INVARIANT GIBBS DYNAMICS FOR TWO-DIMENSIONAL FRACTIONAL WAVE EQUATIONS IN NEGATIVE SOBOLEV SPACES

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Abstract. We consider a fractional nonlinear wave equations (fNLW) with a general power-type nonlinearity, on the two-dimensional torus. Our main goal is to construct invariant global-in-time Gibbs dynamics for a renormalized fNLW. We first construct the Gibbs measure associated with this equation by using the variational approach of Barashkov and Gubinelli. We then prove almost sure local well-posedness with respect to Gibbsian initial data, by exploiting the second order expansion. Finally, we extend solutions globally in time using Bourgain's invariant measure argument.

1. Introduction

We consider the defocusing fractional nonlinear wave equations (fNLW) posed on the two-dimensional flat torus:

$$\begin{cases} \partial_t^2 u + (1-\Delta)^\alpha u + u^{2m+1} = 0 \\ (u, \partial_t u)|_{t=0} = (\phi_0, \phi_1), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^2, \tag{1.1}$$

where $\alpha \in (\alpha_0(m), 1)$, for a parameter $\alpha_0 \in (0,1)$ depending on m to be specified later on in the paper, $m \in \mathbb{N}$ is a positive integer, and $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$. Despite the presence of the mass term, we refer to (1.1) as fNLW, though it is usually referred to as the nonlinear Klein-Gordon equation. We restrict our attention to the case of real-valued solutions $u : \mathbb{R} \times \mathbb{T}^2 \mapsto \mathbb{R}$.

The purpose of this paper is twofold. Firstly, we are interested in the construction of invariant Gibbs measures for a renormalization of equation (1.1). Secondly, we prove almost sure global well-posedness of the latter renormalized NLW. The a.s. global well-posedness is a consequence of an a.s. local well-posedness theory, followed by a Bourgain's invariant measure argument. Hence, our main task will be the proof of an almost sure local well-posedness theory.

Both the construction of Gibbs measures and the local well-posedness of the (renormalized) NLW equation, strongly depend on the value α of the fractional power of the elliptic operator $1-\Delta$ appearing in (1.1).

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1.1. **Gibbs measures and Wick renormalization.** With the notation $v = \partial_t u$, we can write the equation (1.1) in the an Hamiltonian formulation as follows:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial H_\alpha}{\partial(u,v)}, \tag{1.2}$$

where $H_\alpha = H_\alpha(u,v)$ is the conserved Hamiltonian (i.e., the energy functional) defined by

$$\begin{aligned} H_\alpha(u,v) &= \frac{1}{2}\int_{\mathbb{T}^2} |\,(1-\Delta)^{\alpha/2}\,u|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2} v^2\,dx + \frac{1}{2m+2}\int_{\mathbb{T}^2} u^{2m+2}\,dx \\ &= \frac{1}{2}\left(\|u\|^2_{H^\alpha_x(\mathbb{T}^2)} + \|v\|^2_{L^2_x(\mathbb{T}^2)}\right) + \frac{1}{2m+2}\|u\|^{2m+2}_{L^{2m+2}_x(\mathbb{T}^2)}. \end{aligned} \tag{1.3}$$

Here we omitted the time-dependence of u and v.

Mimicking the picture of the finite dimensional setting, where a Liouville theorem shows the invariance of the finite dimensional Lebesgue measure on $\mathbb{R}^{2n}$ for an Hamiltonian flow on the Euclidean space of the same dimension, and hence the invariance of the Gibbs measure, one is tempted to define in a similar way the Gibbs measure

$$dP^{2m+2}_{\alpha,2} = Z^{-1}\exp(-\beta H_\alpha(u,v))\,du\otimes dv \tag{1.4}$$

also in the infinite dimensional case, though in the latter setting we cannot invoke the same result we have in the finite dimensional one. Hence, (1.4) is simply a formal expression and we are required to rigorously justify its construction, which is one of the main task of this article. As in the finite dimensional framework, we also would like to have invariance of (1.4) under the dynamics of (1.1). In (1.4), and in similar expression along the article, Z, Z_N, etc. denote various normalizing constants so that the corresponding measures are probability measures, when appropriate. The parameter β stands for the reciprocal temperature, and we fix $\beta = 1$. All the results contained in the paper actually hold for any $\beta > 0$, and the resulting (renormalized) Gibbs measures are mutually singular for different values of $\beta > 0$, see [24]. So, without loss of generality, we reduce to the normalized case. While the α in the left-hand side of (1.4) clearly refers to the weak dispersion exponent appearing in (1.1), the second subscript '2' stands for the dimension of the manifold where the equation is posed; as we work in the bi-dimensional setting, we drop it. From (1.3), we can rewrite the formal expression (1.4) as

$$\begin{aligned} dP^{2m+2}_\alpha &= Z^{-1}e^{-\frac{1}{2m+2}\int u^{2m+2}\,dx}e^{-\frac{1}{2}\int |(1-\Delta)^{\alpha/2}u|^2\,dx}du\otimes e^{-\frac{1}{2}\int v^2\,dx}dv \\ &\sim e^{-\frac{1}{2m+2}\int u^{2m+2}\,dx}d\vec{\mu}_\alpha, \end{aligned} \tag{1.5}$$

where $\vec{\mu}_\alpha$ is the Gaussian measure on $\mathcal{D}'(\mathbb{T}^2)\times\mathcal{D}'(\mathbb{T}^2)$ with density

$$d\vec{\mu}_\alpha = Z^{-1}e^{-\frac{1}{2}\int |(1-\Delta)^{\alpha/2}u|^2\,dx}\,du\otimes e^{-\frac{1}{2}\int v^2\,dx}\,dv. \tag{1.6}$$

Along the paper, we consider the fractional Klein-Gordon operator $(1-\Delta)^\alpha$ in (1.1) instead of the Wave one $(-\Delta)^\alpha$, in order to avoid a problem at the zeroth frequency. See (1.8) below.

Note that $\vec{\mu}_\alpha$ has a tensorial structure: $\vec{\mu}_\alpha = \mu_\alpha \otimes \nu$, where the marginal measures μ_α and ν are given by

$$d\mu_\alpha = Z_0^{-1} e^{-\frac{1}{2}\int |(1-\Delta)^{\alpha/2} u|^2 \, dx} \, du \qquad \text{and} \qquad d\nu = Z_1^{-1} e^{-\frac{1}{2}\int v^2 \, dx} \, dv. \tag{1.7}$$

Namely, μ_α is the Ornstein-Uhlenbeck measure and ν is the white noise measure on $\mathbb{T}^2$.

Note that $\vec{\mu}_\alpha$ is the induced probability measure under the map

$$\omega \in \Omega \longmapsto (u^\omega, v^\omega)(x) = \bigg(\sum_{n\in\mathbb{Z}^2} \frac{g_{0,n}(\omega)}{\langle n \rangle^\alpha} e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} g_{1,n}(\omega) e^{in\cdot x} \bigg), \tag{1.8}$$

where $\langle n \rangle = \sqrt{1+|n|^2}$, and $\{g_{0,n}, g_{1,n}\}_{n\in\mathbb{Z}^2}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ satisfying the condition $g_{j,-n} = \overline{g}_{j,n}$ for $j = 0, 1$. From (1.8), it is straightforward to see that $\vec{\mu}_\alpha$ is supported on

$$\mathcal{H}^s(\mathbb{T}^2) := H^s(\mathbb{T}^2) \times H^{s-\alpha}(\mathbb{T}^2), \quad s < \alpha - 1.$$

Here, $H^s(\mathbb{T}^2)$ is the usual L^2-based Sobolev space. The latter claim on the support of the measure comes by considering the quantity $\mathbb{E}(\|(u^\omega, v^\omega)\|^2_{\mathcal{H}^s})$, obtaining that it is finite provided that the series $\sum_{n\in\mathbb{Z}^2} \langle n \rangle^{2(s-\alpha)}$ is convergent, thus the condition $s < \alpha - 1$ above, and by showing that $\vec{\mu}_\alpha\{(u, v) : \|(u, v)\|_{\mathcal{H}^{\alpha-1}} < \infty\} = 0$. It turns that $\int_{\mathbb{T}^2} u^{2m+2} \, dx = \infty$, $\vec{\mu}_\alpha$-almost surely. Therefore, the right-hand side of (1.5) cannot define a probability measure, thus we are forced to introduce what is called *renormalization* of the potential energy in the Hamiltonian (1.3). In the 2D setting, a Wick ordering is suitable as such renormalization. We refer the reader to [12, 27, 28, 34].

Let us briefly go over the Wick renormalization on $\mathbb{T}^2$. See [9] for more details on $\mathbb{T}^2$, which is the framework of the present article. For a typical element u under μ_α defined in (1.7), since $u \notin L^2(\mathbb{T}^2)$ almost surely we have that

$$\int_{\mathbb{T}^2} u^2 \, dx = \lim_{N\to\infty} \int_{\mathbb{T}^2} (\mathbf{P}_N u)^2 \, dx = \infty$$

almost surely, where $\mathbf{P}_N u$ is the Fourier projector onto frequencies at most N, namely it acts as $\mathbf{P}_N f := \sum_{n\in\mathbb{Z}^2, |n|\le N} \widehat{f}(n) e^{in\cdot x}$. Moreover, we observe that for each $x \in \mathbb{T}^2$, $\mathbf{P}_N u(x)$ is a mean-zero real-valued Gaussian random variable with variance

$$\sigma_{\alpha,N} := \mathbb{E}[(\mathbf{P}_N u)^2(x)] = \sum_{\substack{n\in\mathbb{Z}^2 \\ |n|\le N}} \frac{1}{\langle n \rangle^{2\alpha}}, \tag{1.9}$$

thus $\sigma_{\alpha,N}$ diverges as $N^{2-2\alpha}$ as $N \to \infty$. Observe that the quantity σ_N^α introduced in (1.9) is independent of $x \in \mathbb{T}^2$.

The discussion above leads us to introduce the Wick ordered monomial $:(\mathbf{P}_N u)^k:$ point-wisely defined by

$$:(\mathbf{P}_N u)^k(x): := H_k(\mathbf{P}_N u(x); \sigma_{\alpha,N}), \tag{1.10}$$

where $H_k(x;\sigma)$ is the Hermite polynomial of degree k defined via the generating function

$$F(t,x;\sigma) := e^{tx-\frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x;\sigma). \tag{1.11}$$

Here we set $H_k(x) := H_k(x;1)$. A list of the first Hermite polynomials is given for the sake of clarity and for readers' convenience:

$$\begin{aligned} &H_0(x;\sigma) = 1, \qquad H_1(x;\sigma) = x, \qquad H_2(x;\sigma) = x^2 - \sigma, \\ &H_3(x;\sigma) = x^3 - 3\sigma x, \qquad H_4(x;\sigma) = x^4 - 6\sigma x^2 + 3\sigma^2. \end{aligned} \tag{1.12}$$

Furthermore, we report some useful identities:

$$H_k(\sigma^{1/2}x;\sigma) = \sigma^{k/2} H_k(x), \tag{1.13}$$

$$H_k(x+y;\sigma) = \sum_{\ell=0}^{k} \binom{k}{\ell} x^{k-\ell} H_l(y;\sigma), \tag{1.14}$$

and

$$\partial_x H_k(x;\sigma) = k H_{k-1}(x;\sigma). \tag{1.15}$$

Our aim is to show that given $m \in \mathbb{N}$, then the limit

$$\int_{\mathbb{T}^2} :u^{2m+2}: dx = \lim_{N\to\infty} \int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}: dx \tag{1.16}$$

exists in $L^p(\mu_\alpha)$ for any finite $p \geq 1$, where $\alpha \in (\alpha_0(m), 1)$ for some suitable $\alpha_0(m)$ (hence the range depends on m), see Proposition 1.1 below. Furthermore, one can show that the truncated partition function $e^{-\frac{1}{2m+2}\int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}: dx}$ admits a limit in $L^p(\mu_\alpha)$ as well, in some (possibly smaller) range of fractional exponents $\alpha \in (\tilde{\alpha}_0(m), 1)$, namely $\tilde{\alpha}_0(m) \geq \alpha_0(m)$. See Theorem 1.2 below for the range of exponential integrability. Note that we have the same range as in Proposition 1.1. We refer to Appendix B for another result about convergence of $R_N(u)$ in a smaller range, using a weaker approach than the one used to prove Theorem 1.2. Indeed, for the functions $G_N(u)$ and $R_N(u)$ defined by

$$G_N(u) := \int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}: dx \tag{1.17}$$

and

$$R_N(u) := e^{-\frac{1}{2m+2}\int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}: dx} = e^{-\frac{1}{2m+2} G_N(u)}, \tag{1.18}$$

we have the following first proposition.

Proposition 1.1. *Let $m \in \mathbb{N}$ and let $\alpha \in \left(1 - \frac{1}{2m+2}, 1\right)$. Let $1 \leq p \leq \infty$. The sequence $\{G_N(u)\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mu_\alpha)$. Specifically, the following bound holds true: there exists a constant $c = c(m)$ such that, for any $1 \leq N \leq M$, and any $p \in [1, \infty]$,*

$$\|G_M(u) - G_N(u)\|_{L^p(\mu_\alpha)} \leq c(m)(p-1)^{m+1} N^{1-2\alpha+\frac{m}{m+1}}. \tag{1.19}$$

We therefore define

$$G(u) := \int_{\mathbb{T}^2} :u^{2m+2}: dx = L^p\text{-}\lim_{N\to\infty} \int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}: \, dx.$$

Note that if $\alpha > 1 - \frac{1}{2m+2}$, the exponent of N in (1.19) is negative, hence $G_N(u)$ is a Cauchy sequence.

The next theorem is the core result to exhibit the existence of the Gibbs measure. It concerns the exponential integrability of $-G_N(u)$.

Theorem 1.2. *Let $m \in \mathbb{N}$ and let $\alpha \in \left(1 - \frac{1}{2m+2}, 1\right)$. Let $1 \leq p \leq \infty$. Then a uniform bound of $R_N(u)$ in $L^p(\mu_\alpha)$ holds true, hence the convergence of $R_N(u)$ to a limit object, say $R(u)$, in $L^p(\mu_\alpha)$.*

With the above theorem and (1.5), we can rigorously define the Gibbs measure

$$dP_\alpha^{2m+2} = Z^{-1} e^{-\frac{1}{2m+2}\int :u^{2m+2}: dx} \, d\vec{\mu}_\alpha.$$

The proof of the above exponential integrability, together with Proposition 1.1, is the main content of Section 3.2.

Remark 1.3. In the case $m = 1$ of the cubic fNLW, the result in theorem Theorem 1.2 as well is its proof overlap with a result by by Sun, Tzvetkov, and Xu from [35]. We proved the result above independently, at the same time. Additionally, it is worth mentioning a few differences between [35] and this paper.

(i) In [35], the authors consider fNLW with a nonlinearity given by a polynomial with prescribed properties; more precisely, the nonlinearity is a linear combination of cubic and higher powers, but which roughly speaking scale as the cubic power. In particular, from the point of view of the measure construction, the nonlinearity in [35] is essentially of cubic type. Consequently, the authors find a lower bound $\alpha > \frac{3}{4}$ for the construction of the Gibbs measure which is independent of the degree of the nonlinearity. In our paper, we consider general defocusing power-type nonlinearities u^{2m+1}, and thus the restriction we have on α for the construction of the Gibbs measure does depend on the power of the nonlinearity, namely on m. See the condition $\alpha > 1 - \frac{1}{2m+2}$ in Proposition 1.1 and Theorem 1.2. This restriction on α coincides with that in [35] when $m = 1$ (cubic nonlinearity).

(ii) Our goal in this paper is different from the goal in [35]. Namely, the aim of [35] is to prove the weak universality of the 2D (renormalized) cubic fNLW (both the convergence of invariant measures and the dynamical weak universality). In this paper, we consider a 2D (renormalized) fNLW with a general power-type nonlinearity, we construct the Gibbs measure, and we prove its almost sure global well-posedness with respect to Gibbsian initial data (see Theorem 1.4 and Theorem 1.7 below).

1.2. **Notion of solution.** We now introduce a notion of solution for the Cauchy problem with random initial data.

