{a_1, a_2\})^{-\frac{p(1-\gamma_p)}{p\gamma_p-2}}.$$

Since $f(p\gamma_p) = 0$, we get

$$(p\gamma_p - 2) = \left(p\gamma_p - \frac{N}{2} \right) \frac{N}{2} \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{-1} \alpha \mathrm{Re} \int u_1 u_2 \overline{u}_3$$

$$\leq \left(p\gamma_p - \frac{N}{2} \right) \frac{N}{2} A_2 \left(\frac{4-N}{\gamma_p(2p\gamma_p - N)C^p(N,p)} \right)^{\frac{N-4}{2p\gamma_p-4}} (\max\{a_1, a_2\})^{\frac{p(1-\gamma_p)(4-N)}{2p\gamma_p-4}},$$

which is a contradiction with respect to the hypothesis $\max\{a_1, a_2\} \leq D < D_0$.

We now prove that $\bar{\mathcal{P}}_{a_1,a_2}$ is a smooth manifold of codimension 1 in $S(a_1,a_2)$. We know that $\bar{\mathcal{P}}_{a_1,a_2}$ is defined by $P(\vec{u}) = 0$, $G_1(u_1, u_2) = 0$, $G_2(u_2, u_3) = 0$ and $G(u_1, u_2, u_3) > 0$ where

$$G_1(u_1, u_3) = \|u_1\|_2^2 + \|u_3\|_2^2 - a_1^2, \quad G_2(u_2, u_3) = \|u_2\|_2^2 + \|u_3\|_2^2 - a_2^2,$$

and

$$G(u_1, u_2, u_3) = \mathrm{Re} \int u_1 u_2 \overline{u}_3.$$

Since P, G_1, G_2, and G are of C^1-class, the proof is complete provided we show that the map $d(P, G_1, G_2, G) : H^1(\mathbb{R}^N, \mathbb{C}^3) \to \mathbb{R}^4$ is surjective, for every

$$(u_1, u_2, u_3) \in \left(G_1^{-1}(0) \times G_2^{-1}(0)\right) \cap G^{-1}(0) \cap P^{-1}(0).$$

If this is not true, $dP(\vec{u})$ has to be linearly dependent from $dG_1(u_1, u_3)$, $dG_2(u_2, u_3)$, and $dG(u_1, u_2, u_3)$, i.e. there exist $\nu_1, \nu_2, \nu_3 \in \mathbb{R}$ such that

$$\begin{cases} \operatorname{Re} \int 2\nabla u_1 \nabla \overline{\varphi}_1 + 2\nu_1 u_1 \overline{\varphi}_1 + \nu_3 \alpha u_3 \overline{u}_2 \overline{\varphi}_1 = \operatorname{Re} \int p\gamma_p |u_1|^{p-2} u_1 \overline{\varphi}_1 + \frac{N\alpha}{2} u_3 \overline{u}_2 \overline{\varphi}_1, \\ \operatorname{Re} \int 2\nabla u_2 \nabla \overline{\varphi}_2 + 2\nu_2 u_2 \overline{\varphi}_2 + \nu_3 \alpha u_3 \overline{u}_1 \overline{\varphi}_2 = \operatorname{Re} \int p\gamma_p |u_2|^{p-2} u_2 \overline{\varphi}_2 + \frac{N\alpha}{2} u_3 \overline{u}_1 \overline{\varphi}_2, \\ \operatorname{Re} \int 2\nabla u_3 \nabla \overline{\varphi}_3 + 2(\nu_1 + \nu_2) u_3 \overline{\varphi}_3 + \nu_3 \alpha u_1 u_2 \overline{\varphi}_3 = \operatorname{Re} \int p\gamma_p |u_3|^{p-2} u_3 \overline{\varphi}_3 + \frac{N\alpha}{2} u_1 u_2 \overline{\varphi}_3. \end{cases}$$

$$(2.18)$$

Testing $(i\varphi_1, i\varphi_2, i\varphi_3)$ instead of $(\varphi_1, \varphi_2, \varphi_3)$ and using the fact $\operatorname{Re}(iz) = -\operatorname{Im}(z)$, we eventually obtain

$$\begin{cases} \int 2\nabla u_1 \nabla \overline{\varphi}_1 + 2\nu_1 u_1 \overline{\varphi}_1 + \nu_3 \alpha u_3 \overline{u}_2 \overline{\varphi}_1 = \int p\gamma_p |u_1|^{p-2} u_1 \overline{\varphi}_1 + \frac{N\alpha}{2} u_3 \overline{u}_2 \overline{\varphi}_1, \\ \int 2\nabla u_2 \nabla \overline{\varphi}_2 + 2\nu_2 u_2 \overline{\varphi}_2 + \nu_3 \alpha u_3 \overline{u}_1 \overline{\varphi}_2 = \int p\gamma_p |u_2|^{p-2} u_2 \overline{\varphi}_2 + \frac{N\alpha}{2} u_3 \overline{u}_1 \overline{\varphi}_2, \\ \int 2\nabla u_3 \nabla \overline{\varphi}_3 + 2(\nu_1 + \nu_2) u_3 \overline{\varphi}_3 + \nu_3 \alpha u_1 u_2 \overline{\varphi}_3 = \int p\gamma_p |u_3|^{p-2} u_3 \overline{\varphi}_3 + \frac{N\alpha}{2} u_1 u_2 \overline{\varphi}_3, \end{cases}$$

for every $(\varphi_1, \varphi_2, \varphi_3) \in C_0^\infty \times C_0^\infty \times C_0^\infty$. Therefore, (u_1, u_2, u_3) satisfies

$$\begin{cases} -2\Delta u_1 + 2\nu_1 u_1 + \nu_3 \alpha u_3 \overline{u}_2 = p\gamma_p |u_1|^{p-2} u_1 + \frac{N\alpha}{2} u_3 \overline{u}_2, \\ -2\Delta u_2 + 2\nu_2 u_2 + \nu_3 \alpha u_3 \overline{u}_1 = p\gamma_p |u_2|^{p-2} u_2 + \frac{N\alpha}{2} u_3 \overline{u}_2, \\ -2\Delta u_3 + 2(\nu_1 + \nu_2) u_3 + \nu_3 \alpha u_1 u_2 = p\gamma_p |u_3|^{p-2} u_3 + \frac{N\alpha}{2} u_1 u_2. \end{cases}$$

The Pohozaev identity for the above system is

$$\sum_{i=1}^3 \int \left(2|\nabla u_i|^2 - p\gamma_p^2 |u_i|^p\right) - \frac{N}{2}\left(\frac{N}{2} - \nu_3\right) \alpha \operatorname{Re} \int u_1 u_2 \overline{u}_3 = 0.$$

Then

$$\sum_{i=1}^3 \int \left(2|\nabla u_i|^2 - p\gamma_p^2 |u_i|^p\right) = 0.$$

Since

$$(u_1, u_2, u_3) \in (G_1^{-1}(0) \times G_2^{-1}(0)) \cap G^{-1}(0) \cap P^{-1}(0),$$

we have

$$\sum_{i=1}^3 \int |\nabla u_i|^2 = \gamma_p \sum_{i=1}^3 \int |u|^p,$$

which is a contradiction. By [34, Proposition A.1], we get that if $\inf\limits_{\vec{u} \in \overline{\mathcal{P}}_{a_1, a_2}} E = E(u_1, u_2, u_3)$, then there exist $\lambda_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$ such that

$$\begin{cases} -(1 + \lambda_4)\Delta u_1 + \lambda_1 u_1 + \lambda_3 \alpha u_3 \overline{u}_2 = \left(1 + \frac{p\gamma_p \lambda_4}{2}\right)|u_1|^{p-2} u_1 + \left(1 + \frac{N\lambda_4}{4}\right)\alpha u_3 \overline{u}_2, \\ -(1 + \lambda_4)\Delta u_2 + \lambda_2 u_2 + \lambda_3 \alpha u_3 \overline{u}_1 = \left(1 + \frac{p\gamma_p \lambda_4}{2}\right)|u_2|^{p-2} u_2 + \left(1 + \frac{N\lambda_4}{4}\right)\alpha u_3 \overline{u}_1, \\ -(1 + \lambda_4)\Delta u_3 + (\lambda_1 + \lambda_2) u_3 + \lambda_3 \alpha u_1 u_2 = \left(1 + \frac{p\gamma_p \lambda_4}{2}\right)|u_3|^{p-2} u_3 + \left(1 + \frac{N\lambda_4}{4}\right)\alpha u_1 u_2. \end{cases}$$

Therefore, by combining $G(u_1, u_2, u_3) > 0$ with [16, Theorem 1] or the proof of [34, Proposition A.1], we have $\lambda_3 = 0$. Thus, we obtain that the restricted set $\mathcal{M}$ does not change the structure of the manifold $\mathcal{P}_{a_1,a_2}$. In conclusion, $\bar{\mathcal{P}}_{a_1,a_2}$ is a smooth manifold of codimension 1 on $S(a_1, a_2)$. $\qquad\square$

Lemma 2.4. *Let $N \leq 3$, $2_* < p < 2^*$, and $\alpha, a_1, a_2 > 0$. If $\max\{a_1, a_2\} < D$, for $\vec{u} \in S(a_1, a_2) \cap \mathcal{M}$, then the function $\Psi_{\vec{u}}(s)$ has exactly two critical points $s_{\vec{u}} < \sigma_{\vec{u}} \in \mathbb{R}$ and two zeros $c_{\vec{u}} < d_{\vec{u}}$ with $s_{\vec{u}} < c_{\vec{u}} < \sigma_{\vec{u}} < d_{\vec{u}}$. Moreover, we have the properties below:*

(i) *$s_{\vec{u}} \star \vec{u} \in \mathcal{P}^+_{a_1,a_2}$ and $\sigma_{\vec{u}} \star \vec{u} \in \mathcal{P}^-_{a_1,a_2}$. Moreover, if $s \star \vec{u} \in \mathcal{P}_{a_1,a_2}$, then either $s = s_{\vec{u}}$ or $s = \sigma_{\vec{u}}$,*

(ii) *$s_{\vec{u}} < R_0 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{-\frac{1}{2}}$ and*

$$\Psi_{\vec{u}}(s_{\vec{u}}) = \inf \left\{ \Psi_{\vec{u}}(s) : s \in \left(0, R_0 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{-\frac{1}{2}} \right) \right\} < 0,$$

(iii) *$E(\sigma_{\vec{u}} \star \vec{u}) = \max_{s \in \mathbb{R}} E(s \star \vec{u}) > 0$,*

(iv) *The maps $\vec{u} \mapsto s_{\vec{u}} \in \mathbb{R}$ and $\vec{u} \mapsto \sigma_{\vec{u}} \in \mathbb{R}$ are of class C^1.*

Proof. Let $\vec{u} \in S(a_1, a_2)$, we have $s \star \vec{u} \in \mathcal{P}_{a_1,a_2}$ if and only if $\Psi'_{\vec{u}}(s) = 0$, Ψ defined in (2.8). By (2.13)-(2.14), we get

$$\Psi_{\vec{u}}(s) = E(s \star \vec{u}) \geq h\left(s \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{\frac{1}{2}} \right).$$

If $\max\{a_1, a_2\} < D$, from Lemma 2.1, point (i), $\Psi_{\vec{u}}(s)$ is positive in the interval

$$\left(R_0 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{-\frac{1}{2}}, R_1 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{-\frac{1}{2}} \right),$$

and we have the asymptotic behavior $\lim_{s \to -\infty} \Psi_{\vec{u}}(s) = 0^-$, $\lim_{s \to +\infty} \Psi_{\vec{u}}(s) = -\infty$, thus we can see that $\Psi_{\vec{u}}(s)$ has a local minimum point $s_{\vec{u}}$ in $\left(0, R_0 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{-\frac{1}{2}} \right)$ and a global maximum point $\sigma_{\vec{u}}$ in $\left(R_0 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{-\frac{1}{2}}, R_1 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{-\frac{1}{2}} \right)$. It follows from Lemma 2.1 that $\Psi_{\vec{u}}(s)$ has no other critical points.

Since $\Psi''_{\vec{u}}(s_{\vec{u}}) \geq 0$, $\Psi''_{\vec{u}}(\sigma_{\vec{u}}) \leq 0$ and $\mathcal{P}^0_{a_1,a_2} = \emptyset$, we know that $s_{\vec{u}} \star \vec{u} \in \mathcal{P}^+_{a_1,a_2}$ and $\sigma_{\vec{u}} \star \vec{u} \in \mathcal{P}^-_{a_1,a_2}$. By the monotonicity and the behavior at infinity of $\Psi_{\vec{u}}(s)$, we get that $\Psi_{\vec{u}}(s)$ has exactly two zeros $c_{\vec{u}} < d_{\vec{u}}$ with $s_{\vec{u}} < c_{\vec{u}} < \sigma_{\vec{u}} < d_{\vec{u}}$. Thus, the conclusions (i)-(iii) follows from the facts above. Finally, applying the Implicit Function Theorem to the C^1 function $g : \mathbb{R} \times S(a_1, a_2) \mapsto \mathbb{R}$ defined by $g = g(s, \vec{u}) = \Psi'_{\vec{u}}(s)$. Therefore, we have that $\vec{u} \mapsto s_{\vec{u}}$ is of class C^1 because $g_{\vec{u}}(s_{\vec{u}}) = 0$ and $\partial_s g_{\vec{u}}(s_{\vec{u}}) = \Psi''_{\vec{u}}(s_{\vec{u}}) > 0$. Similarly, we can prove that $\vec{u} \mapsto \sigma_{\vec{u}} \in \mathbb{R}$ is of class C^1, and (iv) follows. $\qquad\square$

3. Proof of Theorem 1

In this section, we give a proof of Theorem 1, and we divide it into two cases: $p = 2_*$ and $2_* < p < 2^*$. We first prove several results eventually leading to the conclusions of Theorem 1.

3.1. **Mass-energy intracritical case.** Let $c > 0$, and for $N \leq 3$ we consider $2_* < p < 2^*$. We introduce the following complex valued equation:

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u, & u \in H^1(\mathbb{R}^N, \mathbb{C}), \\ \int |u|^2 = c^2. \end{cases} \tag{3.1}$$

From [13, 29, 30, 38], the solutions of (3.1) corresponds to the critical points of the functional $J : H^1(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$,

$$J(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p} \int |u|^p, \tag{3.2}$$

constrained on the sphere

$$S(c) = \{u \in H^1(\mathbb{R}^N) \text{ s.t. } \|u\|_2^2 = c^2\},$$

and the parameter λ appears as a Lagrange multiplier. We introduce the Pohozaev-type constraint for the single equations (3.1)

$$\mathcal{P}_c := \left\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) \cap S(c) \text{ s.t. } \|\nabla u\|_2^2 = \gamma_p \|u\|_p^p \right\}, \tag{3.3}$$

recalling that $\gamma_p = \frac{N(p-2)}{2p}$. Define

$$m(c) = \inf_{\mathcal{P}_c} J(u) > 0. \tag{3.4}$$

The next Lemma, see [23, Lemma 2.3], ensures that the infimum $m(c)$ above is the same if we restrict to real functions.

Lemma 3.1. *Let $c > 0$, $N \leq 3$, and $2_* < p < 2^*$. We have that*

$$m(c) = \inf_{H^1(\mathbb{R}^N, \mathbb{R}) \cap \mathcal{P}_c} J(u),$$

and $m(c)$ is strictly decreasing with respect to c. Moreover, any normalized solution of (3.1) has the form $e^{i\sigma}U$, where $\sigma \in \mathbb{R}$ and U is a positive, radial decreasing normalized solution of (3.1).

