DOUBLE SCATTERING CHANNELS FOR $1D$ NLS IN THE ENERGY SPACE AND ITS GENERALIZATION TO HIGHER DIMENSIONS

LUIGI FORCELLA AND NICOLA VISCIGLIA

Abstract. We consider a class of $1D$ NLS perturbed with a steplike potential. We prove that the nonlinear solutions satisfy the double scattering channels in the energy space. The proof is based on concentration-compactness/rigidity method. We prove moreover that in dimension higher than one, classical scattering holds if the potential is periodic in all but one dimension and is steplike and repulsive in the remaining one.

1. Introduction

The main motivation of this paper is the analysis of the behavior for large times of solutions to the following $1D$ Cauchy problems (see below for a suitable generalization in higher dimensions):

$$\begin{cases} i\partial_t u + \partial_x^2 u - Vu = |u|^\alpha u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \quad \alpha > 4 \\ u(0) = u_0 \in H^1(\mathbb{R}) \end{cases}, \tag{1.1}$$

namely we treat the L^2-supercritical defocusing power nonlinearities, and $V : \mathbb{R} \to \mathbb{R}$ is a real time-independent steplike potential. More precisely we assume that $V(x)$ has two different asymptotic behaviors at $\pm\infty$:

$$a_+ = \lim_{x\to+\infty} V(x) \neq \lim_{x\to-\infty} V(x) = a_-. \tag{1.2}$$

In order to simplify the presentation we shall assume in our treatment

$$a_+ = 1 \quad \text{and} \quad a_- = 0,$$

but of course the arguments and the results below can be extended to the general case $a_+ \neq a_-$. Roughly speaking the Cauchy problem (1.1) looks like the following Cauchy problems respectively for $x >> 0$ and $x << 0$:

$$\begin{cases} i\partial_t v + \partial_x^2 v = |v|^\alpha v \\ v(0) = v_0 \in H^1(\mathbb{R}) \end{cases} \tag{1.3}$$

and

$$\begin{cases} i\partial_t v + (\partial_x^2 - 1)v = |v|^\alpha v \\ v(0) = v_0 \in H^1(\mathbb{R}) \end{cases}. \tag{1.4}$$

We recall that in $1D$, the long time behavior of solutions to (1.3) (and also to (1.4)) has been first obtained in the work by Nakanishi (see [N]), who proved that the solutions to (1.3) (and also (1.4)) scatter to a free wave in $H^1(\mathbb{R})$ (see Definition 1.4 for a precise definition of scattering from nonlinear to linear solutions in a general framework). The Nakanishi argument is a combination of the induction on the energy

in conjunction with a suitable version of Morawetz inequalities with time-dependent weights. Alternative proofs based on the use of the interaction Morawetz estimates, first introduced in [CKSTT], have been obtained later (see [CHVZ, CGT, PV, V] and the references therein).
As far as we know, there are not results available in the literature about the long time behavior of solutions to NLS perturbed by a steplike potential, and this is the main motivation of this paper.

We recall that in physics literature the steplike potentials are called *barrier potentials* and are very useful to study the interactions of particles with the boundary of a solid (see Gesztesy [G] and Gesztesy, Noewll and Pötz [GNP] for more details). We also mention the paper [DS] where, in between other results, it is studied via the twisting trick the long time behavior of solutions to the propagator $e^{it(\partial_x^2-V)}$, where $V(x)$ is steplike (see below for more details on the definition of the double scattering channels). For a more complete list of references devoted to the analysis of steplike potentials we refer to [D'AS]. Neverthless, at the best of our knowledge, no results are available about the long time behavior of solutions to nonlinear Cauchy problem (1.1) with a steplike potential.

It is worth mentioning that in $1D$, we can rely on the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Hence it is straightforward to show that the Cauchy problem (1.1) is locally well posed in the energy space $H^1(\mathbb{R})$. For higher dimensions the local well posedness theory is still well known, see for example Cazenave's book [CT], once the good dispersive properties of the linear flow are established. Moreover, thanks to the defocusing character of the nonlinearity, we can rely on the conservation of the mass and of the energy below, valid in any dimension:

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = \|u(0)\|_{L^2(\mathbb{R}^d)}, \tag{1.5}$$

and

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^d}\left(|\nabla u(t)|^2 + V|u(t)|^2 + \frac{2}{\alpha+2}|u(t)|^{\alpha+2}\right)dx = E(u(0)), \tag{1.6}$$

in order to deduce that the solutions are global. Hence for any initial datum $u_0 \in H^1(\mathbb{R}^d)$ there exists one unique global solution $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ to (1.1) for $d=1$ (and to (1.8) below in higher dimension).