1.2.1. *First order expansion.* From Proposition 1.1 and Theorem 1.2, we can consider the Wick ordered Hamiltonian defined by

$$H_\alpha^W(u,v) = \frac{1}{2}\int_{\mathbb{T}^2} |\,(1-\Delta)^{\alpha/2}\,u|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2} v^2 dx + \frac{1}{2m+2}\int_{\mathbb{T}^2} :u^{2m+2}:\,dx \tag{1.20}$$

and analogously to (1.2), we can consider the Hamiltonian formulation of the equation with the Wick ordered Hamiltonian (1.20), associated to Cauchy data $(\phi_0^\omega, \phi_1^\omega)$ distributed according to P_α^{2m+2}, namely of the form (1.8). At this point we consider the truncated Wick Hamiltonian

$$H_{\alpha,N}^W(u,v) = \frac{1}{2}\int_{\mathbb{T}^2} |\,(1-\Delta)^{\alpha/2}\,u|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2} v^2 dx + \frac{1}{2m+2}\int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}:\,dx, \tag{1.21}$$

and by using (1.10), (1.15), and $\mathbf{P}_N\left(:(\mathbf{P}_N u_N)^{2m+1}:\right) := \mathbf{P}_N\left(H_{2m+1}(\mathbf{P}_N u_N;\sigma_N)\right)$, we get a truncated version of the Wick ordered equation:

$$\begin{cases} \partial_t^2 u_N + (1-\Delta)^\alpha u_N \ + \mathbf{P}_N\left(:\mathbf{P}_N u_N^{2m+1}:\right) = 0 \\ (u_N, \partial_t u_N)|_{t=0} = (\phi_0^\omega, \phi_1^\omega). \end{cases} \tag{1.22}$$

We recall the integral formulation of the fractional NLW equation, i.e. we formally write a solution to (1.1) as

$$u(t) = \cos(t\langle\nabla\rangle^\alpha)\phi_0 + \frac{\sin(t\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha}\phi_1 + \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} u^{2m+1} dt'.$$

The formulation above is the well-known Duhamel formulation of (1.1), and we further denote by $\mathcal{D}$ the Duhamel operator

$$\mathcal{D}(F) := \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} F(t') dt'. \tag{1.23}$$

Going back to (1.22), with such an integral formulation, a solution can be written as

$$u_N = z + w_N,$$

where $z = z(t)$ is the linear part defined by

$$z(t) := \cos(t\langle\nabla\rangle^\alpha)\phi_0 + \frac{\sin(t\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha}\phi_1 \tag{1.24}$$

and $w_N = u_N - z$ is a residual term satisfying the perturbed Wick ordered NLW:

$$\begin{cases} \partial_t^2 w_N + (1-\Delta)^\alpha w_N + :(z+w_N)^{2m+1}: = 0 \\ (w_N, \partial_t w_N)|_{t=0} = (0,0). \end{cases} \tag{1.25}$$

From the property of the Hermite polynomials (1.14), we straightforwardly compute that

$$:(z_N + w_N)^{2m+1}: = \sum_{\ell=0}^{2m+1} \binom{2m+1}{\ell} :z_N^\ell: w_N^{2m+1-\ell}, \tag{1.26}$$

where we also projected onto frequencies at most N also the linear part, namely we rewrite

$$z(t,x) = \sum_{n\in\mathbb{Z}^2} \frac{\cos(t\langle n\rangle^\alpha) g^\omega_{0,n}}{\langle n\rangle^\alpha} e^{in\cdot x} + \sum_{n\in\mathbb{Z}^2} \frac{\sin(t\langle n\rangle^\alpha) g^\omega_{1,n}}{\langle n\rangle^\alpha} e^{in\cdot x} \tag{1.27}$$

and z_N is given by

$$z_N(t) = \sum_{\substack{n\in\mathbb{Z}^2\\ |n|\le N}} \frac{\cos(t\langle n\rangle^\alpha) g^\omega_{0,n}}{\langle n\rangle^\alpha} e^{in\cdot x} + \sum_{\substack{n\in\mathbb{Z}^2\\ |n|\le N}} \frac{\sin(t\langle n\rangle^\alpha) g^\omega_{1,n}}{\langle n\rangle^\alpha} e^{in\cdot x}. \tag{1.28}$$

At this point the strategy is to split a solution into the form $u = z + w$ where z has a negative regularity and w is a regular function: more precisely, our aim is to prove that $:z^\ell:$ is well defined as a limit of stochastic objects in a suitable topology, thus $:z^{2m+1}:$ is well defined and we arrive at the defocusing Wick ordered fNLW:

$$\begin{cases} \partial_t^2 u + (1-\Delta)^\alpha u + :u^{2m+1}: = 0 \\ (u, \partial_t u)|_{t=0} = (\phi_0^\omega, \phi_1^\omega), \end{cases} \tag{1.29}$$

where

$$(\phi_0^\omega, \phi_1^\omega) = \Big(\sum_{n\in\mathbb{Z}^2} \frac{g_{0,n}(\omega)}{\langle n\rangle^\alpha} e^{in\cdot x}, \sum_{n\in\mathbb{Z}^2} g_{1,n}(\omega) e^{in\cdot x} \Big).$$

Note that we only provide an existence theory for (1.29) with solutions of the from $u = z + w$, where the (regular) residual term satisfies

$$\begin{cases} \partial_t^2 w + (1-\Delta)^\alpha w + :(z+w)^{2m+1}: = 0 \\ (w, \partial_t w)|_{t=0} = (0,0). \end{cases} \tag{1.30}$$

We employ a fixed-point argument to exhibit the existence of a solution w to the latter equation, by exploiting the regularity of the stochastic objects $:z^\ell:$. We eventually arrive at the following.

Theorem 1.4. *Fix $m \in \mathbb{N}$ and let $\alpha \in \Big(1 - \frac{1}{4m^2+4m+2}, 1\Big)$. Then (1.29) admits a global solution almost surely in $C_t H_x^{-(1-\alpha)-\varepsilon}$, for any $\varepsilon > 0$. More precisely, the solution belongs to the class*

$$u = z + w \in C_t H_x^{-(1-\alpha)-\varepsilon} + X^s \subset C_t H_x^{-(1-\alpha)-\varepsilon},$$

where X^s is a Strichartz space, for some $s > 0$.

We refer to the proof of Theorem 1.4 for a precise definition of the Strichartz space X^s, see (4.7) and (4.8). The strategy of splitting the solution as a linear (rough) term plus a residual (regular) term is known in the realm of stochastic PDEs under the name of Da Prato-Debussche trick, see [8], and we refer the reader to the early works by McKean [18] and Bourgain [6] where this idea firstly appeared. We also mention [7].

Remark 1.5. The proof of Theorem 1.4 is a combination of an almost sure local well-posedness theory and a globalisation argument which goes back to Bourgain, see [5,6]. The almost sure local well-posedness is precisely the content of Proposition 4.1 in Section 4. The extension of solutions from local to global in time follows by the invariant measure argument of [5,6]. Briefly, it follows by an approximation argument which uses the fact that Proposition 4.1 remains valid for the truncated problem (1.22) with uniform bounds in N, and the invariance of the truncated Gibbs measure defined by $P_{\alpha,N}^{(2m+2)} := Z_N^{-1} R_N(u)\, d\mu_\alpha$. Let us also mention that $P_\alpha^{(2m+2)}$ is invariant under the dynamics of (1.29). We refer to [20] for a general result for the NLW on compact Riemann manifolds without boundary.

Remark 1.6. In [27], Oh and Thomann consider the defocusing nonlinear wave equation, namely (1.1) with $\alpha = 1$, construct the Gibbs measures, and prove an almost sure global well-posedness theory exploiting the first order expansion related to that equation. In our case the main difference is represented by the fact that the fNLW is only α-smoothing, as clear from (1.23), while the wave equation is 1-smoothing. Moreover, the regularity of the stochastic objects $:z^\ell:$, $\ell \in \{0, \dots, 2m+1\}$ (coming from (1.26), see also the proof of Theorem 1.4 in Section 4) is of order $-\ell(1-\alpha) - \varepsilon$, $\varepsilon > 0$, which is responsible for the restriction on α, or equivalently on the nonlinearity u^{2m+1} we can handle. In [27], the stochastic objects have regularity $-\varepsilon$ for any m, which in turn does not imply any restriction of the nonlinearity.

1.2.2. *Second order expansion.* Aiming at widening the range of the fractional exponent α, we further decompose $w = z_2 + w_2$ in the same spirit of the first order expansion. Without repeating the same considerations that apply likewise, here we introduce the solution z_2 to the following Cauchy problem:

$$\begin{cases} \partial_t^2 z_2 + (1-\Delta)^\alpha z_2 + :z^{2m+1}: = 0 \\ (z_2, \partial_t z_2)|_{t=0} = (0,0), \end{cases} \tag{1.31}$$

therefore

$$z_2(t) = \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} :z^{2m+1}: dt'. \tag{1.32}$$

Thus, $u = z + z_2 + w_2$ and the equation solved by w_2 is given by

$$\begin{cases} \partial_t^2 w_2 + (1-\Delta)^\alpha w_2 + \sum_{\ell=0}^{2m} \binom{2m+1}{\ell} :z^\ell:\ (z_2+w_2)^{2m+1-\ell} = 0 \\ (w_2, \partial_t w_2)|_{t=0} = (0,0). \end{cases} \tag{1.33}$$

With the above second order expansion, we enlarge the interval of the exponent α, and we specifically obtain the following improved local well-posedness result.

Theorem 1.7. *Fix $m \in \mathbb{N}$ and let $\alpha \in \left(1 - \frac{1}{4m^2+2m+1}, 1\right)$. Then (1.29) admits a global solution almost surely in $C_t H_x^{-(1-\alpha)-\varepsilon}$, for any $\varepsilon > 0$. More precisely, the solution* *belongs to the class*

$$u = z + z_2 + w_2 \in C_t H_x^{-(1-\alpha)-\varepsilon} + C_t W_x^{-(2m+1)(1-\alpha)+\alpha-\varepsilon,\infty} + X^s \subset C_t H_x^{-(1-\alpha)-\varepsilon},$$

where X^s is a Strichartz space, for some $s > 0$.

See again (4.7) and (4.8) for the definition of the Strichartz space appearing in the statement of Theorem 1.7. The strategy of the proof is analogous to the one for Theorem 1.4. Note that by expanding $(z_2 + w_2)^{2m+1-\ell}$ we end up with terms of the form $: z^\ell : z_2^{2m+1-j-\ell}$ for which we crucially need a control on products of stochastic objects as precisely depicted in Proposition 2.13 (iii), see the next section.

Remark 1.8. A local well-posedness result performing a second order expansion was also given for the defocusing cubic NLW posed on the three-dimensional torus $\mathbb{T}^3$ in [25]. In that paper, Oh, Tzvetkov and the second author only need to control a product of stochastic objects of the from $: z^2 : \mathcal{D}(: z^3 :)$, see Proposition 2.13 below, while here we treat general nonlinearities, so the analysis is more involved and we need the regularity results of Proposition 2.13 for general products of stochastic terms.

2. Preliminary deterministic and stochastic tools

This section is devoted to the statement of useful estimates used along the paper. In particular, Subsection 2.1 concerns deterministic estimates, while Subsection 2.2 contains stochastic tools.

2.1. Deterministic estimates.

We start by recalling the following estimates. They can can be stated in arbitrary dimension, though we specialize to the 2D case. In what follows, we use the standard notation for the fractional Sobolev spaces $W^{s,p}(\mathbb{T}^2)$. $H^s(\mathbb{T}^2)$ corresponds to $p = 2$.

Lemma 2.1. *(Fractional Leibniz rule)* (i) *Let $0 \leq s \leq 1$. Let $1 < p_j, q_j, r < \infty$, $j = 1, 2$, such that $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$. Then, the following holds*

$$\|\langle\nabla\rangle^s (fg)\|_{L^r} \lesssim \Big(\|f\|_{L^{p_1}} \|\langle\nabla\rangle^s g\|_{L^{q_1}} + \|\langle\nabla\rangle^s f\|_{L^{p_2}} \|g\|_{L^{q_2}}\Big) \tag{2.1}$$

for any $f \in L^{p_1}(\mathbb{T}^2) \cap W^{s,p_2}(\mathbb{T}^2)$ and $g \in L^{q_2}(\mathbb{T}^2) \cap W^{s,q_1}(\mathbb{T}^2)$.
(ii) *Suppose that $1 < p, q, r < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{2}$, then*

$$\|\langle\nabla\rangle^{-s} (fg)\|_{L^r} \lesssim \|\langle\nabla\rangle^{-s} f\|_{L^p} \|\langle\nabla\rangle^s g\|_{L^q} \tag{2.2}$$

for any $f \in W^{-s,p}(\mathbb{T}^2)$ and $g \in W^{s,q}(\mathbb{T}^2)$.

Proof. See [13, 15]. □

The following is a classical interpolation inequality.

Lemma 2.2. *For $0 < s_1 < s_2$, the following interpolation holds*

$$\|u\|_{H^{s_1}} \leq \|u\|_{H^{s_2}}^{\frac{s_1}{s_2}} \|u\|_{L^2}^{\frac{s_2 - s_1}{s_2}}$$

for any $u \in H^{s_2}$.

Proof. It easily follows by the definition of the spaces, in conjunction with the Hölder's inequality. □

We proceed by recalling the following estimate on discrete convolution.

Lemma 2.3. *Let $\alpha, \beta \in \mathbb{R}$ satisfy*

$$\alpha + \beta > 2 \qquad \text{and} \qquad \alpha, \beta < 2.$$

Then, the following holds

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z}^2 \\ n = n_1 + n_2}} \frac{1}{\langle n_1 \rangle^\alpha \langle n_2 \rangle^\beta} \lesssim \langle n \rangle^{2-\alpha-\beta}$$

for any $n \in \mathbb{Z}^2$.

Proof. The estimate can be shown by direct computations. See, for example, Lemma 4.1 in [19]. □

We now state the Strichartz estimates used to prove the local well-posedness results. For this purpose, let us introduce some notations. We call a pair (p, q) fractional admissible provided that

$$(p, q) \in [2, \infty] \times [2, \infty], \quad (p, q) \neq (2, \infty), \text{ and } \frac{2}{p} + \frac{2}{q} \leq 1. \tag{2.3}$$

For a fixed α, we define the exponent

$$\gamma_{p,q} = 1 - \frac{2}{q} - \frac{\alpha}{p}. \tag{2.4}$$

Let us consider the Cauchy problem

$$\begin{cases} \partial_t^2 v + (1 - \Delta)^\alpha v = F \\ (v, \partial_t v)|_{t=0} = (v_0, v_1), \end{cases} \qquad (t, x) \in I \times \mathbb{T}^2, \tag{2.5}$$

for $I \subset \mathbb{R}$. We have the following Strichartz estimates, see [10, Corollary 1.4].

Lemma 2.4. *Let I a bounded interval, $\alpha \in (0, 1)$, (p, q) a fractional admissible pair, $(v_0, v_1) \in H^{\gamma_{p,q}} \times H^{\gamma_{p,q}-\alpha}$, and v a weak solution to (2.5). The following estimate holds true:*

$$\|v\|_{L_t^p(I; L_x^q)} \lesssim \|v_0\|_{H_x^{\gamma_{p,q}}} + \|v_1\|_{H_x^{\gamma_{p,q}-\alpha}} + \|F\|_{L_t^1(I; H_x^{\gamma_{p,q}-\alpha})}. \tag{2.6}$$

Note that the Strichartz estimates on $\mathbb{T}^2$ follow from the Strichartz estimates on $\mathbb{R}^2$ by exploiting the finite speed of propagation for the linear fractional wave equation. Beside [10], we also mention the recent paper [35] which includes a self-contained proof.

2.2. **Regularity of stochastic objects.** Firstly, we introduce basic definitions from stochastic analysis, see the monographs [3, 33], aiming at recalling the Wiener chaos estimate (see Lemma 2.5 below).

Let μ be a Gaussian measure on a separable Banach space B, with $H \subset B$ as its Cameron-Martin space. We define the abstract Wiener space as the triple (H, B, μ). Given a complete orthonormal system $\{e_j\}_{j\in\mathbb{N}} \subset B^*$ of $H^* = H$ (here * stands for the dual space), we define a polynomial chaos of order k to be an element of the form $\prod_{j=1}^{\infty} H_{k_j}(\langle x, e_j\rangle)$, where $x \in B$, $k_j \neq 0$ for only finitely many j's, $k = \sum_{j=1}^{\infty} k_j$, H_{k_j} is the Hermite polynomial of degree k_j, and $\langle\cdot,\cdot\rangle = {}_B\langle\cdot,\cdot\rangle_{B^*}$ denotes the B–B^* duality pairing. The closure of polynomial chaoses of order k under $L^2(B,\mu)$ is then denoted by $\mathcal{H}_k$. The elements in $\mathcal{H}_k$ are called homogeneous Wiener chaoses of order k. We also set

$$\mathcal{H}_{\le k} := \bigoplus_{j=0}^{k} \mathcal{H}_j$$

for $k \in \mathbb{N}$. The Wiener chaos estimate can be stated as follows.

Lemma 2.5. *Let $k \in \mathbb{N}$. Then, we have*

$$\|X\|_{L^p(\mu)} \le (p-1)^{\frac{k}{2}} \|X\|_{L^2(\mu)}$$

for any $p \ge 2$ and any $X \in \mathcal{H}_{\le k}$.

The result above comes from the the hypercontractivity of the Ornstein-Uhlenbeck semigroup $U(t) = e^{tL}$ due to Nelson [21], see [34, Theorem I.22], where $L = \Delta - x\cdot\nabla$ is the Ornstein-Uhlenbeck operator. Here, to simplify the exposition, we just gave the definition of the Ornstein-Uhlenbeck operator L when B is the d-dimensional Euclidean space, see also [2] and [32, Section 3]. Let us also recall that any element in $\mathcal{H}_k$ is an eigenfunction of L with corresponding eigenvalue $-k$.

Next we report a lemma which will be crucial for the regularity of the stochastic objects in our analysis. We start by the following definition. Here we specialise to the spatial dimension $d = 2$, even if the same can be stated for any dimension. $B^s_{\infty,\infty} = B^s_{\infty,\infty}(\mathbb{T}^2)$ stands for the classical Besov space.

Definition 2.6. *Given a stochastic process $X : \mathbb{R}^+ \mapsto \mathcal{D}'(\mathbb{T}^2)$, we say it is spatially homogeneous if $\{X(\cdot,t)\}_{t\in\mathbb{R}^+}$ and $\{X(x_0+\cdot,t)\}_{t\in\mathbb{R}^+}$ have the same law for any $x_0 \in \mathbb{T}^2$.*

For an element $h \in \mathbb{R}$, we denote by δ_h the difference (in time) operator acting on a stochastic process $X(t)$, as

$$\delta_h X(t) = X(t+h) - X(t). \tag{2.7}$$

The regularity lemma is as follows.

Lemma 2.7. *Let $\{X_N(t)\}_N$ and $X(t)$ be spatially homogeneous stochastic processes. Suppose that there exists $k \in \mathbb{N}$ such that $X_N(t)$ and $X(t)$ belong to $\mathcal{H}_k$ for any $t \in \mathbb{R}^+$.*

(i) *Let $t \in \mathbb{R}^+$. Suppose that there exists $s_0 \in \mathbb{R}$ such that*

$$\mathbb{E}(|\widehat{X}(n,t)|^2) \lesssim \langle n \rangle^{-2-2s_0}$$

for any $n \in \mathbb{Z}^2$. Then $X(t) \in W^{s,\infty}$ almost surely for any $s < s_0$. In addition, if there exists $\gamma > 0$ such that

$$\mathbb{E}(|\widehat{X}_N(n,t) - \widehat{X}_M(n,t)|^2) \lesssim N^{-\gamma} \langle n \rangle^{-2-2s_0}$$

for any $n \in \mathbb{Z}^2$ and any $M \geq N \geq 1$, then $X_N(t)$ is a Cauchy sequence and converges to some stochastic process $X(t)$ in $W^{s,\infty}(\mathbb{T}^2) \cap B^s_{\infty,\infty}$ almost surely, for $s < s_0$.