Let us introduce the set

$$B_{\rho^*} := \left\{ \vec{u} \in H^1(\mathbb{R}^N, \mathbb{C}^3) \text{ s.t. } \left(\sum_{i=1}^3 \|\nabla u_i\|_2^2 \right)^{\frac{1}{2}} < \rho^* \right\}$$

and

$$V(a_1, a_2) := S(a_1, a_2) \cap B_{\rho^*} \cap \mathcal{M},$$

where $\mathcal{M}$ is defined in (2.10) and ρ^* in (2.15). Thus, we can define the following minimization problem: for any positive a_1 and a_2 such that $\max\{a_1, a_2\} < D$, let

$$m(a_1, a_2) := \inf_{\vec{u} \in V(a_1, a_2)} E(\vec{u}). \tag{3.5}$$

Lemma 3.2. *Let $N \leq 3$, $2_* < p < 2^*$, and $\alpha, a_1, a_2 > 0$. If $\max\{a_1, a_2\} < D$, the set $\mathcal{P}^+_{a_1, a_2}$ is contained in $V(a_1, a_2)$ and*

$$m(a_1, a_2) = m^+(a_1, a_2) := \inf_{\vec{u} \in \mathcal{P}^+_{a_1, a_2} \cap \mathcal{M}} E(\vec{u}) = \inf_{\vec{u} \in \bar{\mathcal{P}}_{a_1, a_2}} E(\vec{u}) < 0. \tag{3.6}$$

Moreover, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$

$$m(a_1, a_2) < \inf_{S(a_1, a_2) \cap (B_{\rho^*} \setminus B_{\rho^* - \varepsilon})} E(\vec{u}).$$

Proof. For $\vec{u} \in V(a_1, a_2)$, we have

$$E(\vec{u}) \geq h\left(\left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2\right)^{\frac{1}{2}}\right) \geq \min_{\rho \in [0, R_0]} h(\rho) > -\infty,$$

where R_0 and h are given in Lemma 2.2. For a function $\vec{u} \in S(a_1, a_2) \cap \mathcal{M}$, there exists $s_0 > 0$ small enough such that $s_0 \star \vec{u} \in B_{\rho_0}$ and $E(s_0 \star \vec{u}) < 0$. Hence, we get $m(a_1, a_2) \in (-\infty, 0)$. From Lemma 2.4, we have $\mathcal{P}^+_{a_1, a_2} \cap \mathcal{M} \subset V(a_1, a_2)$, and then $m(a_1, a_2) \leq \inf_{\mathcal{P}^+_{a_1, a_2} \cap \mathcal{M}} E$. In addition, if $\vec{u} \in V(a_1, a_2)$, $s_{\vec{u}} \star \vec{u} \in \mathcal{P}^+_{a_1, a_2} \subset V(a_1, a_2)$, we get

$$E(s_{\vec{u}} \star \vec{u}) = \min\{E(s \star \vec{u}) : s \in \mathbb{R} \text{ and } s \star \vec{u} \in V(a_1, a_2)\} \leq E(\vec{u}),$$

and it follows that $\inf_{\mathcal{P}^+_{a_1, a_2} \cap \mathcal{M}} E \leq m(a_1, a_2)$. By Lemma 2.4, $E(\vec{u}) > 0$ on $\mathcal{P}^-_{a_1, a_2}$, so we conclude that $m(a_1, a_2) = \inf_{\vec{u} \in \bar{\mathcal{P}}_{a_1, a_2}} E(\vec{u}) = \inf_{\vec{u} \in \mathcal{P}^+_{a_1, a_2} \cap \mathcal{M}} E(\vec{u})$.

There exists $\varepsilon > 0$ small enough such that, if $\rho \in [\rho^* - \varepsilon, \rho^*]$, we have $h(\rho) \geq \frac{m(a_1, a_2)}{2}$, and then

$$E(\vec{u}) \geq h\left(\sum_{j}^{3} \|\nabla u_j\|_2^2\right) \geq \frac{m(a_1, a_2)}{2} > m(a_1, a_2),$$

for any $\vec{u} \in S(a_1, a_2)$ and $\rho^* - \varepsilon \leq \sum_{j}^{3} \|\nabla u_j\|_2^2 \leq \rho^*$, where in the last inequality we used the fact that m is negative. $\square$

Let $\vec{u}$ belong to $H^1(\mathbb{R}^N, \mathbb{C}^3)$. $E(|\vec{u}|) \leq E(\vec{u})$, and by the symmetric rearrangement, see [9, 31],

$$\|\nabla |u_i|^*\|_2 \leq \|\nabla |u_i|\|_2 \leq \|\nabla u_i\|_2, \quad \||u_i|^*\|_p = \|u_i\|_p,$$

and

$$\int |u_1||u_2||u_3| \leq \int |u_1|^* |u_2|^* |u_3|^*,$$

where $|u_i|^*$ is the Schwarz symmetric rearrangement of $|u_i|$, for $i = 1, 2, 3$. Then $E(|\vec{u}|^*) \leq E(|\vec{u}|) \leq E(\vec{u})$, where the short notation $|\vec{u}|^*$ stands for $|\vec{u}|^* = (|u_1|^*, |u_2|^*, |u_3|^*)$. Let us consider $(v_1, v_2, v_3) \in H^1(\mathbb{R}^N, \mathbb{R}^3)$ a solution to the system (1.4), namely

$$\begin{cases} -\Delta v_1 + \lambda_1 v_1 = |v_1|^{p-2} v_1 + \alpha v_3 v_2, \\ -\Delta v_2 + \lambda_2 v_2 = |v_2|^{p-2} v_2 + \alpha v_3 v_1, \\ -\Delta v_3 + (\lambda_1 + \lambda_2) v_3 = |v_3|^{p-2} v_3 + \alpha v_1 v_2. \end{cases} \tag{3.7}$$

Denote

$$\mathcal{P}_{r, a_1, a_2} := \{\vec{v} \in H^1_r(\mathbb{R}^N, \mathbb{R}^3) \cap S(a_1, a_2) \text{ s.t. } P(\vec{v}) = 0\},$$

and

$$\mathcal{P}^+_{r,a_1,a_2} := H^1_r(\mathbb{R}^N, \mathbb{R}^3) \cap \mathcal{P}^+_{a_1,a_2}.$$

The notation $H^1_r(\mathbb{R}^N, \mathbb{R}^3)$ denotes the subspace of functions in $H^1(\mathbb{R}^N, \mathbb{R}^3)$ which are radially symmetric. We set

$$m^+_r(a_1, a_2) := \inf_{\vec{u} \in \mathcal{P}^+_{r,a_1,a_2} \cap \mathcal{M}} E(\vec{u}), \tag{3.8}$$

and

$$W^+_r := \left\{ \vec{u} \in H^1_r(\mathbb{R}^N, \mathbb{R}^3) \cap S(a_1, a_2) \text{ s.t. } E(\vec{u}) = m^+_r(a_1, a_2) \right\}.$$

We have the following.

Lemma 3.3. *Let $N \leq 3$, $2_* < p < 2^*$, and $\alpha, a_1, a_2 > 0$. If $\max\{a_1, a_2\} < D$, then*

$$m^+_r(a_1, a_2) = \inf_{\vec{u} \in \mathcal{P}^+_{r,a_1,a_2} \cap \mathcal{M}} E(\vec{u}) = \inf_{\vec{u} \in \mathcal{P}^+_{a_1,a_2} \cap \mathcal{M}} E(\vec{u}).$$

Moreover, if $\inf_{\mathcal{P}^+_{a_1,a_2} \cap \mathcal{M}} E$ is reached, it is reached by a Schwartz radially symmetric function. More precisely, $\inf_{\mathcal{P}^+_{a_1,a_2} \cap \mathcal{M}} E$ is reached by the vector function $(e^{i\theta_1}w_1, e^{i\theta_1}w_2, e^{i(\theta_1+\theta_2)}w_3)$ where $\vec{w}$ is the minimizer for $\inf_{\mathcal{P}^+_{r,a_1,a_2}} E$ and $(\theta_1, \theta_2) \in \mathbb{R}^2$.

Proof. It follows from $\mathcal{P}^+_{r,a_1,a_2} \subset \mathcal{P}^+_{a_1,a_2}$ that $\inf_{\mathcal{P}^+_{r,a_1,a_2} \cap \mathcal{M}} E \geq \inf_{\mathcal{P}^+_{a_1,a_2} \cap \mathcal{M}} E$. From Lemma 2.4, for any $\vec{u} \in S(a_1, a_2) \cap \mathcal{M}$, there exists $s^+_{\vec{u}} \in \mathbb{R}$ such that $s^+_{\vec{u}} \star \vec{u} \in \mathcal{P}^+_{a_1,a_2}$, and

$$\inf_{\vec{u} \in \mathcal{P}^+_{a_1,a_2} \cap \mathcal{M}} E(\vec{u}) = \inf_{\vec{u} \in S(a_1,a_2) \cap \mathcal{M}} \min_{-\infty < \sigma \leq s^+_{\vec{u}}} E(\sigma \star \vec{u}).$$

For $\vec{u} \in S(a_1, a_2)$, let $\vec{w} \in S_r(a_1, a_2)$ be the Schwarz rearrangement of $(|u_1|, |u_2|, |u_3|)$, i.e. $(w_1, w_2, w_3) := (|u_1|^*, |u_2|^*, |u_3|^*)$. Then, we have for all $\sigma > 0$, $E(\sigma \star \vec{w}) \leq E(\sigma \star \vec{u})$. Recalling that $\Psi'_{\vec{u}}(\sigma) = P(\sigma \star \vec{u})$ (see (2.8)), we have

$$\lim_{\sigma \to -\infty} \Psi'_{\vec{w}}(\sigma) \leq \lim_{\sigma \to -\infty} \Psi'_{\vec{u}}(\sigma) < 0 \quad \text{and} \quad \Psi''_{\vec{w}}(\sigma) \leq \Psi''_{\vec{u}}(\sigma) \quad \forall \, \sigma > -\infty.$$

It follows that $-\infty < s^+_{\vec{u}} \leq s^+_{\vec{w}}$. Therefore, we have

$$\min_{-\infty < \sigma < s^+_{\vec{w}}} E(\sigma \star \vec{w}) \leq \min_{-\infty < \sigma < s^+_{\vec{u}}} E(\sigma \star \vec{u}),$$

and then $\inf_{\mathcal{P}^+_{r,a_1,a_2} \cap \mathcal{M}} E \leq \inf_{\mathcal{P}^+_{a_1,a_2} \cap \mathcal{M}} E$.

First, we set $\vec{v} := (e^{i\theta_1}w_1, e^{i\theta_2}w_2, e^{i(\theta_1+\theta_2)}w_3)$, where $\theta_1, \theta_2 \in \mathbb{R}$ and $E(\vec{w}) = m^+_r(a_1, a_2)$. Then, $\vec{v} \in S(a_1, a_2)$ and

$$E(\vec{v}) = \frac{1}{2} \sum_{j=1}^3 \|\nabla w_i\|^2_2 - \frac{1}{p} \sum_{j=1}^3 \|w_i\|^p_p - \alpha \text{Re} \int e^{i\theta_1} w_1 e^{i\theta_2} w_2 e^{-i(\theta_1+\theta_2)} w_3 = E(\vec{w}).$$

Thus, $\left\{ (e^{i\theta_1}w_1, e^{i\theta_2}w_2, e^{i(\theta_1+\theta_2)}w_3) \text{ s.t. } \theta_1, \theta_2 \in \mathbb{R}, \vec{w} \in W^+_r \right\} \subset \mathcal{G}$.

We claim that for any $\vec{u} \in \mathcal{G}$, $\vec{w} := (|u_1|^*, |u_2|^*, |u_3|^*) \in H_r^1(\mathbb{R}^N, \mathbb{R}^3) \cap S(a_1, a_2)$, the Schwarz rearrangement of $(|u_1|, |u_2|, |u_3|)$, belongs to $\mathcal{P}_{r,a_1,a_2}^+$. Indeed, if $\sum_{i=1}^3 \|\nabla w_i\|_2^2 < \sum_{j=1}^3 \|\nabla u_i\|_2^2$ or $\operatorname{Re} \int u_1 u_2 \overline{u}_3 > \operatorname{Re} \int w_1 w_2 \overline{w}_3$, then $E(\vec{u}) < E(\vec{w})$. We have

$$\inf_{\vec{u} \in \mathcal{P}_{a_1,a_2}^+ \cap \mathcal{M}} E(\vec{u}) = \inf_{\vec{u} \in S(a_1,a_2) \cap \mathcal{M}} \min_{-\infty < \sigma \le s_{\vec{u}}^+} E(\sigma \star \vec{u})$$

$$\le \min_{-\infty < \sigma \le s_{\vec{w}}^+} E(\sigma \star \vec{w}) < \min_{-\infty < \sigma \le s_{\vec{w}}^+} E(\sigma \star \vec{u})$$

$$= \inf_{\vec{u} \in \mathcal{P}_{a_1,a_2}^+ \cap \mathcal{M}} E(\vec{u}),$$

which is a contradiction. In the chain of relations above, we used in order: the definition, the fact that $\vec{w} \in S(a_1, a_2)$, the relation $E(\vec{u}) < E(\vec{w})$, in the last identity we employed the inequality $s_{\vec{u}}^+ \le s_{\vec{w}}^+$, and the fact that u is in the set of ground states $\mathcal{G}$. Therefore, $\vec{w} \in \mathcal{P}_{r,a_1,a_2}^+$ and $E(\vec{w}) = E(\vec{u})$. We set $\tilde{u}_j(x) := \frac{u_j(x)}{|u_j(x)|}$, $j = 1, 2, 3$. Since $|\tilde{u}_j| = 1$, we get $\operatorname{Re}(\overline{\tilde{u}}_j \nabla \tilde{u}_j) = 0$,

$$\nabla u_j = (\nabla(|u_j|))\tilde{u}_j + |u_j| \nabla \tilde{u}_j = \tilde{u}_j \left(\nabla(|u_j|) + |u_j| \overline{\tilde{u}}_j \nabla \tilde{u}_j \right),$$

and $|\nabla u_j|^2 = |\nabla(|u_j|)|^2 + |u_j|^2 |\nabla \tilde{u}_j|^2$. Since $E(\vec{w}) = E(\vec{u})$, we have $E(\vec{u}_0) = E(|u_1|, |u_2|, |u_3|)$, and it follows that

$$\sum_{j=1}^3 \|\nabla |u_j|\|_2^2 - \alpha \int |u_1||u_2||u_3| = \sum_{j=1}^3 \|\nabla u_j\|_2^2 - \alpha \operatorname{Re} \int u_1 u_2 \overline{u}_3.$$

Since $\|\nabla |u_j|\|_2^2 \le \|\nabla u_j\|_2^2$ and $\operatorname{Re} \int u_1 u_2 \overline{u}_3 \le \int |u_1||u_2||u_3|$, we see that $\|\nabla |u_j|\|_2^2 = \|\nabla u_j\|_2^2$ for $j = 1, 2, 3$, and

$$\operatorname{Re} \int u_1 u_2 \overline{u}_3 = \int |u_1||u_2||u_3|. \tag{3.9}$$

By a direct calculation $\int |u_j|^2 |\nabla \tilde{u}_j|^2 = 0$, and for all $x \in \mathbb{R}^N$ we have $\tilde{u}_j(x) \equiv 1$, then there exists $\theta_j \in \mathbb{R}$ such that $u_j(x) = e^{i\theta_j} \rho_j(x)$ on $\mathbb{R}^N$, where $\rho_j(x) = |u_j(x)|$. Notice that

$$\int |u_1||u_2||u_3| = \int \rho_1(x) \rho_2(x) \rho_3(x) > 0. \tag{3.10}$$

Then, combining (3.9) and (3.10), we obtain that $\operatorname{Re}\left(e^{i(\theta_1 + \theta_2 - \theta_3)}\right) = 1$, it follows that $e^{i(\theta_1 + \theta_2)} = e^{i\theta_3}$. Similar arguments apply to the Schwarz symmetric rearrangement of the vector function (ρ_1, ρ_2, ρ_3). Hence, we can prove that $(\rho_1, \rho_2, \rho_3) \in W_r^+$. Therefore, for any $\vec{u} \in \mathcal{G}$, we have $\vec{u} = (e^{i\theta_1} v_1, e^{i\theta_2} v_2, e^{i(\theta_1 + \theta_2)} v_3)$ and $\vec{v} \in W_r^+$. $\qquad \square$

Lemma 3.4. *Let $N \le 3$, $2_* < p < 2^*$, and $\alpha, a_1, a_2 > 0$. If $\max\{a_1, a_2\} < D$, then (1.4) has a ground state solution $(\lambda_1, \lambda_2, u_1, u_2, u_3)$ with $\lambda_1, \lambda_2 > 0$, and $\vec{u} \in S(a_1, a_2)$ is positive, radially symmetric, and decreasing.*

Proof. By Lemma 3.3, we only need to show that $m_r^+(a_1, a_2)$ is attained. Since $m_r^+(a_1, a_2) = \inf_{V(a_1,a_2)} E$, and using the symmetric decreasing rearrangement, we obtain a minimizing sequence $\{\vec{w}_n\}$ with $\vec{w}_n \in H_r^1(\mathbb{R}^N, \mathbb{R}^3) \cap V(a_1, a_2)$ is decreasing for every n. Moreover, by Lemma 3.2, $E(s_{\vec{w}_n} \star \vec{w}_n) \le E(\vec{w}_n)$ and $s_{\vec{w}_n} \star \vec{w}_n \in V(a_1, a_2)$. Replacing $\vec{w}_n$ by $s_{\vec{w}_n} \star \vec{w}_n$, we have a new minimizing sequence $s_{\vec{w}_n} \star \vec{w}_n \in \mathcal{P}_{a_1,a_2}^+ \cap \mathcal{M}$. Thus, by the Ekeland's variational principle, we can choose a nonnegative radial Palais-Smale sequence $\{\vec{u}_n\}$ for $E|_{S(a_1,a_2)}$ at

level $m_r^+(a_1, a_2)$ with $P(\vec{u}_n) = o_n(1)$ such that $\lim\limits_{n \to \infty} E(\vec{u}_n) = m_r^+(a_1, a_2)$ and $E'|_{S(a_1,a_2)} \to 0$ as $n \to \infty$ (see also [27, Lemma 3.7]). Since

$$m_r^+(a_1, a_2) + o_n(1) = E(\vec{u}_n) = \left(\frac{1}{2} - \frac{1}{p\gamma_p} \right) \sum_{i=1}^{3} \int |\nabla u_{i,n}|^2 - \left(1 - \frac{N}{2p\gamma_p} \right) \alpha \int u_{1,n} u_{2,n} u_{3,n},$$

we have that the sequence $\{\vec{u}_n\}$ is bounded in $H_r^1(\mathbb{R}^N, \mathbb{R}^3)$. Indeed, using that $m_r^+(a_1, a_2) < 0$, by the Hölder and the Gagliardo-Nirenberg inequalities,

$$\sum_{i=1}^{3} \int |\nabla u_{i,n}|^2 \leq \frac{2p\gamma_p - N}{p\gamma_p - 2} \alpha \int u_{1,n} u_{2,n} u_{3,n}$$

$$\leq \frac{2p\gamma_p - N}{3(p\gamma_p - 2)} \alpha \max\{a_1^{\frac{6-N}{2}}, a_2^{\frac{6-N}{2}}\} C(N, p)^3 \left(\sum_{i=1}^{3} \|\nabla u_{i,n}\|_2^{\frac{N}{2}} \right).$$

As $2_* < p < 2^*$, we have $\frac{N}{2} < 2 < p\gamma_p$, hence the boundedness. Then there exists (u_1, u_2, u_3) such that $(u_{1,n}, u_{2,n}, u_{3,n}) \rightharpoonup (u_1, u_2, u_3)$ weakly in $H_r^1(\mathbb{R}^N, \mathbb{R}^3)$, $(u_{1,n}, u_{2,n}, u_{3,n}) \to (u_1, u_2, u_3)$ strongly in $L^r \times L^r \times L^r$ for $r \in (2, 2^*)$, and a.e. in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ as $n \to \infty$. Therefore, $u_i \geq 0$ are radial functions for $i = 1, 2, 3$.

By the Lagrange multiplier's rule (see [7, Lemma 3]), we know that there exists a sequence $\{(\lambda_{1,n}, \lambda_{2,n})\} \subset \mathbb{R} \times \mathbb{R}$ such that

$$\begin{cases} \int \nabla u_{1,n} \nabla \phi_1 + \lambda_{1,n} u_{1,n} \phi_1 - |u_{1,n}|^{p-2} u_{1,n} \phi_1 - \alpha u_{3,n} u_{2,n} \phi_1 = o_n(1) \|\phi_1\|_{H^1(\mathbb{R}^N)}, \\ \int \nabla u_{2,n} \nabla \phi_2 + \lambda_{2,n} u_{2,n} \phi_2 - |u_{2,n}|^{p-2} u_{2,n} \phi_2 - \alpha u_{3,n} u_{1,n} \phi_2 = o_n(1) \|\phi_2\|_{H^1(\mathbb{R}^N)}, \\ \int \nabla u_{3,n} \nabla \phi_3 + (\lambda_{1,n} + \lambda_{2,n}) u_{3,n} \phi_3 - |u_{3,n}|^{p-2} u_{3,n} \phi_3 - \alpha u_{1,n} u_{2,n} \phi_2 = o_n(1) \|\phi_3\|_{H^1(\mathbb{R}^N)}, \end{cases}$$
(3.11)

as $n \to \infty$, for every $\phi_i \in H^1(\mathbb{R}^N, \mathbb{R})$ $(i = 1, 2, 3)$. In particular, if we take $(\phi_1, \phi_2, \phi_3) = (u_{1,n}, u_{2,n}, u_{3,n})$, we have that $(\lambda_{1,n}, \lambda_{2,n})$ is bounded, therefore up to a subsequence we have convergence $(\lambda_{1,n}, \lambda_{2,n}) \to (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Passing to the limit in (3.11), we get that (u_1, u_2, u_3) satisfies

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = |u_1|^{p-2} u_1 + \alpha u_3 u_2, \\ -\Delta u_2 + \lambda_2 u_2 = |u_2|^{p-2} u_2 + \alpha u_3 u_1, \\ -\Delta u_3 + (\lambda_1 + \lambda_2) u_3 = |u_3|^{p-2} u_3 + \alpha u_1 u_2. \end{cases}$$

In addition, we claim that $\mathrm{Re} \int u_1 u_2 \overline{u}_3 > 0$. If not, we have

$$\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \leq \gamma_p \sum_{i=1}^{3} \|u_i\|_p^p \leq p\gamma_p A_1 \left(\sum_{i=1}^{3} \|\nabla u_i\|_2^2 \right)^{\frac{p\gamma_p}{2}},$$

and then $(p\gamma_p A_1)^{-\frac{2}{p\gamma_p - 2}} \leq \sum_{i=1}^{3} \|\nabla u_i\|_2^2$. Moreover, as $\mathcal{P}_{a_1,a_2}^+ \subset V(a_1, a_2)$, we get $\vec{u} \in B_{\rho^*}$, and this is a contradiction with $\max\{a_1, a_2\} < D$. From $P(\vec{u}) = 0$, we conclude that

$$\lambda_1 \|u_1\|_2^2 + \lambda_2 \|u_2\|_2^2 + (\lambda_1 + \lambda_2) \|u_3\|_2^2 = \sum_{i=1}^{3} (1 - \gamma_p) \|u_i\|_p^p + \left(3 - \frac{N}{2} \right) \alpha \int u_1 u_2 u_3. \quad (3.12)$$

By $P(\vec{u}_n) = o_n(1)$, we obtain

$$
\begin{aligned}
\lambda_1 a_1^2 + \lambda_2 a_2^2 &= \lim_{n\to\infty} \left(\lambda_1 \|u_{1,n}\|_2^2 + \lambda_2 \|u_{2,n}\|_2^2 + (\lambda_1 + \lambda_2) \|u_{3,n}\|_2^2 \right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^3 (1 - \gamma_p) \|u_{i,n}\|_p^p + \left(3 - \frac{N}{2} \right) \alpha \int u_{1,n} u_{2,n} u_{3,n} \right) \\
&= (1 - \gamma_p) \|u_i\|_p^p + \left(3 - \frac{N}{2} \right) \alpha \int u_1 u_2 u_3.
\end{aligned}
\tag{3.13}
$$

We claim that $u_1 \not\equiv 0$, $u_2 \not\equiv 0$ and $u_3 \not\equiv 0$.

Case 1. If $u_i = 0$ for any $i = 1, 2, 3$, then $\int |u_{i,n}|^p \to 0$, $\int u_{1,n} u_{2,n} u_{3,n} \to 0$, we have

$$
P(\vec{u}_n) = \sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 = o_n(1).
$$

Therefore,

$$
m_r^+(a_1, a_2) + o_n(1) = E(\vec{u}_n) = o_n(1),
$$

and this contradicts the fact that $m_r^+(a_1, a_2) < 0$.

Case 2. If $u_i \not\equiv 0$, $u_j = 0$ and $u_l = 0$, $i, j, l \in \{1, 2, 3\}$, then $u_{j,n} \to 0$ and $u_{l,n} \to 0$ in L^p. Let $\tilde{u}_{i,n} = u_{i,n} - u_i$, $\tilde{u}_{i,n} \to 0$ in L^p. By the Brezis-Lieb Lemma [8], we deduce that

$$
\begin{aligned}
P(\vec{u}_n) &= \sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 - \gamma_p \|u_{i,n}\|_p^p + o_n(1) \\
&= \|\nabla \tilde{u}_{i,n}\|_2^2 + \|\nabla u_{j,n}\|_2^2 + \|\nabla u_{l,n}\|_2^2 + \|\nabla u_i\|_2^2 - \gamma_p \|u_i\|_p^p + o_n(1).
\end{aligned}
$$

Thus,

$$
m_r^+(a_1, a_2) + o_n(1) = E(\vec{u}_n) = \left(\frac{\gamma_p}{2} - \frac{1}{p} \right) \|u_i\|_p^p + o_n(1) \geq 0,
$$

which contradicts our assumption $m_r^+(a_1, a_2) < 0$.

Case 3. If $u_i \not\equiv 0$, $u_j \not\equiv 0$ and $u_l = 0$. By the structure of system (1.4), we get $u_i = 0$ or $u_j = 0$, so Case 3 does not happen.

Therefore, $u_i \not\equiv 0$ for all $i = 1, 2, 3$. It remains to show that $m_r^+(a_1, a_2)$ is achieved. From [26, Lemma A.2], we get $\lambda_1, \lambda_2 > 0$. Moreover, combining (3.12) with (3.13), we have

$$
\lambda_1 a_1^2 + \lambda_2 a_2^2 = \lambda_1 \|u_1\|_2^2 + \lambda_2 \|u_2\|_2^2 + (\lambda_1 + \lambda_2) \|u_3\|_2^2.
\tag{3.14}
$$

Since $\|u_1\|_2^2 + \|u_3\|_2^2 \leq a_1^2$ and $\|u_2\|_2^2 + \|u_3\|_2^2 \leq a_2^2$, it follows from (3.14) that $\|u_1\|_2^2 + \|u_3\|_2^2 = a_1^2$ and $\|u_2\|_2^2 + \|u_3\|_2^2 = a_2^2$, and hence $\vec{u} \in \mathcal{P}_{r,a_1,a_2}$. By the maximum principle (see [24, Theorem 2.10]), $u_i > 0$ ($i = 1, 2, 3$). We then conclude that $\vec{u}_n \to \vec{u}$ in $H_r^1(\mathbb{R}^N, \mathbb{R}^3)$ and $E(\vec{u}) = m_r^+(a_1, a_2)$.

In conclusion, we have proved that $m_r^+(a_1, a_2)$ is attained by a function $\vec{u}$ which is positive, radially symmetric, and decreasing in $r = |x|$. Therefore, the proof is complete. $\square$

We look for the existence of $(\omega_1, \omega_2, \vec{v}) \in \mathbb{R}^2 \times H^1(\mathbb{R}^N, \mathbb{C}^3)$ satisfying (1.10) (see also [28, 32]) and $Q_1(\vec{v}) = a_1^2, Q_2(\vec{v}) = a_2^2$. It is important to our purpose to study the asymptotic behavior of minimizers for $m^+(a_1, a_2)$ because somehow (1.10) can be seen as a limiting

equation of problem (1.4), see Proposition 3.10 below. Then, we find the critical points of
$E_0 : H^1(\mathbb{R}^N, \mathbb{C}^3) \to \mathbb{R}$

$$E_0(\vec{v}) := \frac{1}{2} \sum_{i=1}^{3} \|\nabla v_i\|_2^2 - \text{Re} \int v_1 v_2 \overline{v}_3 \qquad (3.15)$$

constrained on $S(a_1, a_2)$. Let us observe that in [28, Theorem 1.3], only the case $N \leq 2$ is considered. See also Remark 1.8 in the Introduction. Define

$$0 > m_0(a_1, a_2) := \inf_{\vec{v} \in S(a_1, a_2)} E_0(\vec{v}) > -\infty. \qquad (3.16)$$

We have the following Lemmas.

Lemma 3.5. *Let $N \leq 3$. For any $a_1, a_2 > 0$*

$$m_0(a_1, a_2) = m_{0,r}(a_1, a_2) := \inf_{S(a_1,a_2) \cap H_r^1(\mathbb{R}^N, \mathbb{R}^3)} E.$$

In addition, $m_0(a_1, a_2)$ is reached by the vector function $(e^{i\theta_1} w_1, e^{i\theta_1} w_2, e^{i(\theta_1+\theta_2)} w_3)$ where $E(\vec{w}) = \inf_{S(a_1,a_2) \cap H_r^1(\mathbb{R}^N, \mathbb{R}^3)} E$, for some $(\theta_1, \theta_2) \in \mathbb{R}^2$.