(ii) *Suppose that the hypothesis of the previous statement hold true for any $t \in [0,T]$, $T > 0$. Suppose moreover that there exists $\sigma \in (0,1)$ such that*

$$\mathbb{E}(|\delta_h \widehat{X}(n,t)|^2) \lesssim \langle n \rangle^{-2-2s_0+\alpha\sigma} |h|^\sigma \tag{2.8}$$

for any $n \in \mathbb{Z}^2$, $t \in [0,T]$, and $h \in [-1,1]$. Then the stochastic process $X(t)$ enjoys a continuity regularity in time as well, namely $X \in C_t([0,T]; W^{s,\infty}_x \cap B^s_{\infty,\infty})$ for any $s < s_0 - \frac{\alpha\sigma}{2}$. If, in addition, there exists $\gamma > 0$ such that

$$\mathbb{E}(|\widehat{X}_N(n,t) - \widehat{X}_M(n,t)|^2) \lesssim N^{-\gamma} \langle n \rangle^{-2-2s_0+\alpha\sigma} |h|^\sigma$$

for any $n \in \mathbb{Z}^2$, $t \in [0,T]$, $h \in [-1,1]$, and $M \geq N \geq 1$, then $X_N(t)$ is a Cauchy sequence and converges to some stochastic process $X(t)$ in $C_t([0,T]; W^{s,\infty}_x \cap B^s_{\infty,\infty})$ almost surely, for $s < s_0 - \frac{\alpha\sigma}{2}$.

Lemma 2.7 is proved by straightforwardly employing the Wiener chaos estimate, i.e. Lemma 2.5 above. We refer to [19, Proposition 5] and [23, Appendix A] for the proof. We need the type of results in Lemma 2.7 above for a proof of Proposition 2.13. See also [13, 14, 22, 35] for the use of Lemma 2.7 in the context of other singular stochastic PDEs.

Remark 2.8. In Lemma 2.7 above, we specialize to the weakly dispersive NLW, which reflects the choice of some parameters appearing in the estimates. In particular, in the time regularity estimates, see (2.8), the product $\alpha\sigma$ in the power of decay and σ in the power of $|h|$ reflect the scaling of the (linear) fractional wave equation. We refer the reader to [30] for a generalization.

Before giving the main regularity results for stochastic objects that we will use for the proof of a local well-posedness theory in next sections, we introduce some notions, borrowing from [30].

Structure of the stochastic objects. For a family of stochastic processes $F_j : \mathbb{R}^+ \mapsto \mathcal{D}'(\mathbb{T}^2)$, $j = 1, \dots, J$ we suppose that there exists $k_j \in \mathbb{N}$ such that $F_j(t) \in \mathcal{H}_{k_j}$ for any $t \in \mathbb{R}^+$. We assume moreover that the Fourier coefficients of $F_j(t)$ are given by

$$\widehat{F}_j(n,t) = \sum_{\substack{n_1,\dots,n_{k_j} \in \mathbb{Z}^2 \\ n = n_1 + \dots + n_{k_j}}} \mathfrak{F}_j(n_1, \dots, n_{k_j}, t) \tag{2.9}$$

for any $n \in \mathbb{Z}^2$, and each $\mathfrak{F}_j(n_1, \dots, n_{k_j}, t)$ satisfies the condition

$$\mathbb{E}\Big[\mathfrak{F}_j(n_1, \dots, n_{k_j}, t_1)\overline{\mathfrak{F}}_j(m_1, \dots, m_{k_j}, t_2)\Big] = 0 \tag{2.10}$$

for any $t_1, t_2 \in \mathbb{R}^+$, when $\{n_1, \dots, n_{k_j}\} \neq \{m_1, \dots, m_{k_j}\}$, or equivalently when we do not have that $m_\ell = n_{\sigma(\ell)}$ for some $\sigma \in S_{k_j}$, the latter being the symmetric group of k_j elements.

Let $K_0 = 0$ and denote by K_j the sum

$$K_j = k_1 + \cdots + k_j \tag{2.11}$$

for $j = 1, \dots, J$. Let π_k denote the orthogonal projection onto the space $\mathcal{H}_k$, under the assumption that the covariance

$$\mathbb{E}\bigg[\pi_{K_J}\Big(\prod_{j=1}^{J}\mathfrak{F}_j(n_{K_{j-1}+1}, \dots, n_{K_j}, t_1)\Big)\pi_{K_J}\Big(\prod_{j=1}^{J}\overline{\mathfrak{F}}_j(m_{K_{j-1}+1}, \dots, m_{K_j}, t_2)\Big)\bigg] \tag{2.12}$$

does not vanish, namely (2.12) $\neq 0$, then we have (counting multiplicity) that $\{n_1, \dots, n_{K_J}\} = \{m_1, \dots, m_{K_J}\}$ and

$$\begin{aligned}(2.12) = \sum_{\sigma \in S_{K_J}} \mathbb{E}\bigg[&\pi_{K_J}\Big(\prod_{j=1}^{J}\mathfrak{F}_j(n_{K_{j-1}+1}, \dots, n_{K_j}, t_1)\Big) \\ &\times \pi_{K_J}\Big(\prod_{j=1}^{J}\overline{\mathfrak{F}}_j(n_{\sigma(K_{j-1}+1)}, \dots, n_{\sigma(K_j)}, t_2)\Big)\bigg]\end{aligned} \tag{2.13}$$

for any $t_1, t_2 \in \mathbb{R}^+$, with implicit constants not depending on $n_j \in \mathbb{Z}^2$ and $t_1, t_2 \in \mathbb{R}^+$.

Note that $F_j \in \mathcal{H}_{k_j}$ and that $\prod_{j=1}^{J} F_j \in \mathcal{H}_{\leq K_J}$. Subject to the assumptions (2.12) and (2.13), the highest order contribution belonging to $\mathcal{H}_{K_J}$ is estimated as in the following Proposition 2.9. See [30].

Proposition 2.9. *Let $F_j : \mathbb{R}^+ \mapsto \mathcal{D}'(\mathbb{T}^2)$, $j = 1, \dots, J$ be stochastic processes satisfying* (2.9) *and* (2.10)*. Suppose that, for each $j = 1, \dots, J$, the term $\mathfrak{F}_j(n_1, \dots, n_{k_j}, t)$ is symmetric in $n_1, \dots, n_{k_j}$ for any $t \in \mathbb{R}^+$ and satisfy* (2.13)*. Then, the following estimates hold*:

$$\mathbb{E}\bigg[\Big|\pi_{K_J}\mathcal{F}_x\Big(\prod_{j=1}^{J}F_j\Big)(n, t)\Big|^2\bigg] \lesssim \sum_{\substack{n_1, \dots, n_J \in \mathbb{Z}^2 \\ n = n_1 + \cdots + n_J}} \prod_{j=1}^{J}\mathbb{E}\Big[|\widehat{F}_j(n_j, t)|^2\Big] \tag{2.14}$$

and

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\delta_h \pi_{K_J}\mathcal{F}_x\Big(\prod_{j=1}^J F_j\Big)(n,t)\Big|^2\Big] \\
&\quad\lesssim \sum_{\ell=1}^J \sum_{\substack{n_1,\dots,n_J\in\mathbb{Z}^2\\ n=n_1+\cdots+n_J}} \mathbb{E}\Big[|\delta_h \widehat{F}_\ell(n_\ell,t)|^2\Big] \\
&\qquad\quad \times\Big(\prod_{j=1}^{\ell-1}\mathbb{E}\Big[|\widehat{F}_j(n_j,t+h)|^2\Big]\Big)\Big(\prod_{j=\ell+1}^{J}\mathbb{E}\Big[|\widehat{F}_j(n_j,t)|^2\Big]\Big)
\end{aligned} \tag{2.15}$$

for any $n\in\mathbb{Z}^2$*,* $t\in\mathbb{R}^+$*, and* $h\in\mathbb{R}$ *(with* $h\geq -t$ *such that* $t+h\geq 0$*), where* δ_h *is as in* (2.7)*.*

Proof (from [30]*).* For convenience, we first consider a fixed time $t\in\mathbb{R}^+$ thus omitting the the time dependence. We also omit to write that each frequency n_h or m_h (for some index h) appearing below belongs to $\mathbb{Z}^2$. For K_j defined as in (2.11), by using (2.9), (2.13), the Cauchy-Schwarz's inequality in ω, and the Cauchy's inequality in conjunction with the Pythagor's Theorem, the latter enabling us to remove π_{K_J}, we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\pi_{K_J}\mathcal{F}_x\Big(\prod_{j=1}^J F_j\Big)(n)\Big|^2\Big] \\
&\quad= \sum_{n=n_1+\cdots+n_{K_J}}\ \sum_{n=m_1+\cdots+m_{K_J}} \mathbb{E}\Big[\pi_{K_J}\Big(\prod_{j=1}^J \mathfrak{F}_j(n_{K_{j-1}+1},\dots,n_{K_j})\Big) \\
&\qquad\qquad \times \pi_{K_J}\Big(\prod_{j=1}^J \overline{\mathfrak{F}}_j(m_{K_{j-1}+1},\dots,m_{K_j})\Big)\Big] \\
&\quad= \sum_{\sigma\in S_{K_J}}\ \sum_{n=n_1+\cdots+n_{K_J}} \mathbb{E}\Big[\pi_{K_J}\Big(\prod_{j=1}^J \mathfrak{F}_j(n_{K_{j-1}+1},\dots,n_{K_j})\Big) \\
&\qquad\qquad \times \pi_{K_J}\Big(\prod_{j=1}^J \overline{\mathfrak{F}}_j(n_{\sigma(K_{j-1}+1)},\dots,n_{\sigma(K_j)})\Big)\Big] \\
&\quad\lesssim \sum_{n=n_1+\cdots+n_{K_J}} \mathbb{E}\Big[\Big|\prod_{j=1}^J \mathfrak{F}_j(n_{K_{j-1}+1},\dots,n_{K_j})\Big|^2\Big] \\
&\qquad\qquad + \sum_{\sigma\in S_{K_J}}\ \sum_{n=n_1+\cdots+n_{K_J}} \mathbb{E}\Big[\Big|\prod_{j=1}^J \mathfrak{F}_j(n_{\sigma(K_{j-1}+1)},\dots,n_{\sigma(K_j)})\Big|^2\Big] \\
&\quad\lesssim \sum_{n=n_1+\cdots+n_{K_J}} \mathbb{E}\Big[\prod_{j=1}^J |\mathfrak{F}_j(n_{K_{j-1}+1},\dots,n_{K_j})|^2\Big]
\end{aligned} \tag{2.16}$$

for any $n \in \mathbb{Z}^2$. Note that the last step follows by noting that the action of the permutation σ simply amounts to relabelling the indices. From (2.16), Hölder's inequality, Minkowski's integral inequality, and the Wiener chaos estimate (Lemma 2.5), we continue the estimate as

$$\begin{aligned}
\text{RHS of (2.16)} &= \sum_{n=m_1+\cdots+m_J} \mathbb{E}\Big[\prod_{j=1}^{J} \big\| \mathbf{1}_{m_j = n_{K_{j-1}+1}+\cdots+n_{K_j}} \\
&\qquad\qquad \times \mathfrak{F}_j(n_{K_{j-1}+1}, \dots, n_{K_j})\big\|^2_{\ell^2_{n_{K_{j-1}+1},\dots,n_{K_j}}}\Big] \\
&\le \sum_{n=m_1+\cdots+m_J} \prod_{j=1}^{J} \Big\| \big\| \mathbf{1}_{m_j = n_{K_{j-1}+1}+\cdots+n_{K_j}} \\
&\qquad\qquad \times \mathfrak{F}_j(n_{K_{j-1}+1}, \dots, n_{K_j})\big\|_{L^{2J}(\Omega)} \Big\|^2_{\ell^2_{n_{K_{j-1}+1},\dots,n_{K_j}}} \\
&\lesssim \sum_{n=m_1+\cdots+m_J} \Big(\prod_{j=1}^{J} \sum_{m_j = n_{K_{j-1}+1}+\cdots+n_{K_j}} \mathbb{E}\Big[|\mathfrak{F}_j(n_{K_{j-1}+1}, \dots, n_{K_j})|^2\Big]\Big).
\end{aligned} \tag{2.17}$$

In a symmetric framework as in the hypothesis of the Proposition, condition (2.10) reduces to

$$\mathbb{E}\Big[\widehat{F}_j(n,t_1)\overline{\widehat{F}_j(n,t_2)}\Big] \sim \sum_{n=n_1+\cdots+n_{k_j}} \mathbb{E}\Big[\mathfrak{F}_j(n_1,\dots,n_{k_j},t_1)\overline{\mathfrak{F}}(n_1,\dots,n_{k_j},t_2)\Big]. \tag{2.18}$$

Hence, by (2.18) with $t_1 = t_2 = t$, we obtain (2.14). Similarly to the computations leading to (2.16), with (2.7) we get

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\delta_h \pi_{K_J} \mathcal{F}_x\Big(\prod_{j=1}^{J} F_j\Big)(n,t)\Big|^2\Big] \\
&\quad \lesssim \sum_{n=n_1+\cdots+n_{K_J}} \mathbb{E}\Big[\Big|\delta_h\Big(\prod_{j=1}^{J}\mathfrak{F}_j(n_{K_{j-1}+1},\dots,n_{K_j},t)\Big)\Big|^2\Big] \\
&\quad \lesssim \sum_{\ell=1}^{J} \sum_{n=n_1+\cdots+n_{K_J}} \mathbb{E}\Big[\Big|\delta_h \mathfrak{F}_\ell(n_{K_{\ell-1}+1},\dots,n_{K_\ell},t) \\
&\qquad \times \Big(\prod_{j=1}^{\ell-1}\mathfrak{F}_j(n_{K_{j-1}+1},\dots,n_{K_j},t+h)\Big)\Big(\prod_{j=\ell+1}^{J}\mathfrak{F}_j(n_{K_{j-1}+1},\dots,n_{K_j},t)\Big)\Big|^2\Big].
\end{aligned} \tag{2.19}$$

By proceeding as in (2.17) to control the terms on the right-hand side of (2.19), we eventually obtain (2.15). □

Remark 2.10. The hypothesis above on the structure of the stochastic object are used to estimate the highest order contribution (belonging to $\mathcal{H}_{K_J} = \mathcal{H}_{k_1+\cdots k_J}$) of the product of J homogeneous stochastic objects. If one aims to study the regularity for a general product of stochastic objects, one needs to assume further structural hypothesis on the coefficients $\mathfrak{F}_j$. We refer the reader to the paper [30] for a general treatment.

Remark 2.11. The structural hypotheses (2.9), (2.10), and (2.13) given above are sufficient to control the contribution of the highest order Wiener chaos term. To control the lower order Wiener chaos terms, some further assumptions for a general product of stochastic objects must be introduced. We refer to [30]. In the latter paper, Oh and Zine show the desired regularity properties of general products of stochastic objects as above avoiding the use of multiple integral stochastic representation (see for example [29, Appendix B] for a treatment using multiple integral stochastic representation).
In this paper, for reader's convenience, we show the analysis regarding the highest order contribution. We refer the readers to [30] for a proof in a whole generality, both for highest and lower order Wiener chaoses.

Remark 2.12. The symmetry assumption in Proposition 2.9 is not restrictive. Indeed, if this is not the case, we can lie in this setting using a symmetrization argument. Given F_j in (2.9),

$$\begin{aligned}\widehat{F}_j(n,t) &= \sum_{n=n_1+\cdots+n_{k_j}} \mathfrak{F}_j(n_1,\ldots,n_{k_j},t) = \frac{1}{k_j!}\sum_{\sigma\in S_{k_j}}\sum_{n=n_1+\cdots+n_{k_j}} \mathfrak{F}_j(n_{\sigma(1)},\ldots,n_{\sigma(k_j)},t)\\ &= \sum_{n=n_1+\cdots+n_{k_j}} \texttt{Sym}(\mathfrak{F}_j)(n_1,\ldots,n_{k_j},t),\end{aligned}$$

where $\texttt{Sym}(\mathfrak{F}_j)$ is the symmetrization of $\mathfrak{F}_j$ defined by

$$\texttt{Sym}(\mathfrak{F}_j)(n_1,\ldots,n_k,t) = \frac{1}{k_j!}\sum_{\sigma\in S_{k_j}} \mathfrak{F}_j(n_{\sigma(1)},\ldots,n_{\sigma(k_j)},t).$$

Consequently, we can assume, without loss of generality, that $\mathfrak{F}_j$ is symmetric.

Next, we give the regularity results needed for the local Cauchy theory. To this aim, we recall that we may write the solution to the linear fractional wave equation with initial data $(g^\omega_{0,n}, g^\omega_{1,n})$ by (1.27), and we moreover denote, as in the introduction, z_N its projection onto Fourier frequencies $\{|n|\le N\}$. We have the following (see [30] for a generalization). See (1.23) for the definition of $\mathcal{D}$.

Proposition 2.13. *Let $\alpha > 0$ and $k\in\mathbb{N}$.*
(i) *Given $\ell\in\mathbb{N}$, let $\alpha > 1-\frac{1}{\ell}$. Then, for $s<\ell(\alpha-1)$, $\{:\!z_N^\ell\!:\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C_t(\mathbb{R}^+;W^{s,\infty}_x(\mathbb{T}^2))$, almost surely. In particular, denoting the limit by $:\!z^\ell\!:$, we have*

$$:\!z^\ell\!:\,\in C_t(\mathbb{R}^+;W_x^{-\ell(1-\alpha)-\varepsilon,\infty}(\mathbb{T}^2))$$

for any $\varepsilon>0$, almost surely.

(ii) *Assume $\alpha > 1-\frac{1}{k}$. Then, for $s < k(\alpha-1)+\alpha$, $\{\mathcal{D}(:z_N^k:)\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C_t(\mathbb{R}^+;W_x^{s,\infty}(\mathbb{T}^2))$, almost surely. In particular, denoting the limit by $\mathcal{D}(:z^k:)$, we have*

$$\mathcal{D}(:z^k:) \in C_t(\mathbb{R}^+;W_x^{-k(1-\alpha)+\alpha-\varepsilon,\infty}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

Furthermore, suppose that $\alpha > 1-\frac{1}{k+1}$. Let $k_2 \in \mathbb{N}$. Then, for $s < k(\alpha-1)+\alpha$, $\big\{\big(\mathcal{D}(:z_N^k:)\big)^{k_2}\big\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C_t(\mathbb{R}^+;W_x^{s,\infty}(\mathbb{T}^2))$, almost surely. In particular, denoting the limit by $\big(\mathcal{D}(:z^k:)\big)^{k_2}$, we have

$$\big(\mathcal{D}(:z^k:)\big)^{k_2} \in C_t(\mathbb{R}^+;W_x^{-k(1-\alpha)+\alpha-\varepsilon,\infty}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

(iii) *Fix integers $0 \le k_1 \le k-1$ and $0 \le k_2 \le k$. Assume that*

$$\alpha > 1 - \frac{1}{2k+1}. \tag{2.20}$$

Given $N \in \mathbb{N}$, define Y_N by

$$Y_N = :z_N^{k_1}: \big(\mathcal{D}(:z_N^k:)\big)^{k_2}.$$

Then, for $s < -k_1(1-\alpha)$, $\{Y_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C_t(\mathbb{R}^+;W_x^{s,\infty}(\mathbb{T}^2))$, almost surely. In particular, denoting its limit by $Y = :z^{k_1}: \big(\mathcal{D}(:z^k:)\big)^{k_2}$, we have

$$Y =:z^{k_1}: \big(\mathcal{D}(:z^k:)\big)^{k_2} \in C_t(\mathbb{R}^+;W_x^{-k_1(1-\alpha)-\varepsilon,\infty}(\mathbb{T}^2))$$

for any $\varepsilon > 0$, almost surely.

Remark 2.14. Let us note that the condition (2.20) implies the conditions in (i) (namely $\alpha > 1-\frac{1}{k}$) and (ii) (namely $\alpha > 1-\frac{1}{k+1}$).

Proof of Proposition 2.13. (i) Recall that $:z_N^\ell: = \pi_\ell(z_N^\ell)$. Proposition 2.9 gives the estimate

$$\begin{aligned}\mathbb{E}\Big[|\widehat{:z_N^\ell:}(n,t)|^2\Big] &\lesssim \sum_{\substack{n=n_1+\cdots+n_\ell\\ |n_j|\le N}} \prod_{j=1}^{\ell} \mathbb{E}\Big[|\widehat{z}_N(n_j,t)|^2\Big]\\ &\lesssim \sum_{\substack{n=n_1+\cdots+n_\ell\\ |n_j|\le N}} \prod_{j=1}^{\ell} \frac{1}{\langle n_j\rangle^{2\alpha}},\end{aligned}$$

and iterating the summation by means of Lemma 2.3, we get

$$\mathbb{E}\Big[|\widehat{:z_N^\ell:}(n,t)|^2\Big] \lesssim \langle n\rangle^{2(\ell-1)-2\ell\alpha} = \langle n\rangle^{-2+2\ell(1-\alpha)}, \tag{2.21}$$

as $1-\frac{1}{\ell} < \alpha < 1$. With slight changes, we can easily obtain the estimate for the difference at two truncations

$$\mathbb{E}\Big[|\widehat{:z_N^\ell:}(n,t) - \widehat{:z_M^\ell:}(n,t)|^2\Big] \lesssim N^{-\gamma}\langle n\rangle^{-2+2\ell(1-\alpha)},$$

where $\gamma > 0$ is a small parameter and $M \geq N \geq 1$ are arbitrary integers. Hence, from Lemma 2.7 we can infer the convergence, almost surely, of $: z_N^\ell(t):$ to some limit in $W^{s,\infty}(\mathbb{T}^2)$ for $s < -\ell(1-\alpha)$. The convergence, almost surely, of $: z_N^\ell :$ to some limit in $C_t(\mathbb{R}^+; W_x^{s,\infty}(\mathbb{T}^2))$, for $s < -\ell(1-\alpha)$ is done by invoking the mean value theorem and analogous computations yielding the difference estimates in Lemma 2.7 from which we get the desired result.

(ii) We prove here a uniform bound in N of the stochastic objects for a fixed time $t \in \mathbb{R}^+$, the time continuity property and the convergence claims then follow by minor modifications. Note that the constants appearing in estimating the Duhamel operator $\mathcal{D}$ may depend on time, though this dependence does not play any significant role, thus we do not track it in the constants. In what follows, we omit the N dependence. Similar comments apply for the subsequent point (iii).

By setting $\ell = k$, and by using the gain of α-derivative via the Duhamel operator $\mathcal{D}$, the first estimate straightforwardly follows by point (i). Specifically, provided that $1 - \frac{1}{k} < \alpha < 1$, we get from (2.21)

$$\mathbb{E}\Big[|\widehat{\mathcal{D}(: z^k :)}(n,t)|^2\Big] \lesssim \langle n\rangle^{-2-2(\alpha - k(1-\alpha))}. \tag{2.22}$$

From Lemma 2.7, we obtain $\mathcal{D}(: z^k :)(t) \in W^{s,\infty}(\mathbb{T}^2)$ for any $s < -k(1-\alpha) + \alpha$, almost surely.
At this point suppose that $\alpha - k(1-\alpha) > 0$. Then, there exist $s > 0$ and $\varepsilon > 0$ with $0 < s < s + \varepsilon < \alpha - k(1-\alpha)$ such that $\mathcal{D}(: z^k :)(t) \in W^{s+\varepsilon,\infty}(\mathbb{T}^2)$, almost surely. The Sobolev's inequality, with $\varepsilon r > 2$, jointly with the estimate in Lemma 2.1 (i), yield the bound

$$\big\| \big(\mathcal{D}(: z^k :)\big)^{k_2}(t) \big\|_{W^{s,\infty}} \lesssim \big\| \big(\mathcal{D}(: z^k :)\big)^{k_2}(t) \big\|_{W^{s+\varepsilon,r}} \lesssim \| \mathcal{D}(: z^k :)(t) \|_{W^{s+\varepsilon,\infty}}^{k_2} < \infty,$$

almost surely.

(iii) We conclude with the proof of the last point. By (1.10) and (1.27), we get $: z^{k_1}(t) : \in \mathcal{H}_{k_1}$ and $\mathcal{D}(: z^k :)(t) \in \mathcal{H}_k$. Hence, the stochastic object defined by $Y(t) := \, : z^{k_1} : \big(\mathcal{D}(: z^k :)\big)^{k_2}(t) \in \mathcal{H}_{\leq k_1 + kk_2}$. To control the highest order contribution of $Y(t)$ belonging to $\mathcal{H}_{k_1+kk_2}$, we proceed in the following manner. By Proposition 2.9 together with (2.21), (2.22), and Lemma 2.3, we have

$$\begin{aligned}
\mathbb{E}\Big[|\pi_{k_1+kk_2}\widehat{Y}(n,t)|^2\Big] &\lesssim \sum_{n = n_1 + \cdots + n_{k_2+1}} \mathbb{E}\Big[|\widehat{: z^{k_1} :}(n_1,t)|^2\Big] \prod_{j=2}^{k_2+1} \mathbb{E}\Big[|\widehat{\mathcal{D}(: z^k :)}(n_j,t)|^2\Big] \\
&\lesssim \sum_{n = n_1 + \cdots + n_{k_2+1}} \langle n_1\rangle^{-2+2k_1(1-\alpha)} \prod_{j=2}^{k_2+1} \langle n_j\rangle^{-2-2(\alpha - k(1-\alpha))} \\
&\lesssim \langle n\rangle^{-2+2k_1(1-\alpha)+\varepsilon_1}
\end{aligned}$$

for any small $\varepsilon_1 > 0$, provided that $\alpha > 1 - \frac{1}{k}$ and $\alpha - k(1-\alpha) > 0$. Lemma 2.7 then implies $\pi_{k_1+kk_2}Y(t) \in W^{s+\varepsilon,\infty}(\mathbb{T}^2)$ for any $s < -k_1(1-\alpha)$.

□

3. Gibbs measure construction

In this section we construct the Gibbs measure associated to (1.29). We first prove Proposition 1.1, then we prove exponential integrability claimed in Theorem 1.2 by means of the variational approach of Barashkov and Gubinelli, see [1].

3.1. **Convergence of random variables.** We first prove the convergence of the random variables $G_N(u) = \int_{\mathbb{T}^2} :(\mathbf{P}_N u)^{2m+2}: dx$ as in Proposition 1.1. Let us note that it is enough to prove the result for $p = 2$, the general p follows by using the Wiener chaos estimate, see Lemma 2.5. Hence, we prove (1.19) for $p = 2$.

With this aim in mind, we introduce some notations, and some preliminary results. Let $\sigma_{\alpha,N}$ be as in (1.9). For a fixed $x \in \mathbb{T}^2$ and $N \in \mathbb{N}$, we also define

$$\eta_{\alpha,N}(x)(\cdot) := \frac{1}{\sigma_{\alpha,N}^{\frac{1}{2}}} \sum_{|n| \le N} \frac{\overline{e_n(x)}}{\langle n \rangle^{\alpha}} e_n(\cdot) \qquad \text{and} \qquad \gamma_{\alpha,N}(\cdot) := \sum_{|n|\le N} \frac{1}{\langle n \rangle^{2\alpha}} e_n(\cdot), \tag{3.1}$$

where $e_n(y) = e^{in\cdot y}$. Note that $\eta_{\alpha,N}(x)(\cdot)$ is real-valued with unitary $L^2(\mathbb{T}^2)$-norm, namely

$$\|\eta_{\alpha,N}(x)\|_{L^2} = 1$$

for all $x \in \mathbb{T}^2$ and all $N \in \mathbb{N}$. Moreover, we have

$$\langle \eta_{\alpha,M}(x), \eta_{\alpha,N}(y) \rangle_{L^2(\mathbb{T}^2)} = \frac{1}{\sigma_{\alpha,M}^{\frac{1}{2}} \sigma_{\alpha,N}^{\frac{1}{2}}} \gamma_{\alpha,N}(y - x) = \frac{1}{\sigma_{\alpha,M}^{\frac{1}{2}} \sigma_{\alpha,N}^{\frac{1}{2}}} \gamma_{\alpha,N}(x - y), \tag{3.2}$$

for fixed $x, y \in \mathbb{T}^2$ and $M \ge N$.

Furthermore, we introduce the white noise functional. Let $\xi(x;\omega)$ be the (real-valued) mean-zero Gaussian white noise on $\mathbb{T}^2$ defined by

$$\xi(x;\omega) = \sum_{n \in \mathbb{Z}^2} g_n(\omega) e^{in\cdot x},$$

where $\{g_n\}_{n\in\mathbb{Z}^2}$ is a sequence of independent standard complex-valued Gaussian random variables conditioned that $g_{-n} = \overline{g}_n$, $n \in \mathbb{Z}^2$. It is easy to see that $\xi \in \mathcal{H}^s(\mathbb{T}^2) \setminus \mathcal{H}^{-1}(\mathbb{T}^2)$, $s < -1$, almost surely. In particular, ξ is a distribution, acting on smooth functions. In fact, the action of ξ can be defined on $L^2(\mathbb{T}^2)$.

We define the white noise functional $W_{(\cdot)} : L^2(\mathbb{T}^2) \to L^2(\Omega)$ by

$$W_f(\omega) = \langle f, \xi(\omega) \rangle_{L^2} = \sum_{n\in\mathbb{Z}^2} \widehat{f}(n) \overline{g}_n(\omega) \tag{3.3}$$

for a real-valued function $f \in L^2(\mathbb{T}^2)$. Note that $W_f = \xi(f)$ is basically the Wiener integral of f. In particular, W_f is a real-valued Gaussian random variable with mean 0 and variance $\|f\|_{L^2}^2$. Moreover, $W_{(\cdot)}$ is unitary:

$$\mathbb{E}\big[W_f W_h\big] = \langle f, h \rangle_{L^2_x} \tag{3.4}$$

for $f,h\in L^2(\mathbb{T}^2)$. More generally, we have the following, see [28, Lemma 2.1].

Lemma 3.1. *Let $f,h\in L^2(\mathbb{T}^2)$ such that $\|f\|_{L^2}=\|h\|_{L^2}=1$. Then, for $k,m\in\mathbb{N}\cup\{0\}$, we have*

$$\mathbb{E}\big[H_k(W_f)H_m(W_h)\big]=\delta_{km}k![\langle f,h\rangle_{L^2(\mathbb{T}^2)}]^k.$$

Here, δ_{km} denotes the Kronecker's delta function.

For the sake of conciseness, let us denote $u_N=\mathbf{P}_N u$. By using the definitions, we have

$$u_N(x)=\sigma_{\alpha,N}^{1/2}\frac{u_N(x)}{\sigma_{\alpha,N}^{1/2}}=\sigma_{\alpha,N}^{1/2}W_{\eta_{\alpha,N}(x)}, \tag{3.5}$$

and by using the the definition of the Wick ordering and the property (1.13), we easily get from (3.5) that

$$:u_N^{2m+2}(x):=H_{2m+2}(\sigma_{\alpha,N}^{1/2}W_{\eta_{\alpha,N}(x)},\sigma_{\alpha,N})=\sigma_{\alpha,N}^{m+1}H_{2m+2}(W_{\eta_{\alpha,N}(x)}) \tag{3.6}$$

We are in position to prove Proposition 1.1 for $p=2$.

Proof of Proposition 1.1. By definition of (1.17), we write, by expanding the square,

$$\begin{aligned}\|G_M(u)-G_N(u)\|_{L^2(\mu_\alpha)}^2&=\frac{1}{(2m+2)^2}\int_\Omega\Big(\int_{\mathbb{T}^2}:u_M^{2m+2}:\,dx\Big)^2dP(\omega)\\&-\frac{2}{(2m+2)^2}\int_\Omega\iint_{\mathbb{T}^2\times\mathbb{T}^2}:u_N^{2m+2}(x)::u_M^{2m+2}(y):\,dx\,dy\,dP(\omega)\\&+\frac{1}{(2m+2)^2}\int_\Omega\Big(\int_{\mathbb{T}^2}:u_N^{2m+2}:\,dx\Big)^2dP(\omega).\end{aligned} \tag{3.7}$$

At this point we write $(\int_{\mathbb{T}^2}\cdot\,dx)^2=(\int_{\mathbb{T}^2}\cdot\,dx)(\int_{\mathbb{T}^2}\cdot\,dy)$, we rewrite $:u_N^{2m+2}:$ using (3.6), and by means of Lemma 3.1 we obtain

$$\begin{aligned}(3.7)&=\frac{\sigma_{\alpha,M}^{2m+2}}{(2m+2)^2}\int_\Omega\iint_{\mathbb{T}^2\times\mathbb{T}^2}H_{2m+2}(W_{\eta_{\alpha,M}(x)})H_{2m+2}(W_{\eta_{\alpha,M}(y)})\,dx\,dy\,dP(\omega)\\&-\frac{2\sigma_{\alpha,M}^{m+1}\sigma_{\alpha,N}^{m+1}}{(2m+2)^2}\int_\Omega\iint_{\mathbb{T}^2\times\mathbb{T}^2}H_{2m+2}(W_{\eta_{\alpha,M}(x)})H_{2m+2}(W_{\eta_{\alpha,N}(y)})\,dx\,dy\,dP(\omega)\\&+\frac{\sigma_{\alpha,N}^{2m+2}}{(2m+2)^2}\int_\Omega\iint_{\mathbb{T}^2\times\mathbb{T}^2}H_{2m+2}(W_{\eta_{\alpha,N}(x)})H_{2m+2}(W_{\eta_{\alpha,N}(y)})\,dx\,dy\,dP(\omega)\\&=\frac{(2m+2)!\sigma_{\alpha,M}^{2m+2}}{(2m+2)^2}\iint_{\mathbb{T}^2\times\mathbb{T}^2}\langle\eta_{\alpha,M}(x),\eta_{\alpha,M}(y)\rangle^{2m+2}\,dx\,dy\\&-\frac{(2m+2)!\sigma_{\alpha,N}^{2m+2}}{(2m+2)^2}\iint_{\mathbb{T}^2\times\mathbb{T}^2}\langle\eta_{\alpha,N}(x),\eta_{\alpha,N}(y)\rangle^{2m+2}\,dx\,dy\\&=\frac{(2m+2)!}{(2m+2)^2}\iint_{\mathbb{T}^2\times\mathbb{T}^2}\gamma_{\alpha,M}^{2m+2}(x-y)-\gamma_{\alpha,N}^{2m+2}(x-y)\,dx\,dy.\end{aligned}$$

By using the finiteness of the measure of the toroidal domain, the bound $||a|^\beta - |b|^\beta| \leq c(\beta)|a-b|(|a|^{\beta-1} + |b|^{\beta-1})$ for $\beta > 1$, and the Hölder's inequality with conjugate exponents $p = 2m+2$ and $p' = \frac{2m+2}{2m+1}$, we estimate

$$\begin{aligned}
&\iint_{\mathbb{T}^2\times\mathbb{T}^2} \gamma_{\alpha,M}^{2m+2}(x-y) - \gamma_{\alpha,N}^{2m+2}(x-y)\, dx\, dy = c\int_{\mathbb{T}^2} \gamma_{\alpha,M}^{2m+2}(x) - \gamma_{\alpha,N}^{2m+2}(x)\, dx \\
&\leq \tilde{c}(m)\int_{\mathbb{T}^2} (|\gamma_{\alpha,M}(x)|^{2m+1} + |\gamma_{\alpha,N}(x)|^{2m+1})(|\gamma_{\alpha,M}(x) - \gamma_{\alpha,N}(x)|)\, dx \\
&\leq \tilde{c}(m)\|\gamma_{\alpha,M} - \gamma_{\alpha,N}\|_{L^{2m+2}} \left(\|\gamma_{\alpha,M}\|_{L^{2m+2}}^{2m+1} + \|\gamma_{\alpha,N}\|_{L^{2m+2}}^{2m+1}\right). \qquad (3.8)
\end{aligned}$$

By using the Hausdorff-Young's inequality to control the L^{2m+2}-norms, and the Sobolev's embedding $H^{m/(m+1)} \hookrightarrow L^{2m+2}$ to estimate the difference norm, we obtain

$$\begin{aligned}
(3.8) &\leq 2\tilde{c}(m)\left(\sum_{|n|\geq N} \frac{1}{\langle n\rangle^{4\alpha - 2m/(m+1)}}\right)^{1/2} \left(\sum_{n\in\mathbb{Z}^2} \frac{1}{\langle n\rangle^{\frac{4\alpha(m+1)}{2m+1}}}\right)^{(2m+1)^2/(2m+2)} \\
&\lesssim N^{1-2\alpha+\frac{m}{m+1}},
\end{aligned}$$

and the last bound goes to zero provided that

$$\alpha > 1 - \frac{1}{2m+2} := \alpha(m), \qquad (3.9)$$

indeed $\sum \langle n\rangle^{-\frac{4\alpha(m+1)}{2m+1}} < \infty$ for $\alpha > 1 - \frac{1}{2m+2}$. By summarizing the above calculations, we get that under the condition (3.9) on α,

$$\|G_M(u) - G_N(u)\|_{L^2(\mu_\alpha)}^2 \lesssim \tilde{c}(m)N^{1-2\alpha+\frac{m}{m+1}},$$

which in turn easily implies (1.19) when $p = 2$. □

Remark 3.2. Let us emphasize that the key point in Lemma 2.5 is that the constant appearing in estimate does not depend on $d \in \mathbb{N}$. The proof of the Lemma 2.5 is a consequence of the hypercontractivity of the Ornstein-Uhlenbeck semigroup due to Nelson [21].