Proof. It is standard to get that $m_0(a_1, a_2) \leq m_{0,r}(a_1, a_2)$. For any $(u_1, u_2, u_3) \in S(a_1, a_2)$, we have $(|u_1|^*, |u_2|^*, |u_3|^*) \in S(a_1, a_2)$. Moreover, we also have

$$E_0(\vec{u}) = \sum_{i=1}^{3} \frac{1}{2} \|\nabla u_i\|_2^2 - \text{Re} \int u_1 u_2 \overline{u}_3 \geq \sum_{i=1}^{3} \frac{1}{2} \|\nabla |u_i|^*\|_2^2 - \int |u_1|^* |u_2|^* |u_3|^*$$

$$= E_0(|u_1|^*, |u_2|^*, |u_3|^*).$$

Then,

$$E_0(\vec{u}) \geq E_0(|u_1|^*, |u_2|^*, |u_3|^*) \geq m_{0,r}(a_1, a_2),$$

for any $\vec{u} \in S(a_1, a_2)$. Therefore, $m_0(a_1, a_2) \geq m_{0,r}(a_1, a_2)$. Arguing as in the proof of Lemma 3.3, we obtain $m_0(a_1, a_2)$ is reached by the vector function $(e^{i\theta_1} w_1, e^{i\theta_1} w_2, e^{i(\theta_1+\theta_2)} w_3)$ where $E_0(\vec{w}) = \inf_{S(a_1,a_2) \cap H_r^1(\mathbb{R}^N, \mathbb{R}^3)} E_0$ and $(\theta_1, \theta_2) \in \mathbb{R}^2$. $\square$

Lemma 3.6. *Let $N \leq 3$. For any $a_1, a_2 > 0$, $m_0(a_1, a_2)$ is reached by a real valued, positive, radially symmetric, and decreasing function.*

Proof. From Lemma 3.3, we consider a minimizing sequence $\{\vec{u}_n\}$ for $m_{0,r}(a_1, a_2)$. We assume that $\vec{u}_n \in H_r^1(\mathbb{R}^N, \mathbb{R}^3) \cap S(a_1, a_2)$ is radially decreasing for any n (since we can replace u_n with $|u_n|^*$, the Schwarz rearrangement of $|u_n|$). By the Ekeland's variational principle, there exists a Palais-Smale sequence $\{(v_{1,n}, v_{2,n}, v_{3,n})\} \subset S(a_1, a_2) \cap H_r^1(\mathbb{R}^N, \mathbb{R}^3)$, $v_{i,n} \geq 0$ ($i = 1, 2, 3$) such that $\|\vec{u}_n - \vec{v}_n\|_{H^1(\mathbb{R}^N, \mathbb{R}^3)}$ as $n \to \infty$, $E_0(\vec{v}_n) \to m_0(a_1, a_2)$ and

$$dE_0|_{S(a_1,a_2) \cap H_r^1(\mathbb{R}^N, \mathbb{R}^3)}(\vec{v}_n) \to 0 \text{ in } H^{-1}(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N).$$

Since $\{\vec{v}_n\}$ is bounded in $H_r^1(\mathbb{R}^N, \mathbb{R}^3)$, up to a subsequence, we assume that $(v_{1,n}, v_{2,n}, v_{3,n}) \rightharpoonup (v_1, v_2, v_3)$ weakly in $H_r^1(\mathbb{R}^N, \mathbb{R}^3)$ as $n \to \infty$. For $N \leq 3$, $\vec{v}_n \to \vec{v}$ in $L^3 \times L^3 \times L^3$ by compact embedding of $H_r^1(\mathbb{R}^N) \hookrightarrow L^3(\mathbb{R}^N)$. See [11, Proposition 1.7.1] for the 1D case and sequences of non-increasing functions. By the Lagrange multiplier's rule (see [7, Lemma 3]) there exist $\{(\omega_{1,n}, \omega_{2,n})\} \subset \mathbb{R} \times \mathbb{R}$ such that

$$dE_0(\vec{v}_n) - (\omega_{1,n} v_{1,n}, \omega_{2,n} v_{2,n}, (\omega_{1,n} + \omega_{2,n}) v_{3,n}) \to 0 \text{ in } H^{-1}(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N) \times H^{-1}(\mathbb{R}^N),$$

and then $\{(\omega_{1,n}, \omega_{2,n})\}$ is bounded, and $\omega_{i,n} \to \omega_i$, $i = 1, 2$ as $n \to \infty$. Therefore, $\vec{v}$ is a nonnegative solution of

$$\begin{cases} -\Delta v_1 + \omega_1 v_1 = v_3 v_2, \\ -\Delta v_2 + \omega_2 v_2 = v_3 v_1, \\ -\Delta v_3 + (\omega_1 + \omega_2) v_3 = v_1 v_2. \end{cases} \tag{3.17}$$

Moreover, the Pohozaev identity for solutions of (3.17) is given by

$$P_0(\vec{v}) := \sum_{i=1}^{3} \|\nabla v_i\|_2^2 - \frac{N}{2} \int v_1 v_2 v_3 = 0,$$

then we have

$$\omega_1 a_1^2 + \omega_2 a_2^2 = \left(3 - \frac{N}{2}\right) \int v_1 v_2 v_3 = \omega_1 \left(\|v_1\|_2^2 + \|v_3\|_2^2\right) + \omega_2 \left(\|v_2\|_2^2 + \|v_3\|_2^2\right). \tag{3.18}$$

We show that $v_1 \not\equiv 0$, $v_2 \not\equiv 0$ and $v_3 \not\equiv 0$.

Case 1. If $v_i = 0$ for any $i = 1, 2, 3$. By $P_0(\vec{v}_n) \to 0$, we have

$$0 > m_0(a_1, a_2) = \lim_{n\to\infty} \frac{N-4}{2N} \sum_{i=1}^{3} \|\nabla v_{i,n}\|_2^2 \to 0,$$

which is a contradiction.

Case 2. If $v_i \not\equiv 0$, $v_j = v_l = 0$, $i, j, l \in \{1, 2, 3\}$. Then, if $\omega_i > 0$,

$$-\Delta v_i + \omega_i v_i = 0,$$

a contradiction. If $\omega_i \leq 0$, we get

$$-\Delta v_i \geq 0, \quad v_i \in L^2.$$

It follows from [26, Lemma A.2] that $v_i \equiv 0$, which is a contradiction with respect to the assumption.

Case 3. If $u_i \not\equiv 0$, $u_j \not\equiv 0$ and $u_l = 0$. By the structure of system (3.17), we get $u_i = 0$ or $u_j = 0$, so Case 3 does not happen.

By the same argument as in the proof of Lemma 3.4, we have $\omega_1, \omega_2 > 0$. Then, by the strong maximal principle, $\vec{v}$ is a positive solution of (1.10). It follows from (3.18) that $\vec{v} \in S(a_1, a_2)$. Hence, $E_0(\vec{v}) = m_0(a_1, a_2)$. $\qquad\square$

Remark 3.7. As already mentioned in Remark 1.8, a straightforward modification of the proof of Lemma 3.6 solves a problem left open in [28] in the case $N = 3$. Indeed, instead of considering the minimization problem (3.16), we consider as in [28] the problem

$$\Sigma_0(\gamma, \mu, s) := \inf \left\{ E_0(\vec{u}) \text{ s.t. } \vec{u} \in H^1(\mathbb{R}^N, \mathbb{C}^3), \|u_1\|_2^2 = \gamma, \|u_2\|_2^2 = \mu, \|u_3\|_2^2 = \nu \right\},$$

and a similar analysis as the one in Lemma 3.6 gives a positive answer to [28, Theorem 1.3 (ii)] in the three-dimensional case. In addition, if $2_* < p < 2^*$, under scaling transformation, $\alpha^{-\frac{N}{4-N}} \vec{u} \left(\alpha^{-\frac{2}{4-N}} x\right) \to \vec{v}$ in $H^1(\mathbb{R}^N, \mathbb{C}^3)$ as $\alpha \to 0$, where $\vec{u} \in \mathcal{G}$ and $\vec{v}$ is a ground state of (1.10) on $S(a_1, a_2)$ (see Proposition 3.10).

In the following, we derive an improved upper bound of $m_r^+(a_1, a_2)$ when $a_1 = a_2$. Indeed, we show in Lemma 3.8 below, that $m_r^+(a_1, a_1)$ is not only negative, but bounded away from zero. Compare (3.6) and (3.23). We consider the problem

$$\begin{cases} -\Delta u + \lambda u = \alpha u^2, \\ \int |u|^2 = a^2, \end{cases} \tag{3.19}$$

where $\alpha, a > 0$. Define

$$J_0(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\alpha}{3}\|u\|_3^3, \tag{3.20}$$

then solutions u of (3.19) can be found, see [38], as minimizers of

$$0 > m_0(a) := \inf_{u \in S(a)} J_0(u) > -\infty, \tag{3.21}$$

where λ is a Lagrange multiplier, and $S(a) := \{u \in H^1(\mathbb{R}^N, \mathbb{R}) \text{ s.t. } \int u^2 = a^2\}$. From [3], we obtain that (3.19) has a unique positive solution (λ, u_α) given by

$$\lambda = \left(\frac{\alpha^2 a^2}{\|w\|_2^2}\right)^{\frac{2}{4-N}}, \quad u_\alpha = \frac{\lambda}{\alpha}w(\lambda^{\frac{1}{2}}x), \quad -\Delta w + w - w^2 = 0, \tag{3.22}$$

and we recall that w is unique and positive. We have

$$m_0(a) = -\frac{4-N}{2(6-N)}\left(\frac{\alpha^2}{\|w\|_2^2}\right)^{\frac{2}{4-N}} a^{\frac{2(6-N)}{4-N}} < 0.$$

Lemma 3.8. *Let* $N \le 3$, $2_* < p < 2^*$, *and* $\alpha, a_1, a_2 > 0$. *If* $a_1 = a_2 < D$, *then*

$$m^+(a_1, a_1) < 3m_0\left(\frac{a_1}{\sqrt{2}}\right) < 0. \tag{3.23}$$

Proof. $m_0\left(\frac{a_1}{\sqrt{2}}\right)$ is achieved by $\tilde{u}_1 \in S\left(\frac{a_1}{\sqrt{2}}\right)$ and $\tilde{u}_1$ is radially symmetric and decreasing, see [12]. By adopting the same notation as in Lemma 2.2, we have

$$h(\rho) < h_1(\rho) := \frac{1}{2}\rho^2 - \frac{\alpha}{3}C^3(N,p)a_1^{\frac{6-N}{2}}\rho^{\frac{N}{2}}, \tag{3.24}$$

where, by Hölder, we have that

$$\alpha\text{Re}\int u_1 u_2 \bar{u}_3 \le \alpha\|u_1\|_3\|u_2\|_3\|u_2\|_3 \le \frac{\alpha}{3}\sum_{i=1}^3 \int |u_i|^3.$$

With calculations similar to the ones in (2.13), $J_0(\vec{u}) \ge h_1\left(\left(\sum_{i=1}^3 \|\nabla u_i\|_2^2\right)^{\frac{1}{2}}\right)$. By direct calculations, there exists $0 < \hat{\rho} < R_0$ such that $h_1(\hat{\rho}) = 0$. Then, we have

$$3\|\nabla\tilde{u}_1\|_2^2 \le \hat{\rho}^2 < R_0^2 < (\rho^*)^2.$$

Since $h(R_0) = h(R_1) = 0$, by the monotonicity of $h(\rho)$, we deduce that $(\tilde{u}_1, \tilde{u}_1, \tilde{u}_1) \in V(a_1, a_1)$. It implies that

$$m^+(a_1, a_1) = \inf_{\vec{u} \in V(a_1, a_1)} E(\vec{u}) \le E(\tilde{u}_1, \tilde{u}_1, \tilde{u}_1)$$

$$= 3J_0(\tilde{u}_1) - \frac{3}{p}\|\tilde{u}_1\|_p^p < 3m_0\left(\frac{a_1}{\sqrt{2}}\right).$$

Hence, the proof is complete. $\qquad\square$

Lemma 3.9. *Let $N \leq 3$, $2_* < p < 2^*$, and $\alpha, a_1, a_2 > 0$. If $a_1 = a_2 < D$, then for any ground state $\vec{u} \in S(a_1, a_1)$ of (1.4), for $a_1 \to 0$ we have, up to a subsequence,*

$$\left(\alpha\kappa^{-1}u_1(\kappa^{-\frac{1}{2}}x), \alpha\kappa^{-1}u_2(\kappa^{-\frac{1}{2}}x), \alpha\kappa^{-1}u_3(\kappa^{-\frac{1}{2}}x) \right) \to \vec{v}_0 \quad in \ H^1(\mathbb{R}^N, \mathbb{C}^3),$$

where $\vec{v}_0$ is a ground state solution of E_0 constrained on $S(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2)$, w is defined in (3.22), and $\kappa = \left(\frac{\alpha a_1}{\sqrt{2}\|w\|_2} \right)^{\frac{4}{4-N}}$.

Proof. Fix $\alpha > 0$. For any $\{a_n\} \in \mathbb{R}^+$ with $a_n \to 0^+$ as $n \to +\infty$, let $\vec{u}_n \in V(a_n, a_n)$ be a minimizer of $m^+(a_n, a_n)$, where $V(a_n, a_n) = \left\{ \vec{u}_n \in S(a_n, a_n) \cap \mathcal{M} : \left(\sum_{i=1}^3 \|\nabla u\|_2^2 \right)^{\frac{1}{2}} < \rho^* \right\}$. By Lemma 3.4, we get that $\vec{u}_n$ is a ground state of $E|_{S(a_n, a_n)}$. Then the Lagrange multipliers rule implies the existence of some $\lambda_{1,a_n}, \lambda_{2,a_n} \in \mathbb{R}$ such that

$$\begin{cases} \int \nabla u_{1,n} \nabla \overline{\phi}_1 + \lambda_{1,a_n} u_{1,n} \overline{\phi}_1 - |u_{1,n}|^{p-2} u_{1,n} \overline{\phi}_1 = \alpha \mathrm{Re} \int u_{3,n} \overline{u}_{2,n} \overline{\phi}_1, \\ \int \nabla u_{2,n} \nabla \overline{\phi}_2 + \lambda_{2,a_n} u_{2,n} \overline{\phi}_2 - |u_{2,n}|^{p-2} u_{2,n} \overline{\phi}_2 = \alpha \mathrm{Re} \int u_{3,n} \overline{u}_{1,n} \overline{\phi}_2, \\ \int \nabla u_{3,n} \nabla \overline{\phi}_3 + (\lambda_{1,a_n} + \lambda_{2,a_n}) u_{3,n} \overline{\phi}_3 - |u_{3,n}|^{p-2} u_{3,n} \overline{\phi}_3 = \alpha \mathrm{Re} \int u_{1,n} u_{2,n} \overline{\phi}_3, \end{cases} \tag{3.25}$$

for each $\vec{\phi} \in H^1(\mathbb{R}^N, \mathbb{C}^3)$.

We claim that

$$\frac{1-\gamma_p}{\gamma_p} \left(\frac{\alpha 2N(p-3)}{N(p-2)-4} C^3(N,p) \right)^{\frac{4}{4-N}} a_n^{\frac{4}{4-N}} > \lambda_{1,a_n} + \lambda_{2,a_n} > 6K_N a_n^{\frac{4}{4-N}}, \tag{3.26}$$

where $K_N := \frac{4-N}{4(6-N)} \left(\frac{\alpha^2}{2\|w\|_2^2} \right)^{\frac{2}{4-N}}$. Indeed, it follows from (3.25) that

$$(\lambda_{1,a_n} + \lambda_{2,a_n}) a_n^2 = -\sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 + \sum_{i=1}^3 \|u_{i,n}\|_p^p + 3\alpha \mathrm{Re} \int u_{1,n} u_{2,n} \overline{u}_{3,n} > 6K_N a_n^{\frac{2(6-N)}{4-N}}.$$

Since $P(\vec{u}_n) = 0$, by Lemma 3.8 we have

$$\begin{aligned} E(\vec{u}_n) &= \left(\frac{1}{2} - \frac{1}{p\gamma_p} \right) \sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 - \frac{p-3}{p-2} \alpha \mathrm{Re} \int u_{1,n} u_{2,n} u_{3,n} \\ &= -\frac{4-N}{2N} \sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 + \gamma_p \left(\frac{2}{N} - \frac{1}{p\gamma_p} \right) \sum_{i=1}^3 \|u_{i,n}\|_p^p \\ &< -3K_N a_n^{\frac{2(6-N)}{4-N}}. \end{aligned} \tag{3.27}$$

It follows immediately that

$$\frac{6N}{4-N} K_N a_n^{\frac{2(6-N)}{4-N}} < \sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 < \left(\frac{\alpha 2N(p-3)}{N(p-2)-4} C^3(N,p) \right)^{\frac{4}{4-N}} a_n^{\frac{2(6-N)}{4-N}}. \tag{3.28}$$

Hence, combining with $P(\vec{u}_n) = 0$, we obtain that

$$(\lambda_{1,a_n} + \lambda_{2,a_n})a_n^2 = \left(\frac{1}{\gamma_p} - 1\right)\sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 - \left(3 - \frac{N}{2\gamma_p}\right)\alpha \mathrm{Re}\int u_{1,n}u_{2,n}u_{3,n}$$

$$< \frac{1 - \gamma_p}{\gamma_p}\left(\frac{\alpha 2N(p-3)}{N(p-2) - 4}C^3(N,p)\right)^{\frac{4}{4-N}}a_n^{\frac{2(6-N)}{4-N}}.$$

The proof of claim (3.26) is complete.

Define now

$$v_{1,n} := \alpha\kappa_n^{-1}u_{1,n}(\kappa_n^{-\frac{1}{2}}x), \quad v_{2,n} := \alpha\kappa_n^{-1}u_{2,n}(\kappa_n^{-\frac{1}{2}}x), \quad \text{and} \quad v_{3,n} := \alpha\kappa_n^{-1}u_{3,n}(\kappa_n^{-\frac{1}{2}}x), \quad (3.29)$$

where $\kappa_n = \left(\frac{\alpha a_n}{\sqrt{2}\|w\|_2}\right)^{\frac{4}{4-N}}$. Then, for $i = 1, 2, 3$,

$$\|\nabla v_{i,n}\|_2^2 = \kappa_n^{\frac{N-6}{2}}\alpha^2\|\nabla u_{i,n}\|_2^2, \quad \|v_{i,n}\|_p^p = \kappa_n^{\frac{N-2p}{2}}\alpha^p\|u_{i,n}\|_p^p, \quad \|v_{i,n}\|_2^2 = \frac{2\|w\|_2^2}{a_n^2}\|u_{i,n}\|_2^2.$$

Therefore, for $a_n \to 0$ as $n \to \infty$, we have

$$m^+(a_n, a_n) + o_n(1) = E(\vec{u}_n) = \kappa_n^{\frac{6-N}{2}}\alpha^{-2}E_0(\vec{v}_n) - \kappa_n^{\frac{2p-N}{2}}\alpha^{-(p-2)}\sum_{i=1}^3 \|v_{i,n}\|_p^p$$

$$\geq \kappa_n^{\frac{6-N}{2}}\alpha^{-2}m_0\left(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2\right) + o\left(a_n^{\frac{2(6-N)}{4-N}}\right),$$

where we used the definition of κ_n to have $\kappa_n^{\frac{2p-N}{2}} \sim a_n^{\frac{2(2p-N)}{4-N}}$, then we can estimate the remainder with $o(a_n^{\frac{2(6-N)}{4-N}})$, as $p > 3$.

From the definition of $m_0(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2)$, for any $\varepsilon > 0$, there exists $\vec{v}_0 \in S(a_1, a_2)$ such that

$$E_0(\vec{v}_0) \leq m_0\left(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2\right) + \varepsilon.$$

Let $u_{i,a_n} := \kappa_n\alpha^{-1}v_{i,0}\left(\kappa_n^{\frac{1}{2}}x\right)$ for $i = 1, 2, 3$. Therefore, $\vec{u}_{a_n} \in V(a_n, a_n)$ for a_n small enough. Then

$$m^+(a_n, a_n) = \inf_{\vec{u} \in V(a_n, a_n)} E(\vec{u}) \leq E(u_{1,a_n}, u_{2,a_n}, u_{3,a_n})$$

$$\leq \kappa_n^{\frac{6-N}{2}}\alpha^{-2}E_0(\vec{v}_0) + \kappa_n^{\frac{2p-N}{2}}\alpha^{-p}\sum_{i=1}^3 \|v_{i,0}\|_p^p$$

$$\leq \kappa_n^{\frac{6-N}{2}}\alpha^{-2}\left(m_0(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2) + \varepsilon\right) + o\left(a_n^{\frac{2(6-N)}{4-N}}\right).$$

for all $\varepsilon > 0$ and $a_n > 0$ small enough. Therefore,

$$m^+(a_n, a_n) = \kappa_n^{\frac{6-N}{2}}\alpha^{-2}m_0(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2) + o\left(a_n^{\frac{2(6-N)}{4-N}}\right).$$

This implies that $\{\vec{v}_n\}$ is a minimizing sequence for $m_0\left(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2\right)$. If $\{u_n\}$ is a minimizing sequence of $m^+(a_n, a_n)$, $E(\vec{u}_n) = m^+(a_n, a_n) + o(1)$. By the definition of $\{\vec{v}_n\}$,

see (3.29), we have

$$E(\vec{v}_n) = E(\alpha \kappa_n^{-1} \vec{u}_n(\kappa_n^{-\frac{1}{2}} x)) = m_0(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2) + o(a_n^{\frac{2(6-n)}{4-n}}),$$

i.e., $\{v_n\}$ is a minimizing sequence of $m_2(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2)$. Up to a subsequence, there exists a radially symmetric Palais-Smale sequence $\{\vec{\tilde{v}}_n\}$ such that $\|\vec{\tilde{v}}_n - \vec{v}_n\|_{H^1(\mathbb{R}^N, \mathbb{C}^3)} = o_n(1)$. Similar to the proof of Lemma 3.6, up to translation, there exists a minimizer $\vec{v}_0$ for $m_0(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2)$ such that $\vec{\tilde{v}}_n \to \vec{v}_0$ in $H^1(\mathbb{R}^N, \mathbb{C}^3)$. Indeed, by Lemma 3.6 for any minimizing sequence of $m_0(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2)$, there exists a compact subsequence. $\qquad\square$

Proposition 3.10. *Let $N \le 3$, $2_* < p < 2^*$, $a_1, a_2 > 0$, and suppose that $\max\{a_1, a_2\} < D$. Let $\{\alpha_n\}$ be a positive sequence with $\alpha_n \to 0$ as $n \to \infty$, and let $\vec{u}_n$ be a minimizer for $m^+(a_1, a_2)$ (with $\alpha = \alpha_n > 0$), up to a subsequence,*

$$\vec{v}_n := \alpha_n^{-\frac{N}{4-N}} \vec{u}_n\left(\alpha_n^{-\frac{2}{4-N}} x\right) \to \vec{v} \qquad in \ \ H^1(\mathbb{R}^N, \mathbb{C}^3),$$

where $\vec{v}$ is a minimizer of $m_0(a_1, a_2)$.

Proof. Let $\{\alpha_n\} \subset (0, \infty)$ with $\alpha_n \to 0$. From the definition of $m_0(a_1, a_2)$, for any $\varepsilon > 0$ sufficiently small, there exists $\vec{v}_0 \in S(a_1, a_2)$ such that $E_0(\vec{v}_0) \le m_0(a_1, a_2) + \varepsilon$. Let $u_{j,\alpha_n}(x) := \alpha_n^{\frac{N}{4-N}} v_{j,0}\left(\alpha_n^{\frac{2}{4-N}} x\right), (j = 1, 2, 3)$. As the calculation in (3.24), we have $(u_{1,\alpha_n}, u_{2,\alpha_n}, u_{3,\alpha_n}) \in V(a_1, a_2)$, and then

$$
\begin{aligned}
m^+(a_1, a_2) &= \inf_{\vec{u} \in V(a_1, a_2)} E(\vec{u}) \le E(u_{1,\alpha_n}, u_{2,\alpha_n}, u_{3,\alpha_n}) \\
&\le \alpha_n^{\frac{4}{4-N}} E_0(\vec{v}_0) + \alpha_n^{\frac{N(p-2)}{4-N}} \sum_{j=1}^3 \|v_{j,0}\|_p^p \le \alpha_n^{\frac{4}{4-N}} (m_0(a_1, a_2) + \varepsilon) + o(\alpha_n^{\frac{4}{4-N}}),
\end{aligned}
\tag{3.30}
$$

for all $\varepsilon > 0$ and $\alpha_n > 0$ small enough.

Let $\vec{u}_n \in V(a_1, a_2)$ be a minimizer of $m^+(a_1, a_2)$ for $\alpha_n > 0$. Then, combining (3.30) and the same argument as in (3.27)-(3.28), we can prove that there exist $C_1, C_2 > 0$ such that $C_1 \alpha_n^{\frac{4}{4-N}} \le \sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 \le C_2 \alpha_n^{\frac{4}{4-N}}$. Define

$$\vec{v}_n := \alpha_n^{-\frac{N}{4-N}} \vec{u}_n\left(\alpha_n^{-\frac{2}{4-N}} x\right).$$

Then, $\vec{v}_n \in S(a_1, a_2)$, and there exists $C > 0$ such that for all $\alpha_n < 1$, $\sum_{i=1}^3 \|\nabla v_{i,n}\|_2^2 \le C$. Hence,

$$
\begin{aligned}
m^+(a_1, a_2) + o_n(1) = E(\vec{u}_n) &= \alpha_n^{\frac{4}{4-N}} \left(E_0(\vec{v}_n) - \frac{\alpha_n^{\frac{N(p-2)-4}{4-N}}}{p} \sum_{i=1}^3 \|v_{i,n}\|_p^p \right) \\
&\ge m_0(a_1, a_2) \alpha_n^{\frac{4}{4-N}} + o\left(\alpha_n^{\frac{4}{4-N}}\right).
\end{aligned}
$$

Thus, it follows that

$$m^+(a_1, a_2) = m_0(a_1, a_2) \alpha_n^{\frac{4}{4-N}} + o\left(\alpha_n^{\frac{4}{4-N}}\right).$$

This implies that $\{\vec{v}_n\}$ is a minimizing sequence for $m_0(a_1, a_2)$. Up to a subsequence, there exists a radially symmetric Palais-Smale sequence $\{\vec{\tilde{v}}_n\}$ such that $\|\vec{\tilde{v}}_n - \vec{v}_n\|_{H^1(\mathbb{R}^N, \mathbb{C}^3)} = o_n(1)$. Similar to Lemma 3.6, there exists a minimizer $\vec{v}$ for $m_0(a_1, a_2)$ such that $\vec{\tilde{v}}_n \to \vec{v}$ in $H^1(\mathbb{R}^N, \mathbb{C}^3)$. $\qquad\square$

3.2. **Mass-critical case.** In this subsection, we deal with the mass critical case $p = 2_* = 2 + \frac{4}{N}$. As in the previous sections, α, a_1, a_2 are positive. We recall the decomposition of $\mathcal{P}_{a_1, a_2} = \mathcal{P}_{a_1, a_2}^+ \cup \mathcal{P}_{a_1, a_2}^0 \cup \mathcal{P}_{a_1, a_2}^-$ as in Section 2, see (2.9). From the definition of $\mathcal{P}_{a_1, a_2}^0$, i.e., $\Psi_{\vec{u}}'(0) = \Psi_{\vec{u}}''(0) = 0$, then necessarily $u_i = 0$ ($i = 1, 2, 3$). Therefore, $\mathcal{P}_{a_1, a_2}^0 = \emptyset$. Similarly to Lemma 2.3, we can also check that $\mathcal{P}_{a_1, a_2} \cap \mathcal{M}$ is a smooth manifold of codimension 1 in $H^1(\mathbb{R}^N, \mathbb{C}^3)$.

Lemma 3.11. *If* $\max\{a_1, a_2\} < \left(\frac{N+2}{N}\right)^{\frac{N}{4}} (C(N, 2_*))^{-\frac{N+2}{2}}$ *and* $a_1, a_2 > 0$, *for all* $\vec{u} \in S(a_1, a_2) \cap \mathcal{M}$, *there exists* $\sigma_{\vec{u}}$ *such that* $\sigma_{\vec{u}} \star \vec{u} \in \mathcal{P}_{a_1, a_2}$. *Further,* $\sigma_{\vec{u}}$ *is the unique critical point of the function* $\Psi_{\vec{u}}$ *and it is a strict minimum point at negative level. Moreover:*

(i) $\Psi_{\vec{u}}$ *is strictly decreasing in* $(0, \sigma_{\vec{u}})$,

(ii) $\mathcal{P}_{a_1, a_2} = \mathcal{P}_{a_1, a_2}^+$ *and* $P(\vec{u}) < 0$ *if and only if* $\sigma_{\vec{u}} < 1$,

(iii) *the map* $\vec{u} \mapsto \sigma_{\vec{u}} \in \mathbb{R}$ *is of class* C^1.

Proof. (i) Since $p = 2_*$, we have, see the definition of Ψ in (2.8),

$$
\Psi_{\vec{u}}(s) = s^2 \sum_{i=1}^3 \left(\frac{1}{2} \|\nabla u_i\|_2^2 - \frac{N}{2N+4} \|u_i\|_{2_*}^{2_*} \right) - s^{\frac{N}{2}} \alpha \mathrm{Re} \int u_1 u_2 \overline{u}_3
$$
$$
\geq \frac{s^2}{2} \left(\sum_{i=1}^3 \|\nabla u_i\|_2^2 \left(1 - \frac{N (C(N, 2_*))^{2+\frac{4}{N}}}{N+2} \max\left\{ a_1^{\frac{4}{N}}, a_2^{\frac{4}{N}} \right\} \right) \right) - s^{\frac{N}{2}} \alpha \mathrm{Re} \int u_1 u_2 \overline{u}_3.
$$
(3.31)

Note that, for any $\vec{u} \in \mathcal{M}$, $s \star \vec{u} \in \mathcal{P}_{a_1, a_2}$ if and only if $\Psi_{\vec{u}}'(s) = 0$. From the latter property, if $1 - \frac{N(C(N, 2_*))^{2+\frac{4}{N}}}{N+2} \max\{a_1^{\frac{4}{N}}, a_2^{\frac{4}{N}}\}$ is positive, then $\Psi_{\vec{u}}(s)$ has a unique critical point $\sigma_{\vec{u}}$, which is a strict minimum point at negative level. Therefore, under the bound condition on $\max\{a_1, a_2\}$ as in the statement of the Lemma, we have that

$$
\sum_{i=1}^3 \left(\frac{1}{2} \|\nabla u_i\|_2^2 - \frac{N}{2N+4} \|u_i\|_{2_*}^{2_*} \right) > 0.
$$

(ii) If $\vec{u} \in \mathcal{P}_{a_1, a_2} \cap \mathcal{M}$, then $\sigma_{\vec{u}}$ is a minimum point, we have that $\Psi_{\vec{u}}''(\sigma_{\vec{u}}) \geq 0$. Since $\mathcal{P}_{a_1, a_2}^0 = \emptyset$, we have $\vec{u} \in \mathcal{P}_{a_1, a_2}^+$. Finally, $\Psi_{\vec{u}}'(s) > 0$ if and only if $s > \sigma_{\vec{u}}$, then $P(\vec{u}) = \Psi_{\vec{u}}'(0) < 0$ if and only if $\sigma_{\vec{u}} < 1$.

(iii) To prove that the map $\vec{u} \in S(a_1, a_2) \cap \mathcal{M} \mapsto \sigma_{\vec{u}} \in \mathbb{R}$ is of class C^1, we can apply the Implicit Function Theorem as in Lemma 2.4. $\qquad\square$

Lemma 3.12. *Let* $N \leq 3$, *assume* $p = 2_*$, *and let* $\alpha, a_1, a_2 > 0$. *We have the followings:*

(i) *if* $\max\{a_1, a_2\} < \left(\frac{N+2}{N}\right)^{\frac{N}{4}} (C(N, 2_*))^{-\frac{N+2}{2}}$, *then*

$$
-\infty < m^+(a_1, a_2) := \inf_{\vec{u} \in \mathcal{P}_{a_1, a_2}^+ \cap \mathcal{M}} E(\vec{u}) = \inf_{\vec{u} \in S(a_1, a_2)} E(\vec{u}) < 0,
$$

(ii) *if* $\min\{a_1, a_2\} \geq \left(\frac{N+2}{N}\right)^{\frac{N}{4}} (C(N, 2_*))^{-\frac{N+2}{2}}$, *then*

$$\inf_{\vec{u} \in S(a_1, a_2)} E(\vec{u}) = -\infty.$$

Proof. (i) If $0 < \max\{a_1, a_2\} < \left(\frac{N+2}{N}\right)^{\frac{N}{4}} (C(N, 2_*))^{-\frac{N+2}{2}}$, from (3.31), we have E is coercive on $S(a_1, a_2)$, it follows that $m^+(a_1, a_2) > -\infty$. For $\vec{u} \in \mathcal{P}_{a_1, a_2}^+$, then

$$E(s \star \vec{u}) = s^2 \sum_{i=1}^{3} \left(\frac{1}{2} \|\nabla u_i\|_2^2 - \frac{N}{2N + 4} \|u_i\|_{2_*}^{2_*}\right) - s^{\frac{N}{2}} \alpha \operatorname{Re} \int u_1 u_2 \overline{u}_3.$$

Hence, $E(s \star \vec{u}) < 0$ for every $u \in \mathcal{P}_{a_1, a_2}^+ \cap \mathcal{M}$ with $s > 0$ small enough. Therefore, we know that $m^+(a_1, a_2) < 0$.