3.2. **Exponential integrability.** This section is devoted to the proof of Theorem 1.2, and in order to show it we will use a variational approach to Quantum Field Theory introduced in [1]. See see also [16, 17, 26, 35] for recent results using this approach.

Let us begin by introducing some notations. Let $W(t)$ be a cylindrical Brownian motion in $L^2(\mathbb{T}^2)$. Namely, we have

$$W(t) = \sum_{n\in\mathbb{Z}^2} B_n(t)e_n, \qquad (3.10)$$

where $\{B_n\}_{n\in\mathbb{Z}^2}$ is a sequence of mutually independent complex-valued Brownian motions such that $B_{-n} = \overline{B}_n$, $n \in \mathbb{Z}^2$. Here, by convention, we normalize B_n such that

$\mathrm{Var}(B_n(t)) = t$. In particular, B_0 is a standard real-valued Brownian motion. Then, we define a centered Gaussian process $Y(t)$ by

$$Y(t) = \langle \nabla \rangle^{-\alpha} W(t). \tag{3.11}$$

Note that we have $\mathrm{Law}(Y(1)) = \mu_\alpha$, where μ_α is the Gaussian measure in (1.7). By setting $Y_N = \mathbf{P}_N Y$, we have $\mathrm{Law}(Y_N(1)) = (\mathbf{P}_N)_* \mu_\alpha$, i.e. the pushforward of μ_α under $\mathbf{P}_N$. In particular, we have $\mathbb{E}[Y_N^2(1)] = \sigma_{\alpha,N}$, where $\sigma_{\alpha,N}$ is as in (1.9).

Next, let $\mathbb{H}_a$ denote the space of drifts, which are progressively measurable processes belonging to $L^2([0,1]; L^2(\mathbb{T}^2))$, P-almost surely. We now state the Boué-Dupuis variational formula [4, 36]. Specifically, we refer to [36, Theorem 7].

Lemma 3.3. *Let Y be as in* (3.11). *Fix $N \in \mathbb{N}$. Suppose that $F : C^\infty(\mathbb{T}^2) \to \mathbb{R}$ is measurable and such that $\mathbb{E}\big[|F(\mathbf{P}_N Y(1))|^p\big] < \infty$ and $\mathbb{E}\big[|e^{-F(\mathbf{P}_N Y(1))}|^q\big] < \infty$ for some $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have*

$$-\log \mathbb{E}\Big[e^{-F(\mathbf{P}_N Y(1))}\Big] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\Big[F(\mathbf{P}_N Y(1) + \mathbf{P}_N I(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 \, dt\Big], \tag{3.12}$$

where $I(\theta)$ is defined by

$$I(\theta)(t) = \int_0^t \langle \nabla \rangle^{-\alpha} \theta(t') \, dt'$$

and the expectation $\mathbb{E} = \mathbb{E}_P$ is an expectation with respect to the underlying probability measure P.

In the following, we construct a drift θ depending on Y, and the Boué-Dupuis variational formula (Lemma 3.3) is suitable for this purpose since an expectation in (3.12) is taken with respect to the underlying probability P.

Before proceeding to the proof of Theorem 1.2, we state a lemma on the path-wise regularity bounds of $Y(1)$ and $I(\theta)(1)$.

Lemma 3.4. (i) *Let $h \in \mathbb{N}$ and $\varepsilon > 0$. Then, given any finite $p \geq 1$, we have*

$$\mathbb{E}\Big[\| :Y_N(1)^h: \|_{W^{-h(1-\alpha)-\varepsilon,\infty}}^p\Big] \leq C_{\varepsilon,p,h,\alpha} < \infty,$$

uniformly in $N \in \mathbb{N}$.

(ii) *For any $\theta \in \mathbb{H}_a$, we have*

$$\|I(\theta)(1)\|_{H^\alpha}^2 \leq \int_0^1 \|\theta(t)\|_{L^2_x}^2 \, dt. \tag{3.13}$$

Proof. (i) It is a straightforward adaptation of [35, Lemma B3]. Recall that the law of $Y_N(1)$ is $\sum_{|n| \leq N} \frac{g_{1,n}^\omega}{\langle n \rangle^\alpha} e^{in \cdot x}$. Note that $:Y_N(1)^h:$ belongs the Wiener chaos of order h. It suffices to consider the case $p = 2$. Direct computations yield

$$\mathbb{E}\Big[|\widehat{:Y_N(1)^h:}(n)|^2\Big] = h! \sum_{n_1+\cdots+n_h=n} \prod_{i=1}^{h} \mathbb{E}\Big[|\widehat{Y_N(1)}(n_i)|^2\Big]$$
$$\lesssim \sum_{n_1+\cdots+n_h=n} \prod_{i=1}^{h} \frac{1}{\langle n_i\rangle^{2\alpha}} \lesssim \langle n\rangle^{-2+2h(1-\alpha)},$$

where we iteratively used Lemma 2.3 under the assumption $\alpha > 1 - \frac{1}{h}$. Lemma 2.7 gives the desired result.

(ii) The second point is a simple application of the definition of the Sobolev Space H^α, the Minkowski's inequality, and the Cauchy-Schwarz's inequality:

$$\|I(\theta)(1)\|_{H^\alpha} = \Big\| \int_0^1 \theta(t)\, dt \Big\|_{L^2} \leq \int_0^1 \|\theta(t)\|_{L^2}\, dt \leq \Big(\int_0^1 \|\theta(t)\|_{L^2}^2\, dt\Big)^{1/2},$$

then the result follows by squaring both sides. □

3.3. **Proof of Theorem 1.2.** We give now the proof of Theorem 1.2. We prove the uniform exponential integrability via the variational formulation, i.e. by means of Lemma 3.3. Since the argument is identical for any finite $p \geq 1$, we only present details for the case $p = 1$.

In view of the Boué-Dupuis formula (Lemma 3.3), it suffices to establish a lower bound on

$$\mathcal{W}_N(\theta) = \mathbb{E}\Big[G_N(Y(1) + I(\theta)(1)) + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2\, dt\Big], \tag{3.14}$$

uniformly in $N \in \mathbb{N}$ and $\theta \in \mathbb{H}_a$. As usual, we set $Y_N = \mathbf{P}_N Y = \mathbf{P}_N Y(1)$ and $\Theta_N = \mathbf{P}_N\Theta = \mathbf{P}_N I(\theta)(1)$. From (1.17) and (1.14), we have

$$G_N(Y_N + \Theta_N) = \sum_{\ell=0}^{2m+2} \binom{2m+2}{\ell} \int_{\mathbb{T}^2} :Y_N^{2m+2-\ell}:\, \Theta_N^\ell\, dx \tag{3.15}$$

Hence, from (3.14) and (3.15), we have

$$\mathcal{W}_N(\theta) = \mathbb{E}\Big[\sum_{\ell=0}^{2m+2} \binom{2m+2}{\ell} \int_{\mathbb{T}^2} :Y_N^{2m+2-\ell}:\, \Theta_N^\ell\, dx + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L_x^2}^2\, dt\Big]. \tag{3.16}$$

In the following, we first state a lemma, controlling the terms appearing in (3.16).

Remark 3.5. For $\ell = 2m+2$, the term $\int_{\mathbb{T}^2} \Theta_N^{2m+2} = \|\Theta_N\|_{L^{2m+2}}^{2m+2} \geq 0$ is positive and well defined, for $\|\Theta_N\|_{L^{2m+2}}^{2m+2} \lesssim \|\Theta_N\|_{H^\alpha}^{2m+2} \lesssim \Big(\int_0^1 \|\theta(t)\|_{L^2}^2 dt\Big)^{\frac{m+2}{2}}$ for any $\alpha \geq 1 - \frac{1}{m+1}$ (see Proposition 1.1, where we have the condition $\alpha > 1 - \frac{1}{2m+2}$).

3.3.1. *The cubic case.* For the sake of clarity, we start with the proof of the cubic nonlinearity (the general case will be done by using an induction argument). Therefore, let us fix $m = 1$, hence $2m + 2 = 4$ and $\ell \in \{0, 1, 2, 3, 4\}$.

Lemma 3.6. *There exists a small $\varepsilon > 0$ such that, for any $\eta \ll 1$ there exists a constant $c = c(\eta) > 0$ such that*

$$\left| \int_{\mathbb{T}^2} :Y_N^3: \Theta_N \, dx \right| \leq c \| :Y_N^3: \|^2_{W^{-3(1-\alpha)-\varepsilon,\infty}} + \eta \|\Theta_N\|^2_{H^\alpha}, \tag{3.17}$$

$$\left| \int_{\mathbb{T}^2} :Y_N^2: \Theta_N^2 \, dx \right| \leq c \| :Y_N^2: \|^\rho_{W^{-2(1-\alpha)-\varepsilon,\infty}} + \eta \left(\|\Theta_N\|^2_{H^\alpha} + \|\Theta_N\|^4_{L^4} \right), \tag{3.18}$$

$$\left| \int_{\mathbb{T}^2} Y_N \Theta_N^3 \, dx \right| \leq c \|Y_N\|^\rho_{W^{-(1-\alpha)-\varepsilon,\infty}} + \eta \left(\|\Theta_N\|^2_{H^\alpha} + \|\Theta_N\|^4_{L^4} \right), \tag{3.19}$$

for some exponent $\rho > 1$.

Proof of Lemma 3.6. As for the estimate (3.17) we have:

$$\begin{aligned} \left| \int_{\mathbb{T}^2} :Y_N^3: \Theta_N \, dx \right| &\lesssim \| :Y_N^3: \|_{W^{-3(1-\alpha)-\varepsilon,\infty}} \|\Theta_N\|_{W^{3(1-\alpha)+\varepsilon,1}} \\ &\lesssim \| :Y_N^3: \|_{W^{-3(1-\alpha)-\varepsilon,\infty}} \|\Theta_N\|_{H^{3(1-\alpha)+\varepsilon}} \\ &\lesssim \| :Y_N^3: \|_{W^{-3(1-\alpha)-\varepsilon,\infty}} \|\Theta_N\|_{H^\alpha}, \end{aligned}$$

where the last inequality holds true for any $\varepsilon \leq 4\alpha - 3$ (hence ε small enough suffices), then the estimate follows by using the generalized Young's inequality, i.e. $ab \leq c(\eta)a^p + \eta b^q$ for any η and $p > 1, q > 1$ conjugate exponents. In this case we use $p = q = 2$.

Fix a $\delta > 0$. As for (3.18) we have as above that

$$\left| \int_{\mathbb{T}^2} :Y_N^2: \Theta_N^2 \, dx \right| \lesssim \| :Y_N^2: \|_{W^{-2(1-\alpha)-\varepsilon,\infty}} \|\Theta_N^2\|_{W^{2(1-\alpha)+\varepsilon,1+\delta}}.$$

By the fractional Leibniz rule (Lemma 2.1), we have

$$\|\Theta_N^2\|_{W^{2(1-\alpha)+\varepsilon,1+\delta}} \lesssim \|\Theta_N\|_{H^{2(1-\alpha)+\varepsilon}} \|\Theta_N\|_{L^{\frac{2(\delta+1)}{1-\delta}}}.$$

By the generalized Young's inequality with conjugate exponents $p = 1 + \gamma$ and $q = \frac{1+\gamma}{\gamma}$, we get

$$\left| \int_{\mathbb{T}^2} :Y_N^2: \Theta_N^2 \, dx \right| \lesssim c(\eta) \| :Y_N^2: \|^{(1+\gamma)/\gamma}_{W^{-2(1-\alpha)-\varepsilon,\infty}} + \eta \left(\|\Theta_N\|_{H^{2(1-\alpha)+\varepsilon}} \|\Theta_N\|_{L^{\frac{2(\delta+1)}{1-\delta}}} \right)^{1+\gamma}.$$

We take δ small enough, in order to have $\frac{2(\delta+1)}{1-\delta} \leq 4$ ($\delta \leq \frac{1}{3}$ suffices), ε small enough ($\varepsilon \leq 3\alpha - 2$ suffices), and by using again the Young's inequality (with exponents

$p = 4/(1+\gamma)$ and $q = 4/(3-\gamma)$) we obtain

$$\left|\int_{\mathbb{T}^2} :Y_N^2\colon \Theta_N^2\,dx\right| \lesssim c(\eta)\|\,:Y_N^2\colon\|_{W^{-2(1-\alpha)-\varepsilon,\infty}}^{(1+\gamma)/\gamma} + \eta\left(\|\Theta_N\|_{H^\alpha}^{\frac{4(1+\gamma)}{3-\gamma}} + \|\Theta_N\|_{L^4}^4\right)$$
$$\lesssim c(\eta)\|\,:Y_N^2\colon\|_{W^{-2(1-\alpha)-\varepsilon,\infty}}^{(1+\gamma)/\gamma} + \eta\left(\|\Theta_N\|_{H^\alpha}^2 + \|\Theta_N\|_{L^4}^4\right)$$

where the last inequality comes by choosing $\gamma \le \frac{1}{3}$ and by applying again the Young's inequality. We set $\rho = (1+\gamma)/\gamma$ and the proof of (3.18) is complete.

Let us prove (3.19). As above we start with the inequality

$$\left|\int_{\mathbb{T}^2} Y_N\Theta_N^3\,dx\right| \lesssim \|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}\|\Theta_N^3\|_{W^{1-\alpha+\varepsilon,1+\delta}}. \tag{3.20}$$

From the Leibniz inequality we get

$$\|\Theta_N^3\|_{W^{1-\alpha+\varepsilon,1+\delta}} \lesssim \|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N^2\|_{L^q}\|\Theta_N\|_{L^p} + \|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{L^{\tilde q}}\|\Theta_N^2\|_{L^2}$$

where $\frac{1}{1+\delta} = \frac{1}{q} + \frac{1}{p} = \frac{1}{\tilde q} + \frac{1}{2}$. Hence, by explicitly giving $\tilde q = \frac{2(1+\delta)}{1-\delta}$

$$\|\Theta_N^3\|_{W^{1-\alpha+\varepsilon,1+\delta}} \lesssim \|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N^2\|_{L^q}\|\Theta_N\|_{L^p} + \|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{L^{\frac{2(1+\delta)}{1-\delta}}}\|\Theta_N\|_{L^4}^2,$$

and hence from (3.20) we get

$$\left|\int_{\mathbb{T}^2} Y_N\Theta_N^3\,dx\right| \lesssim \|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N^2\|_{L^q}\|\Theta_N\|_{L^p} \tag{3.21}$$
$$+ \|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{L^{\frac{2(1+\delta)}{1-\delta}}}\|\Theta_N\|_{L^4}^2. \tag{3.22}$$

Let us focus on (3.22). By applying the Young's inequality twice, we have

$$(3.22) \lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{L^{\frac{2(1+\delta)}{1-\delta}}}^{1+\gamma}\|\Theta_N\|_{L^4}^{2(1+\gamma)}$$
$$\lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{L^{\frac{2(1+\delta)}{1-\delta}}}^{\frac{2(1+\gamma)}{1-\gamma}} + \|\Theta_N\|_{L^4}^4\right)$$
$$\lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\Theta_N\|_{H^{1-\alpha+2\varepsilon}}^{\frac{2(1+\gamma)}{1-\gamma}} + \|\Theta_N\|_{L^4}^4\right)$$

where for the last inequality we used the following Sobolev estimate

$$\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{L^{\frac{2(1+\delta)}{1-\delta}}} \lesssim \|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{H^{\frac{2\delta}{1+\delta}}} = \|\Theta_N\|_{H^{1-\alpha+2\varepsilon}},$$

provided that δ is small enough. By means of Lemma 2.2, with $s_1 = 1-\alpha+2\varepsilon$ and $s_2 = \alpha$, we continue

$$(3.22) \lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\Theta_N\|_{H^\alpha}^{\frac{2(1+\gamma)}{1-\gamma}\cdot\frac{1-\alpha+2\varepsilon}{\alpha}}\|\Theta_N\|_{L^2}^{\frac{2(1+\gamma)}{1-\gamma}\cdot\frac{2\alpha-1-2\varepsilon}{\alpha}} + \|\Theta_N\|_{L^4}^4\right)$$
$$\lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\Theta_N\|_{H^\alpha}^2 + \|\Theta_N\|_{L^2}^\beta + \|\Theta_N\|_{L^4}^4\right)$$

where we have used again the Young's inequality with conjugate exponents $p = \frac{\alpha(1-\gamma)}{(1+\gamma)(1-\alpha+2\varepsilon)}$ and $q = \frac{\alpha(1-\gamma)}{\alpha(1-\gamma)-(1-\alpha+2\varepsilon)(1+\gamma)}$, hence the exponent β of the L^2-norm of Θ_N is defined by

$$\beta = \frac{2q(1+\gamma)}{1-\gamma}\cdot\frac{2\alpha-1-2\varepsilon}{\alpha} = \frac{2(1+\gamma)(2\alpha-1-2\varepsilon)}{\alpha(1-\gamma)-(1-\alpha+2\varepsilon)(1+\gamma)}.$$

At this point, we see that if $\beta \leq 4$, namely if $\frac{(1+\gamma)(1+2\varepsilon)}{2(1+\gamma)} \leq \alpha$, the term $\|\Theta_N\|_{L^2}^{\beta}$ is controlled by $\|\Theta_N\|_{L^4}^4$ and hence we conclude with

$$(3.22) \lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\Theta_N\|_{H^\alpha}^2 + \|\Theta_N\|_{L^4}^4\right). \tag{3.23}$$

Let us point-out that to have the bound $\beta \leq 4$ is enough to consider γ, ε small enough.