(ii) If $\min\{a_1, a_2\} \geq \left(\frac{N+2}{N}\right)^{\frac{N}{4}} (C(N, 2_*))^{-\frac{N+2}{2}}$, there exists $\vec{u} \in S(a_1, a_2)$ such that $E(\vec{u}) \leq 0$ (see also [38, Section 3]). Using (3.31), we deduce by taking the limit, that $\inf\limits_{S(a_1, a_2)} E = -\infty$. $\qquad \square$

We state the following Lemmas, whose proofs are similar to the ones for Lemmas 3.3, 3.5, and Lemma 3.9, respectively.

Lemma 3.13. *Let $N \leq 3$. For $p = 2_*$, $m^+(a_1, a_2) = m_r^+(a_1, a_2)$, where $m_r^+(a_1, a_2)$ is given by (3.8). In addition, $\inf\limits_{\mathcal{P}_{a_1, a_2}^+ \cap \mathcal{M}} E(\vec{u})$ is attained by $(e^{i\theta_1} w_1, e^{i\theta_1} w_2, e^{i(\theta_1 + \theta_2)} w_3)$ where $E(\vec{w}) = \inf\limits_{\vec{u} \in \mathcal{P}_{r, a_1, a_2}^+ \cap \mathcal{M}} E(\vec{u})$ and $(\theta_1, \theta_2) \in \mathbb{R}^2$.*

Lemma 3.14. *Let $N \leq 3$, assume $p = 2_*$, and let $\alpha, a_1, a_2 > 0$. If $\max\{a_1, a_2\} < \left(\frac{N+2}{N}\right)^{\frac{N}{4}} (C(N, 2_*))^{-\frac{N+2}{2}}$, then $E|_{S(a_1, a_2)}$ has a critical point $\vec{u}$ at $m^+(a_1, a_2)$, and $\vec{u}$ is real valued, positive, and radially symmetric for some $\lambda_1, \lambda_2 > 0$.*

Lemma 3.15. *Let $N \leq 3$, assume $p = 2_*$, and let $\alpha > 0$ and $a_1 = a_2 > 0$. If $0 < a_1 < \left(\frac{N+2}{N}\right)^{\frac{N}{4}} (C(N, 2_*))^{-\frac{N+2}{2}}$, then for any ground state $\vec{u} \in S(a_1, a_1)$ of (1.4), let $a_1 \to 0$, we have*

$$\left(\alpha \kappa^{-1} u_1(\kappa^{-\frac{1}{2}} x), \alpha \kappa^{-1} u_2(\kappa^{-\frac{1}{2}} x), \alpha \kappa^{-1} u_3(\kappa^{-\frac{1}{2}} x)\right) \to \vec{v}_0 \quad \text{in } H^1(\mathbb{R}^N, \mathbb{C}^3),$$

where $\vec{v}_0$ is a minimizer of $m_0(\sqrt{2}\|w\|_2, \sqrt{2}\|w\|_2)$, w is given in (3.22), and $\kappa = \left(\frac{\alpha a_1}{\sqrt{2}\|w\|_2}\right)^{\frac{4}{4-N}}$.

We are now in position to prove the first main result of the paper, namely Theorem 1.

3.3. **Proof of Theorem 1.** We start with the intracritical case, $2_* < p < 2^*$.

(i) It follows from Lemmas 3.3 and 3.4 that there is a local minimizer of E on $V(a_1, a_2)$.

(ii) We shall prove that the set $\mathcal{G}$ defined in Introduction is orbitally stable. By contradiction, suppose that there exist $\varepsilon_0 > 0$, a sequence of times $\{t_n\} \subset \mathbb{R}^+$, and a sequence of initial data $\{\vec{\psi}_{0,n}\} \subset H^1(\mathbb{R}^N, \mathbb{C}^3)$ such that the unique (for n fixed) solution $\vec{\psi}_{\psi_{0,n}}(t)$ to the problem (1.1) with initial datum $\vec{\psi}_{\psi_{0,n}}(0) = \vec{\psi}_{0,n}$ satisfies

$$\operatorname{dist}_{H^1(\mathbb{R}^N, \mathbb{C}^3)}\left(\vec{\psi}_{0,n}, \mathcal{G}\right) < \frac{1}{n} \quad \text{and} \quad \operatorname{dist}_{H^1(\mathbb{R}^N, \mathbb{C}^3)}\left(\vec{\psi}_{\psi_{0,n}}(t_n), \mathcal{G}\right) \geq \varepsilon_0.$$

Without loss of generality, we assume $\vec{\psi}_{0,n} \in S(a_1, a_2)$. Denote $\vec{\psi}_{\psi_{0,n}}(t_n)$ by $\vec{u}_n$. Then by the conservation laws (1.2) and (1.3), $\{\vec{u}_n\} \subset H^1(\mathbb{R}^N, \mathbb{C}^3)$ satisfies $Q_1(\vec{u}_n) = a_1^2$, $Q_2(\vec{u}_n) = a_2^2$ and $E(\vec{u}_n) \to m^+(a_1, a_2)$.

We shall prove that for any $n \in \mathbb{N}$, $\vec{\psi}_{\psi_{0,n}}(t)$ is globally defined in time and $\vec{\psi}_{\psi_{0,n}}(t) \in B_{\rho^*}$ for any $t > 0$, recalling that ρ^* is given in Lemma 2.2. Since $\vec{\psi}_{0,n} \in B_{\rho^*}$, if $\vec{\psi}_{\psi_{0,n}}(t)$ leaves the set B_{ρ^*}, there exists $t_1 \in (0, T_{\max})$ such that $\vec{\psi}_{\psi_{0,n}}(t_1) \in \partial B_{\rho^*}$, where $T_{\max}$ is the maximal forward time of existence for the solution $\vec{\psi}_{\psi_{0,n}}$. By (2.13), we have $E(\vec{\psi}_{\psi_{0,n}}(t_1)) \geq h(\rho^*) \geq 0$, contradicting the conservation of the energy. If $T_{\max} < \infty$, by the blow-up alternative $\lim_{t \to T_{\max}^-} \left(\sum_{i=1}^3 \|\vec{\psi}_{\psi_{0,n}}(t)\|_{L^2(\mathbb{R}^N)}^2 \right) = \infty$, then there also exists $t_2 \in (0, T_{\max})$ such that $\vec{\psi}_{\psi_{0,n}}(t_2) \in \partial B_{\rho^*}$. Analogously to the proof of the fact that $\vec{\psi}_{\psi_{0,n}}(t) \in B_{\rho^*}$, one shows that $T_{\max} = +\infty$. This implies that solutions starting in B_{ρ^*} are globally defined in time. By Lemmas 2.2 and 3.2, if $\max\{a_1, a_2\} < D$, we thus get

$$m^+(a_1, a_2) = \inf_{\vec{u} \in S(a_1,a_2) \cap B_{\rho^*} \cap \mathcal{M}} E(\vec{u}) = \inf \left\{ E(\vec{u}) \text{ s.t. } \vec{u} \in S(a_1, a_2) \cap B_{\max\{a_1,a_2\}D^{-1}\rho^*} \cap \mathcal{M} \right\}.$$

A similar analysis to that in the proof of [28, Theorem 1.2] and [38, Theorem 1.4], yields strict sub-additivity of E on $V(a_1, a_2) = S(a_1, a_2) \cap B_{\rho^*} \cap \mathcal{M}$. Moreover, combining $m^+(a_1, a_2) < 0$ with $E(\vec{u}_n) \to m^+(a_1, a_2)$, we have that $\vec{u}_n \in \mathcal{M}$. Therefore, there exists $\vec{u} \in \mathcal{G}$ such that $\vec{u}_n \to \vec{u}$ in $H^1(\mathbb{R}^N, \mathbb{C}^3)$. Since the set of ground states $\mathcal{G}$ is invariant under translations, this contradicts the equality $\text{dist}_{H^1(\mathbb{R}^N, \mathbb{C}^3)}(\vec{u}_n, \mathcal{G}) \geq \varepsilon_0 > 0$.

(iii) The third point of the Theorem follows from Lemma 3.9.

(iv) The last point follows from (3.28).

We turn now the attention to the critical case. For $p = 2_*$ we have that (i), i.e., existence of minimizer of $m^+(a_1, a_2)$, follows from Lemmas 3.13 and 3.14; the orbital stability of $\mathcal{G}$ as in (ii) can be proved following [1, Theorem 1.4] or [28, Theorem 1.2]; (iii) follows from Lemma 3.15.

We conclude with the proof of (iv). By recalling that the ground state has a negative energy, by using the estimate in (2.13) with $p = 2_*$ we obtain

$$0 > E(\vec{u}) \geq \frac{1}{2} \sum_{i=1}^3 \|\nabla u_i\|_2^2 \left(1 - \frac{N\left(C(N, 2_*)\right)^{2+\frac{4}{N}}}{N+2} \max\left\{ a_1^{\frac{4}{N}}, a_2^{\frac{4}{N}} \right\} \right)$$
$$- \frac{\alpha}{3} C^3(N, 2_*) \max\left\{ a_1^{\frac{6-N}{2}}, a_2^{\frac{6-N}{2}} \right\} \left(\sum_{i=1}^3 \|\nabla u_i\|_2^2 \right)^{\frac{N}{4}},$$

then, if $\alpha \to 0$, we have

$$\frac{1}{2} \left(\sum_{i=1}^3 \|\nabla u_i\|_2^2 \right)^{1-\frac{N}{4}} \left(1 - \frac{N\left(C(N, 2_*)\right)^{2+\frac{4}{N}}}{N+2} \max\left\{ a_1^{\frac{4}{N}}, a_2^{\frac{4}{N}} \right\} \right)$$
$$< \frac{\alpha}{3} C^3(N, 2_*) \max\left\{ a_1^{\frac{6-N}{2}}, a_2^{\frac{6-N}{2}} \right\} \to 0.$$

The proof is complete.

4. Proof of Theorem 2

In this section, for $\alpha, a_1, a_2 > 0$, $2_* < p < 2^*$ for $N \leq 3$, we study the existence and properties of the second[1] standing wave solution of (1.4). Define

$$m^-(a_1, a_2) = \inf_{\vec{u} \in \mathcal{P}^-_{a_1,a_2}} E(\vec{u}).$$

By Lemma 2.3, if $\max\{a_1, a_2\} < D$, we check that $\mathcal{P}^0_{a_1,a_2}$ is empty. Similar to the proof of Lemma 3.3, we get that if $\max\{a_1, a_2\} < D$, then $\inf_{\vec{u} \in \mathcal{P}^-_{r,a_1,a_2}} E(\vec{u}) = \inf_{\vec{u} \in \mathcal{P}^-_{a_1,a_2}} E(\vec{u})$. Furthermore, $\inf_{\mathcal{P}^-_{a_1,a_2}} E(\vec{u})$ is reached by the vector function $(e^{i\theta_1} w_1, e^{i\theta_1} w_2, e^{i(\theta_1+\theta_2)} w_3)$ where $\vec{w}$ is the minimizer for $\inf_{\mathcal{P}^-_{r,a_1,a_2}} E$ and $(\theta_1, \theta_2) \in \mathbb{R}^2$ and $\mathcal{P}^-_{r,a_1,a_2} = \mathcal{P}^-_{a_1,a_2} \cap H^1_r(\mathbb{R}^N, \mathbb{R}^3)$.

Lemma 4.1. *Suppose that* $\max\{a_1, a_2\} < D$, $\alpha, a_1, a_2 > 0$, $2_* < p < 2^*$ *and* $N \leq 3$, *then there exists* $\alpha_0 > 0$ *such that*

$$0 < m^-(a_1, a_2) := \inf_{\vec{u} \in \mathcal{P}^-_{a_1,a_2}} E(\vec{u}) < \min\{m(a_1), m(a_2)\},$$

for any $\alpha > \alpha_0$, *where* $m(a_1)$ *and* $m(a_2)$ *are defined in* (3.4).

Proof. For any $\vec{u} \in \mathcal{P}^-_{a_1,a_2}$, see Lemma 2.4, we have

$$E(\vec{u}) = \Psi_{\vec{u}}(0) \geq \Psi_{\vec{u}}(\sigma_{\vec{u}}) = E(\sigma_{\vec{u}} \star \vec{u}) = h\left(\sigma_{\vec{u}} \left(\sum_{i=1}^3 \|\nabla u_i\|_2^2\right)^{\frac{1}{2}}\right) > 0,$$

and then $\inf_{\mathcal{P}^-_{a_1,a_2}} E \geq \max_{\mathbb{R}} h > 0$. For fixed $a_1, a_2 > 0$, by Lemma 3.1, $m(b)$ is achieved by $u_0 \in S(b) \cap H^1(\mathbb{R}^N, \mathbb{R}^3)$ for any $0 < b$. Let u_1 be the positive solution of (3.1) with parameter $\|u_1\|_2^2 = b^2$, u_2 be the positive solution of (3.1) with $\|u_2\|_2^2 = a_1^2 - b^2$ and u_3 be the positive solution of (3.1) with $\|u_3\|_2^2 = a_2^2 - b^2$. We have $(u_1, u_2, u_3) \in S(a_1, a_2)$, and it is easy to see that

$$J(s \star u_1) \to 0 \quad \text{and} \quad J(s \star u_2) \to 0 \quad \text{and} \quad J(s \star u_3) \to 0, \quad \text{as } s \to 0,$$

see (3.2) for the definition of J. Therefore, there exists an $s_0 > 0$ small enough which is independent of α such that

$$\max_{s < s_0} E(s \star (u_1, u_2, u_3)) < \max_{s < s_0} J(s \star u_1) + J(s \star u_2) + J(s \star u_3)$$
$$< \min\{m(a_1), m(a_2)\},$$

as both $m(a_1)$ and $m(a_2)$ are strictly positive. If $s \geq s_0$, then the interaction term is bounded from below as in the following:

$$\int (s \star u_1)(s \star u_2)(s \star u_3) = s^{\frac{N}{2}} \int u_1 u_2 u_3 \geq K s_0^{\frac{N}{2}},$$

---

[1]The first solution obtained in Theorem 1 is a ground state solution (local minimum point). The second solution we are going to prove the existence, is a mountain pass solution.

where $K > 0$. Thus, we have

$$\max_{s \geq s_0} E\left(s \star (u_1, u_2, u_3)\right) \leq \max_{s \geq s_0} J(s \star u_1) + J(s \star u_2) + J(s \star u_3) - \alpha K s_0^{\frac{N}{2}}$$

$$\leq m(b) + m\left(\sqrt{a_1^2 - b^2}\right) + m\left(\sqrt{a_2^2 - b^2}\right) - \alpha K s_0^{\frac{N}{2}}.$$

From Lemma 3.1, $m(b)$ is strictly decreasing for $b > 0$, then $m(b) \geq \max\{m(a_1), m(a_2)\}$. It is clear that there exists $\alpha_0 > 0$ such that

$$\max_{s \geq s_0} E(s \star (u_1, u_2, u_3)) < \min\{m(a_1), m(a_2)\} \quad \text{for all} \quad \alpha > \alpha_0.$$

Hence, the proof is complete. $\qquad\square$

Lemma 4.2. *Let* $\max\{a_1, a_2\} < D$, $a_1, a_2 > 0$, $2_* < p < 2^*$ *and* $N \leq 3$. *There exists* $\alpha_0 > 0$ *such that for all* $\alpha > \alpha_0$, $m^-(a_1, a_2)$ *is achieved by some* $\vec{v}$, *which is real-valued, positive, radially symmetric and decreasing.*

Proof. Similarly, we only need to show that $m_r^-(a_1, a_2)$ is attained. If $N = 2, 3$, for $E|_{S(a_1,a_2)}$ with a radially Palais-Smale sequence at level $m_r^-(a_1, a_2)$ and $P(\vec{u}_n) \to 0$, we refer to [3, 13, 27, 38]. If $N = 1$, combine [38, Remark 5.2] with [13, Lemma 3.1] with the necessary modifications. Therefore, we can choose a nonnegative and radially symmetric Palais-Smale sequence $\{\vec{u}_n\}$ for $m_r^-(a_1, a_2)$ with $P(\vec{u}_n) = o_n(1)$, i.e. $\lim_{n \to \infty} E(\vec{u}_n) = m_r^-(a_1, a_2)$ and $E'|_{S(a_1,a_2)} \to 0$ as $n \to \infty$. Similar to the proof of Lemma 3.4, we have that sequence $\{\vec{u}_n\}$ is bounded in $H^1(\mathbb{R}^N, \mathbb{C}^3)$, and there exists (u_1, u_2, u_3) such that $(u_{1,n}, u_{2,n}, u_{3,n}) \rightharpoonup (u_1, u_2, u_3)$ in $H^1(\mathbb{R}^N, \mathbb{C}^3)$. Hence, $u_i \geq 0$ are radial functions for all $i = 1, 2, 3$.