It remains to estimate

$$(3.21) = \|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N^2\|_{L^q}\|\Theta_N\|_{L^p}$$

where $\frac{1}{1+\delta} = \frac{1}{p}+\frac{1}{q}$. Set $p=4$ and $q = \frac{4(1+\delta)}{3-\delta}$. By the generalized Young's inequality we have, as above,

$$(3.21) \lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N^2\|_{L^{\frac{4(1+\delta)}{3-\delta}}}\|\Theta_N\|_{L^4}\right)^{1+\gamma}.$$

By the Leibniz rule,

$$\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N^2\|_{L^{\frac{4(1+\delta)}{3-\delta}}} \lesssim \|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{L^{\tilde q}}\|\Theta_N\|_{L^{\tilde p}}$$

for $\frac{3-\delta}{4(1+\delta)} = \frac{1}{\tilde q}+\frac{1}{\tilde p}$. We chose $\tilde p = 4$ and therefore $\tilde q = \frac{2(1+\delta)}{1-\delta}$. We continue the estimate above as

$$(3.21) \lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\langle\nabla\rangle^{1-\alpha+\varepsilon}\Theta_N\|_{L^{\frac{2(1+\delta)}{1-\delta}}}^{1+\gamma}\|\Theta_N\|_{L^4}^{2(1+\gamma)}\right)$$

and by the Sobolev inequality, provided $\frac{2\delta}{1+\delta} \leq \varepsilon$ we have

$$(3.21) \lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\Theta_N\|_{H^{1-\alpha+2\varepsilon}}^{1+\gamma}\|\Theta_N\|_{L^4}^{2(1+\gamma)}\right).$$

We use the Young's inequality to get

$$(3.21) \lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\Theta_N\|_{H^{1-\alpha+2\varepsilon}}^{\frac{2(1+\gamma)}{1-\gamma}} + \|\Theta_N\|_{L^4}^4\right),$$

and by repeating the same interpolation argument as for the proof of (3.22), we finish the estimate

$$(3.21) \lesssim c(\eta)\|Y_N\|_{W^{-(1-\alpha)-\varepsilon,\infty}}^{\frac{1+\gamma}{\gamma}} + \eta\left(\|\Theta_N\|_{H^\alpha}^2 + \|\Theta_N\|_{L^4}^4\right). \tag{3.24}$$

By combining (3.20), (3.21), (3.22), (3.23), and (3.24) we conclude with the proof of (3.19), with $\rho = (1+\gamma)/\gamma$. □

We can now prove Theorem 1.2 in the cubic case.

Proof of Theorem 1.2. We recall the definition of

$$\mathcal{W}_N(\theta) = \mathbb{E}\Big[\sum_{\ell=0}^{4}\binom{4}{\ell}\int_{\mathbb{T}^2} :Y_N^{4-\ell}: \Theta_N^\ell \, dx + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 \, dt\Big].$$

and we observe that

$$\begin{aligned}\mathcal{W}_N(\theta) &= \mathbb{E}\Big[\int_{\mathbb{T}^2} :Y_N^4: dx\Big] + \mathbb{E}\Big[\sum_{\ell=1}^{3}\binom{4}{\ell}\int_{\mathbb{T}^2} :Y_N^{4-\ell}: \Theta_N^\ell \, dx\Big] \\ &\quad + \mathbb{E}\Big[\|\Theta_N\|_{L^4}^4 + \frac{1}{2}\int_0^1 \|\theta(t)\|_{L^2_x}^2 \, dt\Big] \\ &= \mathcal{A} + \mathcal{B} + \mathcal{C}.\end{aligned}$$

Note that the term $\mathcal{C}$ is positive. The first term $\mathcal{A}$ is uniformly bounded in N, by Proposition 1.1, while by using Lemma 3.4 in conjunction with the estimates in Lemma 3.6, we obtain that

$$\mathcal{A} + \mathcal{B} \gtrsim -1 - \eta\, \mathcal{C}$$

and hence,

$$\inf_{N,\theta} \mathcal{W}_N(\theta) \gtrsim -1 + (1-\eta)\,\mathcal{C} \gtrsim -1.$$

The uniform (in N and θ) bound above, jointly with (3.12) gives the desired result. □

3.3.2. *Higher nonlinearities.* We turn now the attention to the general case of an arbitrary defocusing nonlinearity. Therefore, let us fix $m \geq 1$, hence $2m+2 \geq 4$ and $\ell \in \{0, 1, \dots, 2m+2\}$. We aim at generalizing the estimates in Lemma 3.6.

Lemma 3.7. *Fix $m \geq 1$ and let $\ell \in \{1, \dots, 2m+1\}$. There exist $\varepsilon > 0$ and exponents $\rho_\ell > 1$ such that, for $\ell = 1$,*

$$\Big|\int_{\mathbb{T}^2} :Y_N^{2m+1}: \Theta_N \, dx\Big| \leq c\|Y_N\|_{W^{-(2m+1)(1-\alpha)-\varepsilon,\infty}}^{\rho_1} + \eta\|\Theta_N\|_{H^\alpha}^2 \tag{3.25}$$

and, for $\ell \geq 2$

$$\Big|\int_{\mathbb{T}^2} :Y_N^{2m+2-\ell}: \Theta_N^\ell \, dx\Big| \leq c\|Y_N\|_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}}^{\rho_\ell} + \eta\left(\|\Theta_N\|_{H^\alpha}^2 + \|\Theta_N\|_{L^{2m+2}}^{2m+2}\right) \tag{3.26}$$

for any $\eta \ll 1$ and $c = c(\eta)$.

Proof. For $\ell = 1$ the estimate is as in (3.17). Namely,

$$\begin{aligned}\Big|\int_{\mathbb{T}^2} :Y_N^{2m+1}: \Theta_N \, dx\Big| &\lesssim \|:Y_N^{2m+1}:\|_{W^{-(2m+1)(1-\alpha)-\varepsilon,\infty}} \|\Theta_N\|_{W^{(2m+1)(1-\alpha)+\varepsilon,1}} \\ &\lesssim \|:Y_N^{2m+1}:\|_{W^{-(2m+1)(1-\alpha)-\varepsilon,\infty}} \|\Theta_N\|_{H^{(2m+1)(1-\alpha)+\varepsilon}} \\ &\lesssim \|:Y_N^{2m+1}:\|_{W^{-(2m+1)(1-\alpha)-\varepsilon,\infty}} \|\Theta_N\|_{H^\alpha},\end{aligned}$$

where the last estimate holds true for any $\varepsilon \leq 1 - \frac{m+1}{2m+1}$, which guarantees the embedding $H^{(2m+1)(1-\alpha)+\varepsilon} \subset H^\alpha$. Then the estimate (3.25) follows by using the generalized Young's inequality $ab \lesssim c(\eta)a^2 + \eta b^2$ and $\rho_1 = 2$.

Let us consider the case $\ell = 2$. Fix a $\delta > 0$. Similarly to the proof of (3.18), we have

$$\left| \int_{\mathbb{T}^2} :Y_N^{2m}: \Theta_N^2 \, dx \right| \lesssim \| :Y_N^{2m}: \|_{W^{-2m(1-\alpha)-\varepsilon,\infty}} \|\Theta_N^2\|_{W^{2m(1-\alpha)+\varepsilon,1+\delta}}.$$

By the fractional Leibniz rule (Lemma 2.1), we have

$$\|\Theta_N^2\|_{W^{2m(1-\alpha)+\varepsilon,1+\delta}} \lesssim \|\Theta_N\|_{H^{2m(1-\alpha)+\varepsilon}} \|\Theta_N\|_{L^{\frac{2(\delta+1)}{1-\delta}}}.$$

By the generalized Young's inequality $ab \lesssim c(\eta)a^p + \eta b^q$ with conjugate exponents $p = 1+\gamma$ and $q = \frac{1+\gamma}{\gamma}$, we get

$$\begin{aligned}\left| \int_{\mathbb{T}^2} :Y_N^{2m}: \Theta_N^2 \, dx \right| &\lesssim c(\eta) \| :Y_N^{2m}: \|^{(1+\gamma)/\gamma}_{W^{-2m(1-\alpha)-\varepsilon,\infty}} \\ &\quad + \eta \left(\|\Theta_N\|_{H^{2m(1-\alpha)+\varepsilon}} \|\Theta_N\|_{L^{\frac{2(\delta+1)}{1-\delta}}} \right)^{1+\gamma}.\end{aligned}$$

By taking δ and ε small enough, and by using again the Young's inequality (with exponents $p = (2m+2)/(1+\gamma)$ and $q = \frac{(2m+2)}{(2m+2)-(1+\gamma)}$) we obtain

$$\begin{aligned}\left| \int_{\mathbb{T}^2} :Y_N^{2m}: \Theta_N^2 \, dx \right| &\lesssim c(\eta) \| :Y_N^{2m}: \|^{(1+\gamma)/\gamma}_{W^{-2m(1-\alpha)-\varepsilon,\infty}} \\ &\quad + \eta \left(\|\Theta_N\|_{H^\alpha}^{\frac{(1+\gamma)(2m+2)}{(2m+2)-(1+\gamma)}} + \|\Theta_N\|^{2m+2}_{L^{2m+2}} \right) \\ &\lesssim c(\eta) \| :Y_N^{2m}: \|^{(1+\gamma)/\gamma}_{W^{-2m(1-\alpha)-\varepsilon,\infty}} + \eta \left(\|\Theta_N\|^2_{H^\alpha} + \|\Theta_N\|^{2m+2}_{L^{2m+2}} \right)\end{aligned}$$

where the last inequality comes by choosing γ small enough ($\gamma < \frac{m}{m+2}$ suffices) in order to have $\frac{(1+\gamma)(2m+2)}{(2m+2)-(1+\gamma)} \leq 2$ and by applying again the Young's inequality. We set $\rho_2 = (1+\gamma)/\gamma$ and the proof of (3.26) for $\ell = 2$ is complete.

The general case is done by induction on ℓ. So suppose that at the $(\ell-1)$-th step we have that

$$\exists\, \delta \ll 1 \text{ s.t. } \|\langle \nabla \rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon} \Theta_N^{\ell-1}\|_{L^{\frac{(2m+2)(1+\delta)}{2m+1-\delta}}} \lesssim \|\Theta_N\|_{H^{(2m+2-\ell)(1-\alpha)+c\varepsilon}} \|\Theta_N\|^{\ell-2}_{L^{2m+2}}$$

for some $c = c(\delta) > 1$.

$$\left| \int_{\mathbb{T}^2} :Y_N^{2m+2-\ell}: \Theta_N^\ell \, dx \right| \leq \| :Y_N^{2m+2-\ell}: \|_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} \|\Theta_N^\ell\|_{W^{(2m+2-\ell)(1-\alpha)+\varepsilon,1+\delta}}. \tag{3.27}$$

Let us consider the term $\|\Theta_N^\ell\|_{W^{(2m+2-\ell)(1-\alpha)+\varepsilon,1+\delta}}$. By using the Leibniz rule

$$\|\Theta_N^\ell\|_{W^{(2m+2-\ell)(1-\alpha)+\varepsilon,1+\delta}} \lesssim \underbrace{\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N^{\ell-1}\|_{L^q}\|\Theta_N\|_{L^p}}_{B(\ell)} + \underbrace{\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N\|_{L^r}\|\Theta_N^{\ell-1}\|_{L^{\tilde r}}}_{A(\ell)},$$

where $\frac{1}{1+\delta} = \frac{1}{q} + \frac{1}{p} = \frac{1}{r} + \frac{1}{\tilde r}$. By choosing $\tilde r = 2$ and hence $r = \frac{2(\delta+1)}{1-\delta}$, by applying again the generalized Young's inequality we get

$$A(\ell)\cdot \| :Y_N^{2m+2-\ell}: \|_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} \lesssim \| :Y_N^{2m+2-\ell}: \|^{(1+\gamma)/\gamma}_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} + \eta\left(\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N\|^{1+\gamma}_{L^r}\|\Theta_N\|^{(\ell-1)(1+\gamma)}_{L^{2(\ell-1)}}\right)$$

and by using again the Young's inequality with $p = \frac{2m+2}{(\ell-1)(1+\gamma)}$ and $q = \frac{2m+2}{2m+2-(\ell-1)(1+\gamma)}$, we get

$$A(\ell)\cdot \| :Y_N^{2m+2-\ell}: \|_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} \lesssim \| :Y_N^{2m+2-\ell}: \|^{(1+\gamma)/\gamma}_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} + \eta\left(\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N\|^{\beta}_{L^r} + \|\Theta_N\|^{2m+2}_{L^{2m+2}}\right)$$

where we set

$$\beta = (1+\gamma)q = \frac{(1+\gamma)(2m+2)}{2m+2-(\ell-1)(1+\gamma)}.$$

By the Sobolev embedding, and by choosing δ small enough, we have that

$$\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N\|^{\beta}_{L^{\frac{2(\delta+1)}{1-\delta}}} \lesssim \|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N\|^{\beta}_{H^{\frac{2\delta}{1+\delta}}} \lesssim \|\Theta_N\|^{\beta}_{H^{(2m+2-\ell)(1-\alpha)+2\varepsilon}}.$$

Therefore, by means of the interpolation inequality as in Lemma 2.2 with $s_1 = (2m+2-\ell)(1-\alpha)+2\varepsilon$ and $s_2 = \alpha$ we obtain

$$\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N\|^{\beta}_{L^{\frac{2(\delta+1)}{1-\delta}}} \lesssim \|\Theta_N\|^{\beta\cdot\frac{(2m+2-\ell)(1-\alpha)+2\varepsilon}{\alpha}}_{H^\alpha}\|\Theta_N\|^{\beta\cdot\frac{\alpha-(2m+2-\ell)(1-\alpha)-2\varepsilon}{\alpha}}_{L^2},$$

and by means of the Young's inequality with $p = \frac{2\alpha}{\beta((2m+2-\ell)(1-\alpha)+2\varepsilon)}$ we continue as

$$\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N\|^{\beta}_{L^{\frac{2(\delta+1)}{1-\delta}}} \lesssim \|\Theta_N\|^2_{H^\alpha} + \|\Theta_N\|^{\tilde\beta}_{L^2},$$

where

$$\tilde\beta = \left(p'\beta\frac{\alpha-(2m+2-\ell)(1-\alpha)-2\varepsilon}{\alpha}\right) = \frac{2\beta\left(\alpha-(2m+2-\ell)(1-\alpha)-2\varepsilon\right)}{2\alpha-\beta\left((2m+2-\ell)(1-\alpha)+2\varepsilon\right)}. \tag{3.28}$$

For the sake of brevity, let us set $\tilde m = 2m+2$. We want that $\tilde\beta \le \tilde m$, so that in (3.28) we can apply the Young's inequality once more and conclude the estimate for the term $A(\ell)\cdot\| :Y_N^{2m+2-\ell}: \|_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}}$. After algebraic computations, we get that in order to have $\tilde\beta \le \tilde m$, we must have

$$\alpha\left(2\beta + 2\beta(\tilde m-\ell) - 2\tilde m - \beta\tilde m(\tilde m-\ell)\right) \le -\beta\tilde m(\tilde m-\ell) - 2\beta\varepsilon\tilde m + 2\beta(\tilde m-\ell) + 4\beta\varepsilon.$$

Let us observe that $\beta \to \frac{\tilde{m}}{\tilde{m}-(\ell-1)}$ as $\gamma \to 0$ and that $\varepsilon\beta \to 0$ as $\varepsilon \to 0$. Hence in order to guarantee that $\tilde{\beta} \leq \tilde{m}$ it is enough to show that

$$\left(\frac{2\tilde{m}}{\tilde{m}-(\ell-1)} + \frac{2\tilde{m}(\tilde{m}-\ell)}{\tilde{m}-(\ell-1)} - 2\tilde{m} - \frac{\tilde{m}^2(\tilde{m}-\ell)}{\tilde{m}-(\ell-1)}\right)\alpha < -\frac{\tilde{m}^2(\tilde{m}-\ell)}{\tilde{m}-(\ell-1)} + \frac{2\tilde{m}(\tilde{m}-\ell)}{\tilde{m}-(\ell-1)}.$$

The above inequality is verified provided

$$\alpha > 1 - \frac{2}{\tilde{m}} = 1 - \frac{1}{m+1}$$

which holds true as we are working under the hypothesis $\alpha > 1 - \frac{1}{2m+2}$. In conclusion, for γ and ε small enough we have that

$$\begin{aligned} &A(\ell) \cdot \| :Y_N^{2m+2-\ell}: \|_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} \\ &\lesssim \| :Y_N^{2m+2-\ell}: \|^{(1+\gamma)/\gamma}_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} + \eta\left(\|\Theta_N\|^2_{H^\alpha} + \|\Theta_N\|^{2m+2}_{L^{2m+2}}\right) \end{aligned} \tag{3.29}$$

Let us go back to the term

$$\| :Y_N^{2m+2-\ell}: \|_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} \underbrace{\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N^{\ell-1}\|_{L^q}\|\Theta_N\|_{L^p}}_{B(\ell)}. \tag{3.30}$$

Also in this case we start with a Young's generalized inequality, we set $p = 2m+2$ and $q = \frac{(2m+2)(1+\delta)}{2m+1-\delta}$, and we write

$$\begin{aligned} (3.30) &\lesssim \| :Y_N^{2m+2-\ell}: \|^{(1+\gamma)/\gamma}_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} \\ &\quad + \eta\left(\|\langle\nabla\rangle^{(2m+2-\ell)(1-\alpha)+\varepsilon}\Theta_N^{\ell-1}\|_{L^{\frac{(2m+2)(1+\delta)}{2m+1-\delta}}}\|\Theta_N\|_{L^{2m+2}}\right)^{1+\gamma} \\ &\lesssim \| :Y_N^{2m+2-\ell}: \|^{(1+\gamma)/\gamma}_{W^{-(2m+2-\ell)(1-\alpha)-\varepsilon,\infty}} \\ &\quad + \eta\left(\|\Theta_N\|_{H^{(2m+2-\ell)(1-\alpha)+c\varepsilon}}\|\Theta_N\|^{\ell-1}_{L^{2m+2}}\right)^{1+\gamma} \end{aligned}$$

where for the last inequality we used the inductive hypothesis. At this point the estimate for the latter term can be concluded with the same argument as for the term involving $A(\ell)$, i.e. by using the Young's inequality and interpolation. □

Note that with Lemma 3.7 at hand, the proof of Theorem 1.2 for a general m is concluded as for the cubic case.

4. First order expansion almost sure local well-posedness

In this Section, we prove the almost sure local well-posedness of the defocusing Wick ordered fraction NLW (1.30).