We claim that $u_1 \not\equiv 0$, $u_2 \not\equiv 0$, and $u_3 \not\equiv 0$.

Case 1. If $u_i = 0$ for any $i = 1, 2, 3$, then $\int |u_{i,n}|^p \to 0$, $\int u_{1,n} u_{2,n} u_{3,n} \to 0$, we have

$$P(\vec{u}_n) = \sum_{i=1}^{3} \|\nabla u_{i,n}\|_2^2 = o_n(1).$$

Therefore,

$$m_r^-(a_1, a_2) + o_n(1) = E(\vec{u}_n) = o_n(1),$$

this contradicts the fact that $m^-(a_1, a_2) > 0$.

Case 2. If $u_i \not\equiv 0$, $u_j = 0$ and $u_l = 0$, $i, j, l \in \{1, 2, 3\}$, then $u_{j,n} \to 0$ and $u_{l,n} \to 0$ in L^p. Let $\tilde{u}_{i,n} = u_{i,n} - u_i$, $\tilde{u}_{i,n} \to 0$ in L^p. By the maximum principle (see [24, Theorem 2.10]), u is a positive solution of (3.1). By the Brezis-Lieb Lemma [8], we deduce that

$$P(\vec{u}_n) = \sum_{i=1}^{3} \|\nabla u_{i,n}\|_2^2 - \gamma_p \|u_{i,n}\|_p^p + o_n(1)$$

$$= \|\nabla \tilde{u}_{i,n}\|_2^2 + \|\nabla u_{j,n}\|_2^2 + \|\nabla u_{l,n}\|_2^2 + \|\nabla u_i\|_2^2 - \gamma_p \|u_i\|_p^p + o_n(1).$$

Since u_i satisfies $P(u_i) = \|\nabla u_i\|_2^2 - \gamma_p \|u_i\|_p^p = o_n(1)$, we have

$$m_r^-(a_1, a_2) + o_n(1) = E(\vec{u}_n) = \frac{1}{2}\|\nabla u_i\|_2^2 - \frac{1}{p}\|u_i\|_p^p + o_n(1)$$

$$= \left(\frac{\gamma_p}{2} - \frac{1}{p}\right)\|u_i\|_p^p + o_n(1) \geq m(\|u_i\|_2).$$

which contradicts our assumption $m_r^-(a_1, a_2) < \min\{m(a_1), m(a_2)\}$ if $\alpha > \alpha_0$.

Case 3. If $u_i \not\equiv 0$, $u_j \not\equiv 0$ and $u_l = 0$. By the structure of system (1.4), we get $u_i = 0$ or $u_j = 0$, so Case 3 does not happen.

So we can apply a similar argument as the proof of Lemma 3.4. Therefore, we then conclude that $\vec{u}_n \to \vec{u}$ in $H_r^1(\mathbb{R}^N, \mathbb{R}^3)$ and $E(\vec{u}) = m_r^-(a_1, a_2)$. □

At this point, we study the semi-trivial limit behavior as $a_1 > 0$ and $a_2 \to 0$.

Lemma 4.3. *Let $\alpha, a_1, a_2 > 0$, and $2_* < p < 2^*$ for $N \le 3$. If $a_1 \ne 0$ is fixed and $a_2 \to 0$ ($a_1 \to 0$ and $a_2 \ne 0$), then for the second solution $\vec{v}$ of (1.4), up to a subsequence, we have $m^-(a_1, a_2) \to m(a_1)$, and*

$$\left(\tilde{\kappa}^{-\frac{1}{p-2}} v_1\left(\tilde{\kappa}^{-\frac{1}{2}} x\right), v_2(x), v_3(x)\right) \to (w_p, 0, 0) \quad in \ H^1(\mathbb{R}^N, \mathbb{C}^3),$$

where $\tilde{\kappa} = \left(\frac{a_1^2}{\|w_p\|_2^2}\right)^{\frac{p-2}{2-p\gamma_p}}$ and w_p is the positive radial solution of $-\Delta w + w = |w|^{p-2}w$.

Proof. An analysis similar to that in the proof of [28, Lemma 2.6] show that for $a_1, a_2 \ge 0$, $m^-(a_1, a_2)$ is continuous at (a_1, a_2). By Theorem 2, for $a_{1,n}, a_{2,n} > 0$, there exists $(u_{1,n}, u_{2,n}, u_{3,n}) \in H_r^1(\mathbb{R}^N, \mathbb{C}^3) \cap S(a_{1,n}, a_{2,n})$ such that

$$P(\vec{u}_n) = o_n(1) \text{ and } E(\vec{u}_n) \to m^-(a_{1,n}, a_{2,n})$$

provided α is large enough. We assume that $a_{1,n} \to a_1$ and $a_{2,n} \to 0$. Then we have that $\|u_{1,n}\|_2^2 \to a_1^2$, $\|u_{2,n}\|_2^2 \to 0$ and $\|u_{3,n}\|_2^2 \to 0$, and $\vec{u}_n$ is a bounded sequence in $H^1(\mathbb{R}^N, \mathbb{R}^3)$. There exists $u_1 \in H^1(\mathbb{R}^N, \mathbb{R})$ such that $u_{1,n} \rightharpoonup u_1$ and $u_{2,n} \rightharpoonup 0$ and $u_{3,n} \rightharpoonup 0$. Therefore, we have $\int u_{1,n} u_{2,n} u_{3,n} \to 0$. Moreover, by the Lagrange multipliers rule there exists $\omega_n \in \mathbb{R}$ such that

$$\int \nabla u_{1,n} \nabla \psi - \lambda_n u_{1,n} \psi - |u_{1,n}|^{p-2} u_{1,n} \psi = o_n(1) \|\phi\|_{H^1(\mathbb{R}^N)},$$

for all $\phi \in H^1(\mathbb{R}^N, \mathbb{R})$. The choice $\psi = u_{1,n}$ gives

$$\lambda_n a_1^2 = \|\nabla u_{1,n}\|_2^2 - \|u_{1,n}\|_p^p + o_n(1).$$

And the boundedness of $\{\vec{u}_n\}$ in $H^1(\mathbb{R}^N, \mathbb{R}^3)$ implies that $\{\lambda_n\}$ is bounded as well, thus $\lambda_n \to \lambda_1 \in \mathbb{R}$. Similarly, since $\|u_{2,n}\|_2^2 \to 0$ and $\|u_{3,n}\|_2^2 \to 0$, we have $u_{2,n}, u_{3,n} \to 0$ in $H^1(\mathbb{R}^N, \mathbb{R})$. Recalling that $P(\vec{u}_n) \to 0$,

$$o_n(1) = P(\vec{u}_n) = \sum_{i=1}^3 \|\nabla u_{i,n}\|_2^2 - \gamma_p \|u_{1,n}\|_p^p + o_n(1) = \|u_{1,n}\|_2^2 - \gamma_p \|u_{1,n}\|_p^p + o_n(1),$$

we have

$$\lambda_n a_1^2 = (1 - \gamma_p) \|u_{1,n}\|_p^p + o_n(1).$$

Since $\gamma_p < 1$, we deduce that $\lambda_1 \ge 0$ with equality only if $u_1 \equiv 0$. But u_1 cannot be identically 0 because $E(\vec{u}_n) \not\to 0$. Then, up to a subsequence, $\lambda_n \to \lambda_1 > 0$. By weak convergence, u_1 is a radial weak solution of $-\Delta u + \lambda_1 u = |u|^{p-2}u$. We infer that

$$\int |\nabla(u_{i,n} - u_1)|^2 + \lambda_1 |u_{1,n} - u_1|^2 = o_n(1),$$

then $u_{1,n} \to u_1$ in $H^1_r(\mathbb{R}^N, \mathbb{R})$. In addition,

$$E(\vec{u}_n) = \frac{1}{2}\|u_{1,n}\|_2^2 - \frac{1}{p}\|u_{1,n}\|_p^p + o_n(1) = \frac{1}{2}\left(1 - \frac{1}{\gamma_p}\right)\|u_{1,n}\|_p^p + o_n(1)$$

$$= \frac{1}{2}\left(1 - \frac{1}{\gamma_p}\right)\|u_1\|_p^p + o_n(1) = m(a_1) + o_n(1).$$

By rescaling, $u_1 = \tilde{\kappa}^{\frac{1}{p-2}} w_p(\tilde{\kappa}^{\frac{1}{2}} x)$ where $\tilde{\kappa} = \left(\frac{a_1^2}{\|w_p\|_2^2}\right)^{\frac{p-2}{2-p\gamma_p}}$ and w_p is the positive radial solution of $-\Delta w + w = |w|^{p-2} w$. $\qquad\square$

Proof of Theorem 2. (i) It follows from Lemmas 3.3 and 4.2 that there is a mountain-pass critical point of E on $S(a_1, a_2)$. Therefore, there exists $\vec{v} \in S(a_1, a_2)$ such that $E(\vec{v}) = m^-(a_1, a_2)$.

(ii) It follows from Lemma 4.3.

5. Proof of Theorem 3

In this section we prove the global existence result. We observe that the following identity holds true:

$$E(\vec{\psi}) - \frac{1}{p\gamma_p}P(\vec{\psi}) = \frac{1}{2}\left(1 - \frac{2}{p\gamma_p}\right)\sum_{i=1}^3 \|\nabla\psi_i\|_2^2 - \alpha\left(1 - \frac{1}{p-2}\right)\text{Re}\int \psi_1\psi_2\overline{\psi}_3, \qquad (5.1)$$

recalling the definition of the energy (1.2) and the Pohozaev functions in (1.12).

Proof of Theorem 3. From Chapter 4 in [11], we get that (1.1) is locally well-posed, therefore, $\vec{\psi} \in C\left([0, T_{\max}), H^1(\mathbb{R}^N, \mathbb{C}^3)\right)$ for some $T_{\max} > 0$, and by the blow-up alternative $T_{\max} = +\infty$ or $\sum_{i=1}^3 \|\nabla\psi_i(t)\|_2^2 \to +\infty$ as $t \to T_{\max}^-$. We assume by contradiction that $\sum_{i=1}^3 \|\nabla\psi_i(t)\|_2^2 \to +\infty$ as $t \to T_{\max}^-$. We omit the time dependence when no confusion may arise. By the Gagliardo-Nirenberg inequality,

$$E(\vec{\psi}) - \frac{1}{p\gamma_p}P(\vec{\psi})$$

$$= \frac{1}{2}\left(1 - \frac{2}{p\gamma_p}\right)\sum_{i=1}^3 \|\nabla\psi_i\|_2^2 - \alpha\left(1 - \frac{1}{p-2}\right)\text{Re}\int \psi_1\psi_2\overline{\psi}_3$$

$$\geq \frac{1}{2}\left(1 - \frac{2}{p\gamma_p}\right)\sum_{i=1}^3 \|\nabla\psi_i\|_2^2 - \frac{\alpha}{3}C^3(N,p)\max\left\{a_1^{\frac{6-N}{2}}, a_2^{\frac{6-N}{2}}\right\}\left(\sum_{i=1}^3 \|\nabla\psi_i\|_2^2\right)^{\frac{N}{4}}.$$

Therefore, we have

$$E(\vec{\psi}(t)) - \frac{1}{p\gamma_p}P(\vec{\psi}(t)) \to +\infty \quad \text{as } t \to T_{\max}^-,$$

and by conservation of the energy, it follows that $P(\vec{\psi}(t)) \to -\infty$ as $t \to T_{\max}^-$.

We claim, with a strategy as in [38], that there exists $K > 0$ such that $t_{\vec{\psi}_0} < 1$ for all $\vec{\psi}_0 \in S(a_1, a_2)$ with $P(\vec{\psi}_0) < -K$.

We separate two cases. At first, suppose that $\vec{\psi}_0 \in \mathcal{M}$, then by the Gagliardo-Nirenberg inequality,

$$P(\vec{\psi}_0) \geq \sum_{i=1}^{3} \|\nabla\psi_{0,i}\|_2^2 - \gamma_p C^p(N,p) \max\left\{ a_1^{\frac{p-p\gamma_p}{2}}, a_2^{\frac{p-p\gamma_p}{2}} \right\} \left(\sum_{i=1}^{3} \|\nabla\psi_{0,i}\|_2^2 \right)^{\frac{p\gamma_p}{2}}$$
$$- \frac{N\alpha}{2} C^3(N,p) \max\left\{ a_1^{\frac{6-N}{2}}, a_2^{\frac{6-N}{2}} \right\} \left(\sum_{i=1}^{3} \|\nabla\psi_{0,i}\|_2^2 \right)^{\frac{N}{4}}.$$

This implies that $P(\vec{\psi}_0) \geq g\left(\left(\sum_{i=1}^{3} \|\nabla\psi_{0,i}\|_2^2 \right)^{\frac{1}{2}} \right)$, where

$$g(y) = y^2 - \gamma_p C^p(N,p) \max\left\{ a_1^{\frac{p-p\gamma_p}{2}}, a_2^{\frac{p-p\gamma_p}{2}} \right\} y^{p\gamma_p} - \frac{N\alpha}{2} C^3(N,p) \max\left\{ a_1^{\frac{6-N}{2}}, a_2^{\frac{6-N}{2}} \right\} y^{\frac{N}{2}}.$$

As in the proof of Lemma 2.2, under the assumption of $\max\{a_1, a_2\} < D$, there exists $R_2, R_3 > 0$ such that g is positive on (R_2, R_3). Since $\lim_{y\to 0^+} g(y) = 0^-$ and g is continuous, there exists $K > 0$ such that $g(y) \geq -K$ on $[0, R_2]$. From Lemma 2.4, we get that $s_{\vec{\psi}_0}$ is the local minimizer of $\Psi_{\vec{\psi}_0}$, and hence

$$\inf_{s\in(0,s_{\vec{\psi}_0})} s\Psi'_{\vec{\psi}_0}(s) = \inf_{s\in(0,s_{\vec{\psi}_0})} P(s \star \vec{\psi}_0)$$
$$\geq \inf_{s\in(0,s_{\vec{\psi}_0})} g\left(s \star \left(\sum_{i=1}^{3} \|\nabla\psi_i\|_2^2 \right)^{\frac{1}{2}} \right) \geq \inf_{y\in(0,R_2)} g(y) \geq -K.$$

We assume by contradiction that $P(\vec{\psi}_0) < -K$ but $t_{\vec{\psi}_0} \geq 1$. If $1 \in [s_{\vec{\psi}_0}, t_{\vec{\psi}_0}]$, then we have $P(\vec{\psi}_0) = \Psi'_{\vec{\psi}_0}(1) \geq 0$, which is impossible. If $s_{\vec{\psi}_0} > 1$, it follows that

$$-K > P(\vec{\psi}_0) = \Psi'_{\vec{\psi}_0}(1) \geq \inf_{s\in(0,s_{\vec{\psi}_0})} s\Psi'_{\vec{\psi}_0}(s) \geq -K,$$

which is a contradiction.