By relying on Strichartz estimates, we prove that for some range of the fractional exponent α, the Cauchy problem (1.30) admits a solution, by means of a fixed-point argument. Specifically, the strategy is to use a fixed point argument on the non-homogeneous part of the Duhamel's formulation of (1.30); indeed, for our purposes, we assume zero initial data for the Cauchy problem associated to (1.30). It is worth mentioning that the estimates we are going to use are deterministic, though we deeply use the regularities of the stochastic terms (given in the Subsection 2.2).

Let us consider the non-homogeneous term in the Duhamel's formulation of (1.30), and we define the solution map as follows: we introduce the map $\Psi(w)$ by

$$\Psi(w)(t) = \Psi^{\omega}_{z}(w)(t) := \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^{\alpha})}{\langle\nabla\rangle^{\alpha}} :(z+w)^{2m+1}: dt'. \tag{4.1}$$

We recall from the property of the Hermite polynomials (1.14), that

$$:(z+w)^{2m+1}: = \sum_{\ell=0}^{2m+1} \binom{2m+1}{\ell} :z^{\ell}: w^{2m+1-\ell},$$

then the map $\Psi(w)(t) = \Psi^{\omega}_{\{:z^{\ell}:,\, \ell\in\{0,\dots,2m+1\}\}}(w)(t)$ can be rewritten as

$$\begin{aligned}\Psi(w)(t) &= \sum_{\ell=0}^{2m+1} \binom{2m+1}{\ell} \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^{\alpha})}{\langle\nabla\rangle^{\alpha}} \left(:z^{\ell}: w^{2m+1-\ell}\right) dt' \\ &= \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^{\alpha})}{\langle\nabla\rangle^{\alpha}} w^{2m+1}\, dt' &(4.2)\\ &+ \sum_{\ell=1}^{2m} \binom{2m+1}{\ell} \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^{\alpha})}{\langle\nabla\rangle^{\alpha}} \left(:z^{\ell}: w^{2m+1-\ell}\right) dt' &(4.3)\\ &+ \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^{\alpha})}{\langle\nabla\rangle^{\alpha}} :z^{2m+1}: dt'. &(4.4)\end{aligned}$$

The map Ψ depends on the set of stochastic objects

$$\{:z^{\ell}:\}_{\ell\in\{0,\dots,2m+1\}}, \tag{4.5}$$

and the existence of a fixed point for Ψ will be shown in a deterministic way. Basically, we decompose the ill-posed (due to the rough initial data) solution-map in two steps:

$$\begin{aligned}&(u_0^{\omega}, u_1^{\omega}) \longmapsto \{: z^{\ell} :,\ \ell \in \{0,\dots,2m+1\}\} \longmapsto w \in X^s \\ &\qquad \longmapsto u = z + w \in C_t([0,T]; H_x^{-(1-\alpha)-\varepsilon}) + X^s \subset C_t([0,T]; H_x^{-(1-\alpha)-\varepsilon}),\end{aligned} \tag{4.6}$$

where X^s is a suitable Strichartz space, see (4.8) below. In the first lifting step we generate the enhanced data set of stochastic objects defined in (4.5) with low regularity, and in the second we give the solution in the sense specified in the Introduction, provided by a fixed point argument in a deterministic way.

Let us note that the critical exponent for our equation is $s_c = 1 - \frac{\alpha}{m}$. To start, we fix

$$1 > s > s_c = 1 - \frac{\alpha}{m}. \tag{4.7}$$

We define the space

$$X^s := L_t^{\infty}([0,T]; H_x^s) \cap L_t^p([0,T]; W_x^{s-\gamma_{p,q},q}) \tag{4.8}$$

where T is a given time $T \leq 1$, and (p,q) is a fractional admissible pair depending on m (see (2.3) and (2.4) for the definitions). In particular, we will fix

$$p = 2m+1+\epsilon, \quad \epsilon > 0,$$

and q sufficiently large, to be chosen later on.

We are going to prove the following a.s. local well-posedness result.

Proposition 4.1. *Fix $m \in \mathbb{N}$ and $\alpha \in \left(1 - \frac{1}{4m^2+4m+2}, 1\right)$. The map Ψ defined in (4.1) admits a unique fixed point in the space X^s, for some $s > 0$, on the time interval $[0,T]$, for some $T \leq 1$. Consequently, (1.30) admits a local solution almost surely in the class $C_t([0,T]; H_x^{-(1-\alpha)-\varepsilon})$, for any $0 < \varepsilon \ll 1$. More precisely, the solution is of the form*

$$u = z + w \in C_t([0,T]; H_x^{-(1-\alpha)-\varepsilon}) + X^s.$$

Remark 4.2. Let us observe that for $s \geq 0$, the solution $u = z + w$ to (1.30) has spatial regularity

$$u \in H_x^{-(1-\alpha)-\varepsilon} + X^s \subset H_x^{-(1-\alpha)-\varepsilon}, \quad \varepsilon > 0.$$

Indeed, $-(1-\alpha) - \varepsilon < 0$, hence the claim follows by Sobolev's embeddings. Note that the uniqueness of the solution only holds in the Strichartz space X^s, and we refer to this kind of local well-posedness result as *conditional* (in the remainder w). See Appendix A for an *unconditional* (on the remainder w) local well-posedness theory.

Proof of Proposition 4.1. By recalling that the map (4.1) can be decomposed as the sum

$$(4.1) = (4.2) + (4.3) + (4.4)$$

we distinguish three cases. In what follows, $L_t^p = L_t^p([0,T])$.

• Case 1, $\ell = 0$. In this case, corresponding to the term (4.2), $: z^\ell : \equiv 1$, namely is constant.

By Strichartz estimates, see Lemma 2.4, and the dual Sobolev's embedding, i.e. $L_x^r \subset H_x^{s-\alpha}$ for

$$\frac{\alpha - s}{2} \geq \frac{1}{r} - \frac{1}{2}, \tag{4.9}$$

we get

$$\|(4.2)\|_{X^s} \lesssim \|w^{2m+1}\|_{L_t^1 H_x^{s-\alpha}} \lesssim \|w\|^{2m+1}_{L_t^{2m+1} L_x^{(2m+1)r}}.$$

By the embedding $W_x^{s-\gamma_{p,q},q} \subset L_x^{(2m+1)r}$ with

$$\frac{s - \gamma_{p,q}}{2} \geq \frac{1}{q} - \frac{1}{(2m+1)r}, \tag{4.10}$$

we continue the estimate above as

$$\|(4.2)\|_{X^s} \lesssim \|w\|^{2m+1}_{L_t^{2m+1} W_x^{s-\gamma_{p,q},q}} \lesssim T^\theta \|w\|^{2m+1}_{L_t^p W_x^{s-\gamma_{p,q},q}},$$

where in the last step we used the Lebesgue inclusion as $p > 2m+1$, then $\theta > 0$. We observe that the conditions (4.9) and (4.10), along with definitions (2.3) and (2.4), imply that

$$s > 1 - \frac{\alpha}{m}, \tag{4.11}$$

regardless on q, where q is the spatial integrability of an admissible pair.

• Case 2, $\ell = 2m+1$. In this case, we are considering the term (4.4) in the Duhamel formulation (4.1). By Strichartz estimate

$$\begin{aligned}\|(4.4)\|_{X^s} &\lesssim \| :z^{2m+1}: \|_{L^1_t H^{s-\alpha}_x} \\ &\lesssim T\|\langle\nabla\rangle^{s-\alpha} :z^{2m+1}: \|_{L^\infty_t L^\infty_x} \\ &\lesssim T\|\langle\nabla\rangle^{-(2m+1)(1-\alpha)-\varepsilon} :z^{2m+1}: \|_{L^\infty_t L^\infty_x}\end{aligned}$$

where we used the Hölder's inequality and the Lebesgue inclusion in the second line, and the Sobolev's embedding in the last line, provided

$$s-\alpha < -(2m+1)(1-\alpha) \iff s < \alpha(2m+2)-(2m+1). \tag{4.12}$$

Note that $:z^{2m+1}: \in C_t(\mathbb{R}^+; W_x^{-(2m+1)(1-\alpha)-\varepsilon,\infty}(\mathbb{T}^2))$ by Proposition 2.13 (i), for $\alpha > 1-\frac{1}{2m+1}$. Therefore, in order to have a non-empty condition on the exponents satisfying (4.11) and (4.12), we impose that

$$\alpha > 1-\frac{1}{2m^2+2m+1}.$$

• Case 3, $1 \le \ell \le 2m$. We use a duality argument to estimate

$$\begin{aligned}\|f\|_{L^1_t H^{s-\alpha}_x} &= \sup_{g\in L^\infty_t H^{\alpha-s}_x, \|g\|_{L^\infty_t H^{\alpha-s}_x}\le 1} \left|\iint fg\,dx\,dt\right| \\ &\le \sup_{g\in L^\infty_t H^{\alpha-s}_x, \|g\|_{L^\infty_t H^{\alpha-s}_x}\le 1} \|\langle\nabla\rangle^\eta g\|_{L^{\tilde p_2'}_t L^{\tilde q_2'}_x} \|\langle\nabla\rangle^{-\eta} f\|_{L^{\tilde p_2}_t L^{\tilde q_2}_x}\end{aligned}$$

for $\eta < 1$, with $(\tilde p_2, \tilde p_2')$ and $(\tilde q_2, \tilde q_2')$ two Hölder's conjugate pairs. We make moreover use of the following interpolation inequality:

$$\|\langle\nabla\rangle^\eta h\|_{L^{p_1}L^{q_1}} \le \|h\|^{1-\eta/\sigma}_{L^a L^b} \|h\|^{\eta/\sigma}_{L^\infty H^\sigma}$$

provided $\eta < \sigma$, $\frac{1}{q_1} = \frac{\sigma-\eta}{b\sigma} + \frac{\eta}{2\sigma}$ and $\frac{1}{p_1} = \frac{\sigma-\eta}{a\sigma}$. Hence

$$\begin{aligned}\|f\|_{L^1_t H^{s-\alpha}_x} &\le \sup_{\substack{g\in L^\infty_t H^{\alpha-s}_x \\ \|g\|_{L^\infty_t H^{\alpha-s}_x}\le 1}} \|\langle\nabla\rangle^\delta g\|_{L^{\tilde p_2'}_t L^{\tilde q_2'}_x} \|\langle\nabla\rangle^{-\delta} f\|_{L^{\tilde p_2}_t L^{\tilde q_2}_x} \\ &\le \sup_{\substack{g\in L^\infty_t H^{\alpha-s}_x \\ \|g\|_{L^\infty_t H^{\alpha-s}_x}\le 1}} \|g\|^{1-\delta/(\alpha-s)}_{L^\infty_t L^b_x} \|g\|^{\delta/(\alpha-s)}_{L^\infty_t H^{\alpha-s}_x} \|\langle\nabla\rangle^{-\delta} f\|_{L^{\tilde p_2}_t L^{\tilde q_2}_x}\end{aligned}$$

provided that

$$\delta < \alpha - s, \qquad \tilde p_2 = 1, \qquad \frac{1}{\tilde q_2} = \frac{(\alpha-s)-\delta}{b'(\alpha-s)} + \frac{\delta}{2(\alpha-s)}, \qquad b = \frac{2}{1-(\alpha-s)}.$$

Note that $b = \frac{2}{1-(\alpha-s)}$ is the Sobolev's conjugate exponent for the space $H^{\alpha-s}$ which guarantees that $g \in L^b_x$. Therefore, with $f = :z^\ell: w^{2m+1-\ell}$, and by using the estimates

(2.2) and (2.1)

$$\begin{aligned}\| :z^\ell : w^{2m+1-\ell}\|_{L^1_t H^{s-\alpha}_x} &\leq \|\langle\nabla\rangle^{-\delta}\left(:z^\ell : w^{2m+1-\ell}\right)\|_{L^1_t L^{\tilde q_2}_x}\\ &\lesssim \|\|\langle\nabla\rangle^{-\delta} :z^\ell :\|_{L^\infty_x}\|\langle\nabla\rangle^{\delta} w^{2m+1-\ell}\|_{L^{\tilde q_2}_x}\|_{L^1_t}\\ &\lesssim \|\langle\nabla\rangle^{-\delta} :z^\ell :\|_{L^\infty_t L^\infty_x}\|\langle\nabla\rangle^{\delta} w\|^{2m+1-\ell}_{L^{2m}_t L^{2m\tilde q_2}_x}\\ &\lesssim \|\langle\nabla\rangle^{\delta} w\|^{2m+1-\ell}_{L^{2m}_t L^{2m\tilde q_2}_x}\end{aligned} \tag{4.13}$$

where we impose, to get the last line, $\delta > 2m(1-\alpha)$. By exploiting again the interpolation inequality with

$$\frac{1}{2m} = \frac{1-\delta}{p_3 s} \quad \text{and} \quad \frac{1}{2m\tilde q_2} = \frac{s-\delta}{qs} + \frac{\delta}{2s}$$

we continue with the estimate

$$\| :z^\ell : w^{2m+1-\ell}\|_{L^1_t H^{s-\alpha}_x} \lesssim \left(\|w\|^{1-\delta/s}_{L^{p_3}_t L^q_x}\|w\|^{\delta/s}_{L^\infty_t H^s_x}\right)^{2m+1-\ell}. \tag{4.14}$$

From the definition of $\tilde q_2$ we rewrite

$$\frac{s-\delta}{qs} = \frac{1}{4m}(1+\alpha-s-\delta) - \frac{\delta}{2s}$$

and the right-hand side of the above identity is positive provided $\delta < \frac{s+\alpha s-s^2}{2m+s}$. Therefore we impose

$$2m(1-\alpha) < \delta < \min\left\{s, \alpha - s, \frac{s+\alpha s - s^2}{2m+s}\right\}.$$

At this point we observe that for any positive s, $s > \frac{s+\alpha s-s^2}{2m+s}$, then

$$2m(1-\alpha) < \delta < \min\left\{\alpha - s, \frac{s+\alpha s - s^2}{2m+s}\right\}.$$

Note that $\alpha - s < \frac{s+\alpha s-s^2}{2m+s}$ if and only if $s > \frac{2m\alpha}{2m+1}$. As we are working in the case $s > 1 - \frac{\alpha}{m}$, we have that $1 - \frac{\alpha}{m} < \frac{2m\alpha}{2m+1}$, for $\alpha > 1 - \frac{m+1}{2m^2+2m+1}$ which in turn is implied by $\alpha > 1 - \frac{1}{2m^2+2m+1}$. Coupling together $s > \frac{2m\alpha}{2m+1}$, and (4.12) we obtain that

$$\alpha > 1 - \frac{1}{4m^2+4m+2}.$$

Note that $2m(1-\alpha) < \alpha - s$ when working with the condition (4.12).

From (4.14), the fact that $p > p_3$, and by taking q sufficiently large, we get

$$\| :z^\ell : w^{2m+1-\ell}\|_{L^1_t H^{s-\alpha}_x} \lesssim T^\theta\left(\|w\|^{1-\delta/s}_{L^p_t W^{s-\gamma_{p,q},q}_x}\|w\|^{\delta/s}_{L^\infty_t H^s_x}\right)^{2m+1-\ell},$$

as $s - \gamma_{p,q} > 0$ for $s > \frac{2m\alpha}{2m+1}$. The claim of the Proposition follows by a fixed point argument, with $s \in (s_0, s_1)$, where

$$s_0 = \frac{2m\alpha}{2m+1} \quad \text{and} \quad s_1 = \alpha(2m+2) - (2m+1).$$

By repeating the argument for $\Psi(w-\tilde{w})$ one shows that Φ is a contraction map and the proof is done. $\square$

5. Second order expansion almost sure local well-posedness

We now prove the a.s. local well-posedness for the second order expansion.
As in the proof of the local well-posedness for the first order expansion, we introduce some notation. Similarly to the previous Section, the functional space we are going to work with, is the space $X^s := L^\infty_t H^s_x \cap L^p_t W^{s-\gamma_{p,q},q}_x$ introduced in (4.8). We recall that (p,q) are fractional admissible, see (2.3) and $\gamma_{p,q}$ defined in (2.4). In particular, we take $p = 2m+1+\epsilon$, with q to be chosen later on.

We recall from (1.33) that the equation solved by w_2, is given by

$$\begin{cases} \partial_t^2 w_2 + (1-\Delta)^\alpha w_2 + \sum_{\ell=0}^{2m} \binom{2m+1}{\ell} :z^\ell: (z_2+w_2)^{2m+1-\ell} = 0 \\ (w_2, \partial_t w_2)|_{t=0} = (0,0) \end{cases} \tag{5.1}$$

where $u = z + z_2 + w_2$, see (1.24) and (1.31) for the definition of the components z and z_2.
By expanding the binomial power, and by denoting $b_{m,\ell} = \binom{2m+1}{\ell}$ and $c_{m,\ell,j} = \binom{2m+1-\ell}{j}$, the solution to (5.1) is represented by $\Psi(w_2)(t) = \Psi^\omega_{\{:z^\ell:\, z_2^k\}}(w_2)(t)$ given by

$$\Psi(w_2)(t) = \sum_{\ell=0}^{2m} \sum_{j=0}^{2m+1-\ell} b_{m,\ell} c_{m,\ell,j} \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} \left(:z^\ell:\, z_2^{2m+1-\ell-j} w_2^j\right) dt' \tag{5.2}$$

$$= \sum_{\ell=0}^{2m} b_{m,\ell} c_{m,\ell,0} \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} \left(:z^\ell:\, z_2^{2m+1-\ell}\right) dt' \tag{5.3}$$

$$+ \sum_{\ell=0}^{2m} \sum_{j=1}^{2m+1-\ell} b_{m,\ell} c_{m,\ell,j} \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} \left(:z^\ell:\, z_2^{2m+1-\ell-j} w_2^j\right) dt', \tag{5.4}$$

where we have separated (5.2) as the sum of the source term case (5.3), i.e. no terms in w_2, and terms involving w_2, i.e. (5.4). Also here, as in the previous Section, we create an enhanced data set (with other stochastic objects) and then solve the problem by performing a fixed point argument in a deterministic way, by using the regularity of the stochastic terms. In this case, we further factorize the ill-posed solution map likely to (4.6), but by constructing a larger enhanced data set consisting of

$$:z^\ell: \left(\mathcal{D}(:z^{2m+1}:)\right)^{2m+1-\ell-j}, \quad \ell \in \{0,\dots,2m\}, j \in \{1,\dots,2m+1-\ell\}.$$

Proposition 5.1. *Fix $m \in \mathbb{N}$ and $\alpha \in \left(1 - \frac{1}{4m^2+2m+1}, 1\right)$. Then the map Ψ defined in* (5.2) *admits a unique fixed point in the space X^s, for some $s > 0$, on a time interval $[0,T]$ for some $T \le 1$. Consequently,* (1.29) *admits a local solution almost surely in the class $C_t([0,T]; H_x^{-(1-\alpha)-\varepsilon})$, for any $0 < \varepsilon \ll 1$. More precisely, the solution is of the form*

$$u = z + z_2 + w_2 \in C_t([0,T]; H_x^{-(1-\alpha)-\varepsilon}(\mathbb{T}^2)) + C_t([0,T]; W_x^{-(2m+1)(1-\alpha)+\alpha-\varepsilon,\infty}(\mathbb{T}^2)) + X^s.$$

Proof. We prove the estimates needed to show that the map defined in (5.2) is a contraction.

• Case 1. Estimate on the source term. Let us notice that for $j=0$, in the right-hand side of (5.2) there are no terms involving w_2, see (5.3). By Strichartz estimates and the embedding of Lebesgue spaces on finite measure domains,

$$\|(5.3)\|_{X^s} \lesssim \sum_{\ell=0}^{2m} \| :z^\ell: z_2^{2m+1-\ell}\|_{L^1_t H^{s-\alpha}_x} \lesssim T \sum_{\ell=0}^{2m} \|\langle\nabla\rangle^{s-\alpha}\left(:z^\ell: z_2^{2m+1-\ell}\right)\|_{L^\infty_t L^\infty_x}.$$

By Proposition 2.13 (iii) with $k=2m+1$, and $\ell=1,\dots,2m$, we have that $:z^\ell: z_2^{2m+1-\ell} \in C_t W_x^{-\ell(1-\alpha)-\varepsilon,\infty}$ for $\alpha > 1-\frac{1}{4m+3}$; thus, provided

$$s < \alpha(2m+1)-2m \tag{5.5}$$

we estimate

$$\|(5.3)\|_{X^s} \lesssim T\sum_{\ell=0}^{2m} \| :z^\ell: z_2^{2m+1-\ell}\|_{L^\infty_t W_x^{-\ell(1-\alpha)-\varepsilon,\infty}} \lesssim T.$$

• Case 2. Estimates on (5.4). Let us rewrite (5.4) as the

$$(5.4) = \sum_{\ell=0}^{2m}\sum_{j=1}^{2m+1-\ell} b_{m,\ell}c_{m,\ell,j}\int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha}\left(:z^\ell: z_2^{2m+1-\ell-j}w_2^j\right)dt' \tag{5.6}$$

$$= \sum_{\ell=1}^{2m}\sum_{j=1}^{2m+1-\ell} b_{m,\ell}c_{m,\ell,j}\int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha}\left(:z^\ell: z_2^{2m+1-\ell-j}w_2^j\right)dt' \tag{5.7}$$

$$+ \sum_{j=1}^{2m} b_{m,0}c_{m,0,j}\int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha}\left(z_2^{2m+1-j}w_2^j\right)dt' \tag{5.8}$$

$$+ \int_0^t \frac{\sin((t-t')\langle\nabla\rangle^\alpha)}{\langle\nabla\rangle^\alpha} w_2^{2m+1}dt'. \tag{5.9}$$

Similarly to the proof of Proposition 4.1, Case 1,

$$\|(5.9)\|_{X^s} \lesssim \|w_2\|^{2m+1}_{L^{2m+1}_t W_x^{s-\gamma_{p,q},q}} \lesssim T^\theta \|w_2\|^{2m+1}_{L^p_t W_x^{s-\gamma_{p,q},q}},$$

where in the last step we used the Lebesgue inclusion as $p>2m+1$, then $\theta>0$.

The estimates for (5.7) and (5.8) follows by performing an analogous strategy as in Proposition 4.1, Case 3: as in (4.13)

$$\|(5.7)+(5.8)\|_{X^s} \lesssim \|\langle\nabla\rangle^{-\delta}\left(:z^\ell: z_2^{2m+1-j-\ell}\right)\|_{L^\infty_t L^\infty_x}\|\langle\nabla\rangle^\delta w_2\|^j_{L^{2m}_t L^{2m\tilde q_2}_x}.$$

Hence, for $\delta > (2m+1)\alpha$, we have $\|(5.7)+(5.8)\|_{X^s} \lesssim \|\langle\nabla\rangle^\delta w_2\|^j_{L^{2m}_t L^{2m\tilde q_2}_x}$. At this point, the same considerations on the considered parameters as in Proposition 4.1, Case 3, yield to

$$\|(5.7)+(5.8)\|_{X^s} \lesssim \left(\|w_2\|^{1-\delta/s}_{L^{p_3}_t L^q_x}\|w_2\|^{\delta/s}_{L^\infty_t H^s_x}\right)^j$$

for $\frac{2m\alpha}{2m+1} < \alpha(2m+1) - 2m$, or equivalently

$$\alpha > 1 - \frac{1}{4m^2 + 2m + 1}.$$

The claim of the Proposition follows by a fixed point argument, with $s \in (s_0, s_1)$, where

$$s_0 = \frac{2m}{2m+1}\alpha \quad \text{and} \quad s_1 = \alpha(2m+1) - 2m.$$

By repeating the argument for $\Psi(w_2 - \tilde{w}_2)$ one shows that Φ is a contraction map and the proof is done. □

Appendix A. Unconditional almost sure global well-posedness

Inspired by [15] – where the authors prove almost sure global well-posedness of the hyperbolic Φ_2-model – by a simple Sobolev's embedding argument, we are able to show almost sure local well-posedness for some range of α. Note that the range of α covered by using Sobolev's embeddings instead of Strichartz, is smaller that the one obtained in Proposition (4.1).

We are going to prove that the map Ψ defined in (4.1) is a contraction map for small times, and we are going to prove the latter property for functions w with spatial regularity H^s, where $s \in (s_0, s_1)$ and

$$\begin{aligned} s_0 &= s_0(\alpha, m) = -\frac{1}{4m} + \frac{1+\alpha}{2} \\ s_1 &= s_1(\alpha, m) = -(2m+1) + (2m+2)\alpha, \end{aligned} \tag{A.1}$$

for some $\alpha \geq \bar{\alpha}(m) = 1 - \frac{1}{8m^2+6m}$.

Proposition A.1. *Fix $m \in \mathbb{N}$. Let $\alpha > 1 - \frac{1}{8m^2+6m}$ and $s \in (s_0, s_1)$, s_0, s_1 being defined in* (A.1)*. Then there exists a local solution to* (1.30) *in $C_t([0,T]; H^s_x(\mathbb{T}^2))$, almost surely.*

Remark A.2. Let us observe that for s satisfying (A.1), $s \geq 0$, thus the solution $u = z + w$ to (1.30) has spatial regularity

$$u \in H^{-(1-\alpha)-\varepsilon} + H^s \subset H^{-(1-\alpha)-\varepsilon}, \quad \varepsilon > 0.$$

Indeed, $-(1-\alpha) - \varepsilon < 0$, hence the claim follows by Sobolev's embeddings.

Proof of Proposition A.1 . In order to estimate the $2m+2$ terms appearing in (4.1), we distinguish three cases, i.e. (4.2),(4.3), and (4.4). In what follows, $L^\infty_t = L^\infty_t([0,T])$.

• Case 1, $\ell = 0$. We consider the term (4.2), thus $:z^\ell: \equiv 1$, namely a constant term. It is straightforward to get

$$\|(4.2)\|_{L^\infty_t H^s_x} \lesssim T\|w^{2m+1}\|_{L^\infty_t H^{s-\alpha}_x} \lesssim T\|w\|^{2m+1}_{L^\infty_t L_x^{\frac{2(2m+1)}{1-(s-\alpha)}}} \lesssim T\|w\|^{2m+1}_{L^\infty_t H^s_x}$$

where in the first inequality we used the Minkowski's inequality, along with the definition of the Sobolev spaces, and the Hölder's inequality; for the second estimate we used the (dual) Sobolev's embedding; and for the last inequality we used again the Sobolev's embedding, provided

$$s \geq 1 - \frac{\alpha}{2m}. \tag{A.2}$$

• Case 2, $\ell = 2m+1$. We consider here the term in (4.4). In this case we have to deal only with the stochastic term $:z^{2m+1}:$. We estimate

$$\begin{aligned}\|(4.4)\|_{L_t^\infty H_x^s} &\lesssim T\|\langle\nabla\rangle^{s-\alpha} :z^{2m+1}: \|_{L_t^\infty L_x^2}\\ &\lesssim T\|\langle\nabla\rangle^{-(2m+1)(1-\alpha)-\varepsilon} :z^{2m+1}: \|_{L_t^\infty L_x^\infty}\end{aligned}$$

provided

$$s < (2m+2)\alpha - (2m+1) \tag{A.3}$$

and $\varepsilon \ll 1$, where we used the Minkowski's and the Hölder's inequalities in the first line, the Sobolev's embedding and the inclusion of Lebesgue spaces in the second line.

• Case 3, $1 \leq \ell \leq m$. The last case is the one where we have to deal with both the stochastic terms and the deterministic terms. We estimate the term (4.3) as follows:

$$\begin{aligned}\|(4.3)\|_{L_t^\infty H_x^s} &\lesssim T\|\langle\nabla\rangle^{s-\alpha}\left(w^{2m+1-\ell} :z^\ell:\right)\|_{L_t^\infty L_x^2}\\ &\lesssim T\|\langle\nabla\rangle^{s-\alpha} :z^\ell: \|_{L_t^\infty L_x^\infty}\|\langle\nabla\rangle^{\alpha-s}w^{2m+1-\ell}\|_{L_t^\infty L_x^{2(2m+1-\ell)}}\\ &\lesssim T\|\langle\nabla\rangle^{-\ell(1-\alpha)-\varepsilon} :z^\ell: \|_{L_t^\infty L_x^\infty}\|\langle\nabla\rangle^{\alpha-s}w^{2m+1-\ell}\|_{L_T^\infty L_x^{2(2m+1-\ell)}}\\ &\lesssim T\|\langle\nabla\rangle^{-\ell(1-\alpha)-\varepsilon} :z^\ell: \|_{L_t^\infty L_x^\infty}\|\langle\nabla\rangle^{\alpha-s}w\|^{2m+1-\ell}_{L_t^\infty L_x^{2(2m+1-\ell)}}\\ &\lesssim T\|\langle\nabla\rangle^{-\ell(1-\alpha)-\varepsilon} :z^\ell: \|_{L_t^\infty L_x^\infty}\|\langle\nabla\rangle^{\alpha-s}w\|^{2m+1-\ell}_{L_t^\infty L_x^{2(2m)}}\\ &\lesssim T\|\langle\nabla\rangle^{-\ell(1-\alpha)-\varepsilon} :z^\ell: \|_{L_t^\infty L_x^\infty}\|w\|^{2m+1-\ell}_{L_t^\infty H_x^s}\end{aligned}$$

provided that

$$s < -\ell + \alpha(\ell+1) \tag{A.4}$$

and $\varepsilon \ll 1$, and

$$s \geq -\frac{1}{4m} + \frac{1+\alpha}{2}. \tag{A.5}$$

In the chain of inequalities above, we used the Minkowski's and the Hölder's inequalities in the first line; the estimate (2.2) and the inclusion of Lebesgue spaces in the second line; the Sobolev's embedding in the third line, for s satisfying (A.4); the Leibniz rule (2.1) in the fourth line; the embedding between Lebesgue spaces in the fifth line; and in the last line we used the Sobolev's embedding, provided that s satisfies (A.5).
Let us notice that (A.4) is automatically verified once we restrict the value of s as in (A.3). Hence, by putting together (A.2), (A.3), and (A.5), we constraint the parameters

to fulfil

$$\begin{aligned} &s < -(2m+1) + (2m+2)\alpha \\ &s \geq \max\left\{-\frac{1}{4m} + \frac{1+\alpha}{2}, 1 - \frac{\alpha}{2m}\right\} = -\frac{1}{4m} + \frac{1+\alpha}{2} \end{aligned} \tag{A.6}$$

which in turn implies, in order to guarantee that (A.6) is a non-empty condition, that

$$\alpha > 1 - \frac{1}{8m^2 + 6m}.$$

By repeating the argument for $\Psi(w - \tilde{w})$ one shows that Φ is a contraction map and the proof is done.

Remark A.3. Note that the range $(1 - \frac{1}{8m^2+6m}, 1)$ is smaller compared to the one obtained by performing a fixed point argument in a Strichartz space instead of only Sobolev.

□

Appendix B. Exponential integrability by Nelson-type tail estimate

We show here an exponential integrability result by means of a Nelson-type tail estimate. The obtained result is weaker than the one obtained in Theorem 1.2 using the Barashkov-Gubinelli approach [1]. We report the following result to highlight how the variational approach is stronger to get the exponential integrability property for the construction of the Gibbs measure.

Proposition B.1. *Let $m \in \mathbb{N}$ and let $\alpha \in \left(1 - \frac{1}{4m^2+4m+2}, 1\right)$. For any fixed $1 \leq p \leq \infty$, the sequence $\{R_N(u)\}_{N\in\mathbb{N}}$ is uniformly bounded in $L^p(\mu_\alpha)$. Moreover, $R_N(u)$ converges in $L^p(\mu_\alpha)$, and we denote by $R(u)$ its limit:*

$$R(u) = e^{-\frac{1}{2m+2}G(u)} = e^{-\frac{1}{2m+2}\int_{\mathbb{T}^2} :u^{2m+2}: dx} = L^p\text{-}\lim_{N\to\infty} R_N(u) \tag{B.1}$$

Remark B.2. Note that the range of fractional exponents in Proposition B.1 is smaller compared to the one in Theorem 1.2.

We recall the following Lemma, see [32, Lemma 4.5].

Lemma B.3. *Let F be a real valued measurable function on H. Suppose that there exist $\beta > 0$, $N, k \in \mathbb{N}$, and $C > 0$ such that for every $p \geq 2$*

$$\|F\|_{L^p(\mu_\alpha)} \leq CN^{-\beta} p^{k/2},$$

then, for some $\delta > 0$ and $\tilde{C} > 0$

$$\mu_\alpha\{u \in H \text{ s.t. } |F(u)| > \lambda\} \leq \tilde{C} e^{-\delta N^{2\beta/k}\lambda^{2/k}}.$$

Proof of Proposition B.1. Let us observe that by (1.17), (3.5), and the boundedness from below of the Hermite polynomial H_{2m+2} by a constant denoted by $-c(m)$, we obtain, by using (1.9),

$$-G_N(u) \leq c(m) N^{(2-2\alpha)(m+1)}. \tag{B.2}$$

We use the layer cake representation of the L^p-norm of a function to estimate

$$\|R_N(u)\|_{L^p(\mu_\alpha)}^p = \int_0^\infty \mu_\alpha\{e^{-pG_N(u)} > \tau\}\, d\tau \lesssim 1 + \int_{\tau_0}^\infty \mu_\alpha\{-pG_N(u) > \log\tau\}\, d\tau.$$

The finiteness of the L^p-norm of $R_N(u)$, and its uniform bound in N, follows if we give a suitable decay estimate for the function $f(\tau) = \mu_\alpha\{-pG_N(u) > \log\tau\}$ guaranteeing the convergence of the integral over (τ_0, ∞). Namely we are going to prove a Nelson-type tail estimate.

Let N_0 be an integer such that $\log\tau = 2pc(m)N_0^{(2-2\alpha)(m+1)}$, where $c(m)$ is as in (B.2). Hence it is easy to see that $\mu_\alpha\{-pG_N(u) > \log\tau\} = 0$ for any $N \le N_0$.

Consider now $N > N_0$. As $-2pG_{N_0}(u) \le \log\tau$, it is straightforward to estimate

$$\begin{aligned}\mu_\alpha\{-pG_N(u) > \log\tau\} &\le \mu_\alpha\left\{-pG_N(u) > \log\tau - pG_{N_0}(u) - \frac{\log\tau}{2}\right\}\\ &= \mu_\alpha\left\{-pG_N(u) + pG_{N_0}(u) > \frac{\log\tau}{2}\right\}\\ &\le Ce^{-\delta(\log\tau)^{1/(m+1)}N_0^{(\alpha-1/2)/(m+1)}}\\ &= Ce^{-\delta(\log\tau)^{1/(m+1)}(\log\tau)^{(\alpha-1/2)/(2-2\alpha)(m+1)^2}} := g(\tau),\end{aligned}$$

where we used Lemma B.3 for the second inequality above. Let us denote $c(\alpha) = \frac{2\alpha-1}{4-4\alpha} > 0$. By a change of variable we have:

$$\int_{\tau_0}^\infty g(\tau)\, d\tau \lesssim \int_{s_0}^\infty \frac{e^s}{e^{s^{1/(m+1)}s^{c(\alpha)/(m+1)^2}}}\, ds$$

which is convergent provided we have

$$\frac{1}{m+1} + \frac{c(\alpha)}{(m+1)^2} > 1 \iff \alpha > 1 - \frac{1}{4m^2+4m+2}.$$

This shows the uniform bound of the sequence $R_N(u)$ in L^p. To conclude with the convergence of $R_N(u)$ to $R(u)$ as defined in (B.1), we observe that, as a consequence of the convergence in measure of $G_N(u)$, $R_N(u)$ converges to $R(u)$ in measure as well. Thus, the Hölder's inequality gives the result. Indeed,

$$\begin{aligned}\|R - R_N\|_{L^p} &\le \|(R-R_N)\chi_{\{|R_N(\cdot)-R(\cdot)|\le\varepsilon/2\}}\|_{L^p} + \|(R-R_N)\chi_{\{|R_N(\cdot)-R(\cdot)|\ge\varepsilon/2\}}\|_{L^p}\\ &\le \frac{\varepsilon}{2}(\mu_\alpha\{|R_N(\cdot) - R(\cdot)| \le \varepsilon/2\})^{1/p}\\ &+ (\|R\|_{L^{2p}} + \|R_N\|_{L^{2p}})(\mu_\alpha\{|R_N(\cdot) - R(\cdot)| \le \varepsilon/2\})^{1/2p} \le \varepsilon.\end{aligned}$$

□

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