Secondly, suppose that $\vec{\psi}_0 \notin \mathcal{M}$, let $t_{\vec{\psi}_0}$ be the unique critical point of the function $\Psi_{\vec{u}}$ which is a strict maximum point at positive level. Then $t_{\vec{\psi}_0} < 1$ for $\vec{\psi}_0 \in S(a_1, a_2)$ with $P(\vec{\psi}_0) < -K$. Thus, the proof of the claim is complete.

Since $P\left(\vec{\psi}(t) \right) \to -\infty$ as $t \to T_{\max}^-$, by the above claim and Lemma 2.4, it gives that $t_{\vec{\psi}(T_{\max}-\varepsilon)} < 1$ if ε is small enough. It follows from $P(\vec{\psi}_0) > 0$ that $t_{\vec{\psi}_0} > 1$, and since $\vec{\psi}_0 \mapsto t_{\vec{\psi}_0}$ is continuous in $H^1(\mathbb{R}^N, \mathbb{C}^3)$, then there exists $\tau \in (0, T_{\max})$ such that $t_{\vec{\psi}(\tau)} = 1$, i.e., $\vec{\psi}(\tau) \in \mathcal{P}_{a_1, a_2}^-$. The conservation of the energy and the assumption on $E(\vec{\psi}_0)$ yields

$$\inf_{\mathcal{P}_{a_1,a_2}^-} E > E(\vec{\psi}_0) = E(\vec{\psi}(\tau)) \geq \inf_{\mathcal{P}_{a_1,a_2}^-} E,$$

which is a contradiction.

6. Proof of Theorem 4

In this last section, we prove that the conditions in Theorem 4 are sufficient to have formation of singularities in finite time, as well as the instability result. The Pohozaev function P below is defined in (1.12).

Lemma 6.1. *Under the assumption of Theorem 2, let $\vec{\psi}(t)$ be the solution of (1.1) with initial datum $\vec{\psi}_0 \in S(a_1, a_2)$, $P(\vec{\psi}_0) < 0$ and $E(\vec{\psi}_0) < \inf E(\vec{v})$. Then there exists $\eta > 0$ such that $P(\vec{\psi}(t)) \leq -\eta < 0$ for any t in the maximal time of existence.*

Proof. Similar to the proof of Lemma 2.2, $t_{\vec{\psi}_0}$ is the unique global maximal point of $\Psi_{\vec{\psi}_0}$, and $\Psi_{\vec{\psi}_0}$ is strictly decreasing and concave in $(t_{\vec{\psi}_0}, +\infty)$, see (2.8) for the definition of Ψ. From [38, Section 10], we have the following claim, if $\vec{\psi}_0 \in S(a_1, a_2)$ and $t_{\vec{\psi}_0} \in (0, 1)$, then

$$P(\vec{\psi}_0) \leq E(\vec{\psi}_0) - \inf_{\mathcal{P}^-_{a_1,a_2}} E. \tag{6.1}$$

Let $\vec{\psi}(t)$ be the solution of (1.1) with initial datum $\vec{\psi}(0) := \vec{\psi}_0$, defined on the interval $[0, T_{\max})$. By continuity, and $P(\vec{\psi}_0) < 0$, provided t is sufficiently small we have $P(\vec{\psi}(t)) < 0$. Therefore, from (6.1),

$$P(\vec{\psi}(t)) \leq E(\vec{\psi}(t)) - \inf_{\mathcal{P}^-_{a_1,a_2}} E = E(\vec{\psi}_0) - \inf_{\mathcal{P}^-_{a_1,a_2}} E =: -\eta < 0, \tag{6.2}$$

for any t. Hence, we deduce from the continuity that $P(\vec{\psi}(t)) < -\eta$ for all $t \in [0, T_{\max})$. $\square$

The next result is a refinement of the Lemma 6.1.

Lemma 6.2. *Under the same hypothesis of Lemma 6.1, there exists a positive constant $\delta > 0$ such that*

$$P(\vec{\psi}(t)) \leq -\delta \sum_{i=1}^{3} \|\nabla \psi_i(t)\|_2^2.$$

Proof. From the proof of Lemma 6.1 we already know that there exists a positive $\eta > 0$ such that $P(\vec{\psi}(t)) \leq -\eta$ in the maximal time of existence of the solution, see (6.2). By the algebraic relation (we omit the time dependence on ψ_i)

$$E(\vec{\psi}) - \frac{1}{p\gamma_p} P(\vec{\psi}) = \frac{1}{2}\left(1 - \frac{2}{p\gamma_p}\right) \sum_{i=1}^{3} \|\nabla \psi_i\|_2^2 - \alpha\left(1 - \frac{1}{p-2}\right) \mathrm{Re} \int \psi_1 \psi_2 \overline{\psi}_3,$$

we have that

$$\sum_{i=1}^{3} \|\nabla \psi_i\|_2^2 = 2\frac{p\gamma_p}{p\gamma_p - 2}\left(E(\vec{\psi}) - \frac{1}{p\gamma_p} P(\vec{\psi}) + \alpha\frac{p-3}{p-2}\mathrm{Re} \int \psi_1 \psi_2 \overline{\psi}_3\right).$$

Therefore,

$$P(\vec{\psi}) + \delta\|\nabla\vec{\psi}\|_2^2 = \left(1 - \frac{2\delta}{p\gamma_p - 2}\right) P(\vec{\psi}) + \frac{2\delta p\gamma_p}{p\gamma_p - 2} E(\vec{\psi}) + \frac{2\alpha\delta p\gamma_p}{p\gamma_p - 2}\frac{p-3}{p-2}\mathrm{Re} \int \psi_1 \psi_2 \overline{\psi}_3. \tag{6.3}$$

By the Hölder and the Gagliardo-Nirenberg interpolation inequalities, jointly with the conservation of the masses, see (1.3),

$$\int \psi_1 \psi_2 \overline{\psi}_3 \leq \|\psi_1\|_3 \|\psi_2\|_3 \|\psi_3\|_3 \lesssim (\|\nabla\psi_1\|_2 \|\nabla\psi_2\|_2 \|\nabla\psi_3\|_2)^{N/6} \lesssim \left(\sum_{i=1}^{3} \|\nabla\psi_i\|_2^2 \right)^{N/4}.$$

For $N = 2, 3$, $N/4 < 1$, and hence by the generalized Young's inequality

$$\delta \frac{2\alpha p\gamma_p}{p\gamma_p - 2} \frac{p-3}{p-2} \mathrm{Re} \int \psi_1 \psi_2 \overline{\psi}_3 \leq \frac{\delta}{2} \sum_{i=1}^{3} \|\nabla\psi_i\|_2^2 + C\delta.$$

By inserting the above estimate in (6.3), and using the conservation of the energy, we get

$$P(\vec{\psi}(t)) + \frac{\delta}{2} \sum_{i=1}^{3} \|\nabla\psi_i(t)\|_2^2 \leq -\left(1 - \frac{2\delta}{p\gamma_p - 2} \right) \sigma + \delta C,$$

and by choosing δ sufficiently small, we obtain the desired result, as the right-hand side can be made strictly negative uniformly in time. $\qquad \square$

6.1. **Proof of Theorem 4.** We can now prove the blow-up results. Define

$$I(t) = \sum_{i=1}^{3} \int \varphi |\psi_i(t)|^2 \, dx \qquad (6.4)$$

for a smooth, real, nonnegative, time independent function $\varphi = \varphi(x)$. By differentiating twice in time and using (1.1), we get (we omit the time dependence on ψ_i)

$$I'(t) = \sum_{i=1}^{3} 2\mathrm{Im} \left\{ \int \nabla\varphi \overline{\psi}_i \nabla\psi_i \right\}$$

and

$$I''(t) = \sum_{i=1}^{3} 4\mathrm{Re} \left\{ \int \nabla^2\varphi \nabla\psi_i \nabla\overline{\psi}_i \right\} - \int \Delta^2\varphi |\psi_i|^2 - 2\left(1 - \frac{2}{p} \right) \int \Delta\varphi |\psi_i|^p$$
$$- 2\alpha \mathrm{Re} \int \Delta\varphi \psi_1 \psi_2 \overline{\psi}_3. \qquad (6.5)$$

By plugging $\varphi = |x|^2$ in (6.4), and using (6.5) along with Lemma 6.1, after integrating in time twice we obtain

$$0 \leq I(t) \leq -8\eta t^2 + O(t) \qquad \text{for all } t \in [0, T_{\max}),$$

and a convexity argument gives $T_{\max} < \infty$.

We now consider radial solutions. Let $\chi : [0, \infty) \to [0, \infty)$ be a smooth, nonnegative function satisfying

$$\chi(r) := \begin{cases} r^2 & \text{if } 0 \leq r \leq 1, \\ \text{const.} & \text{if } r \geq 2, \end{cases} \qquad \chi''(r) \leq 2, \quad \forall r \geq 0. \qquad (6.6)$$

Given $R > 1$, we define by rescaling, the radial function $\varphi_R : \mathbb{R}^N \to \mathbb{R}$ by

$$\varphi_R(x) = \varphi_R(r) := R^2 \chi(r/R). \qquad (6.7)$$

If φ is radial and $\vec{\psi}$ is also radial, then

$$I''(t) = \sum_{i=1}^{3} 4 \int \varphi_R''(r)|\nabla\psi_i|^2 - \int \Delta^2\varphi_R|\psi_i|^2 - 2\left(1 - \frac{2}{p}\right)\int \Delta\varphi_R|\psi_i|^p$$

$$- 2\alpha\mathrm{Re}\int \Delta\varphi_R\psi_1\psi_2\overline{\psi}_3$$

$$= \sum_{i=1}^{3} 8 \int |\nabla\psi_i|^2 + 4\sum_{i=1}^{3}\int (\varphi_R''(r) - 2)|\nabla\psi_i|^2 - \int \Delta^2\varphi_R|\psi_i|^2$$

$$+ \sum_{i=1}^{3} 2\left(1 - \frac{2}{p}\right)\int (2N - \Delta\varphi_R)|\psi_i|^p - 4N\sum_{i=1}^{3}\left(1 - \frac{2}{p}\right)\int |\psi_i|^p$$

$$+ 2\alpha\mathrm{Re}\int (2N - \Delta\varphi_R)\psi_1\psi_2\overline{\psi}_3 - 4N\alpha2\mathrm{Re}\int \psi_1\psi_2\overline{\psi}_3.$$

By using the properties of the localisation function, and the conservation of masses (the quantities Q_1 and Q_2, see (1.3)), we estimate

$$I''(t) \le 8\sum_{i=1}^{3}\int |\nabla\psi_i|^2 + CR^{-2} - 4N\sum_{i=1}^{3}\left(1 - \frac{2}{p}\right)\int |\psi_i|^p - 4N\alpha2\mathrm{Re}\int \psi_1\psi_2\overline{\psi}_3$$

$$+ C\sum_{i=1}^{3}\int_{|x|\ge R}|\psi_i|^p + 2\alpha\int_{|x|\ge R}|\psi_1\psi_2\overline{\psi}_3| \tag{6.8}$$

$$= 8P(\vec{\psi}) + CR^{-2} + C\sum_{i=1}^{3}\int_{|x|\ge R}|\psi_i|^p + 2\alpha\int_{|x|\ge R}|\psi_1\psi_2\overline{\psi}_3|.$$

To estimate the last term, we recall the following radial Sobolev embedding (see e.g. [15]): for a radial function $f \in H^1(\mathbb{R}^3)$, we have for $\frac{1}{2} \le s < 1$ and $N \ge 2$,

$$\sup_{x\neq 0}|x|^{\frac{N}{2}-s}|f(x)| \le C\|\nabla f\|_2^s\|f\|_2^{1-s}. \tag{6.9}$$

Thanks to (6.9) and the conservation of mass, we estimate with $s = \frac{1}{2}$,

$$\int_{|x|\ge R}|\psi_i|^p = \int_{|x|\ge R}|\psi_i|^2|\psi_i|^{p-2} \lesssim \left(R^{-\frac{(N-1)}{2}}\|\nabla\psi_i\|_2^{1/2}\|\psi_i\|_2^{1/2}\right)^{p-2}\|\psi_i\|_2^2$$

$$\lesssim R^{-\frac{(N-1)(p-2)}{2}}\|\nabla\psi_i\|_2^{(p-2)/2}. \tag{6.10}$$

Note that by the Hölder and the Cauchy-Schwarz inequalities,

$$\int_{|x|\ge R}|\psi_1\psi_2\overline{\psi}_3| \le \frac{1}{3}\sum_{i=1}^{3}\int_{|x|\ge R}|\psi_i|^3,$$

thus by (6.10) with $p = 3$ we get

$$\int_{|x|\ge R}|\psi_1\psi_2\overline{\psi}_3| \lesssim R^{-\frac{N-1}{2}}\sum_{i=1}^{3}\|\nabla\psi_i\|_2^{1/2}. \tag{6.11}$$

Hence, from (6.8), (6.10), and (6.11) we get

$$I''(t) \leq 8P(\vec{\psi}) + CR^{-2} + CR^{-\frac{(N-1)(p-2)}{2}} \sum_{i=1}^{3} \|\nabla\psi_i\|_2^{(p-2)/2} + R^{-\frac{N-1}{2}} \sum_{i=1}^{3} \|\nabla\psi_i\|_2^{1/2}. \quad (6.12)$$

Let us observe that in dimension $N = 3$ it holds true that $\frac{p-2}{2} < 2$ provided $p < 6 = 2 + \frac{4}{N-2} = p^*$, which fits our assumption in the three-dimensional setting. When $N = 2$, we must restrict the range of the nonlinearity to $p \in (4, 6)$. See also Ogawa and Tsutsumi [35].

A convexity argument yields the blow-up result, by glueing together (6.12), (6.2) and Lemma 6.2, provided R is large enough.

Remark 6.3. In the three-dimensional case, the radial symmetry can be further relaxed to a cylindrical symmetric setting, provided we impose partial weighted L^2-summability of the initial data, see the first author's results in [2, 4, 21].

6.2. **Proof of Corollary 1.** Let $\vec{v}$ be the excited state constructed in Theorem 2, point (i). For any $s > 0$, let $\vec{v}_s := s \star \vec{v}$, and let $\vec{\psi}_s$ be the solution to (1.1) with the initial datum $\vec{v}_s$. Then, $\vec{v}_s \to u$ as $s \to 1^+$. By Lemma 6.1, it is sufficient to prove that $\vec{\psi}_s$ blows-up in finite time. In fact, it follows from [6] that $\vec{v} \in H^1(\mathbb{R}^N, \mathbb{R}^3)$ decays exponentially at infinity, and hence $|x|\vec{v} \in L^2(\mathbb{R}^N, \mathbb{R}^3)$. Let $\sigma_{\vec{v}_s}$ be defined in Lemma 2.4, we have

$$E(\vec{v}_s) = E(s \star \vec{v}) < E(\sigma_{\vec{v}_s} \star \vec{v}) = \inf_{\mathcal{P}_{a_1,a_2}^-} E,$$

because $P(\vec{v}_s) < 0$. The proof of Corollary 1 is completed.

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