It is well-known that a key point in order to study the long time behavior of nonlinear solutions is a good knowledge of the dispersive properties of the linear flow, namely the so called Strichartz estimates. A lot of works have been written in the literature about the topic, both in $1D$ and in higher dimensions. We briefly mention [AY, CK, D'AF, GS, W1, W2, Y] for the one dimensional case and [BPST, GVV, JK, JSS, R, RS] for the higher dimensional case, referring to the bibliographies contained in these papers for a more detailed list of works on the subject. It is worth mentioning that in all the papers mentioned above the potential perturbation is assumed to decay at infinity, hence steplike potential are not allowed. Concerning contributions in the literature to NLS perturbed by a decaying potential we have several results, in between we quote the following most recent ones: [BV, CR, CGV, GHW, H, La, Li, LZ], and all the references therein.

At the best of our knowledge, the unique paper where the dispersive properties of the corresponding $1D$ linear flow perturbed by a steplike potential $V(x)$ have been analyzed is [D'AS], where the $L^1 - L^\infty$ decay estimate in $1D$ is proved:

$$\|e^{it(\partial_x^2 - V)} f\|_{L^\infty(\mathbb{R})} \lesssim |t|^{-1/2} \|f\|_{L^1(\mathbb{R})}, \quad \forall\, t \neq 0 \quad \forall\, f \in L^1(\mathbb{R}). \tag{1.7}$$

We point out that beside the different spatial behavior of $V(x)$ on left and on right of the line, other assumptions must be satisfied by the potential. There is a huge literature devoted to those spectral properties, nevertheless we shall not focus on it since our main point is to show how to go from (1.7) to the analysis of the long time behavior of solutions to (1.1). We will assume therefore as black-box the dispersive relation (1.7) and for its proof, under further assumptions on the steplike potential $V(x)$, we refer to Theorem 1.1 in [D'AS]. Our first aim is to provide a nonlinear version of the *double scattering channels* that has been established in the literature in the linear context (see [DS]).

Definition 1.1. Let $u_0 \in H^1(\mathbb{R})$ be given and $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}))$ be the unique global solution to (1.1) with $V(x)$ that satisfies (1.2) with $a_- = 0$ and $a_+ = 1$. Then we say that $u(t,x)$ satisfies the *double scattering channels* provided that

$$\lim_{t \to \pm\infty} \|u(t,x) - e^{it\partial_x^2}\eta_\pm - e^{it(\partial_x^2 - 1)}\gamma_\pm\|_{H^1(\mathbb{R})} = 0,$$

for suitable $\eta_\pm, \gamma_\pm \in H^1(\mathbb{R})$.

We can now state our first result in $1D$.

Theorem 1.2. *Assume that $V : \mathbb{R} \to \mathbb{R}$ is a bounded, nonnegative potential satisfying (1.2) with $a_- = 0$ and $a_+ = 0$, and (1.7). Furthermore, suppose that:*

- $|\partial_x V(x)| \overset{|x|\to\infty}{\longrightarrow} 0$;
- $\lim_{x\to+\infty} |x|^{1+\varepsilon}|V(x) - 1| = 0$, $\lim_{x\to-\infty} |x|^{1+\varepsilon}|V(x)| = 0$ *for some* $\epsilon > 0$;
- $x \cdot \partial_x V(x) \leq 0$.

Then for every $u_0 \in H^1(\mathbb{R})$ the corresponding unique solution $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ to (1.1) satisfies the double scattering channels *(according to Definition 1.1).*

Remark 1.3. It is worth mentioning that the assumption (1.7) it may look somehow quite strong. However we emphasize that the knowledge of the estimate (1.7) provides for free informations on the long time behavior of nonlinear solutions for small data, but in general it is more complicated to deal with large data, as it is the case in Theorem 1.2. For instance consider the case of $1D$ NLS perturbed by a periodic potential. In this situation it has been established in the literature the validity of the dispersive estimate for the linear propagator (see [Cu]) and also the small data nonlinear scattering ([CuV]). However, at the best of our knowledge, it is unclear how to deal with the large data scattering.

The proof of Theorem 1.2 goes in two steps. The first one is to show that solutions to (1.1) scatter to solutions of the linear problem (see Definition 1.4 for a rigorous definition of scattering in a general framework); the second one is the asymptotic description of solutions to the linear problem associated with (1.1) in the energy space H^1 (see Theorem 1.8). Concerning the first step we use the technique of concentration-compactness/rigidity pioneered by Kenig and Merle (see [KM1, KM2]). Since this argument is rather general, we shall present it in a more

general higher dimensional setting.
More precisely in higher dimension we consider the following family of NLS

$$\begin{cases} i\partial_t u + \Delta u - Vu = |u|^\alpha u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0) = u_0 \in H^1(\mathbb{R}^d) \end{cases}, \tag{1.8}$$

where

$$\begin{cases} \frac{4}{d} < \alpha < \frac{4}{d-2} & \text{if} \quad d \geq 3 \\ \frac{4}{d} < \alpha & \text{if} \quad d \leq 2 \end{cases}.$$

The potential $V(x)$ is assumed to satisfy, uniformly in $\bar{x} \in \mathbb{R}^{d-1}$,

$$a_- = \lim_{x_1 \to -\infty} V(x_1, \bar{x}) \neq \lim_{x_1 \to +\infty} V(x_1, \bar{x}) = a_+, \quad \text{where} \quad x = (x_1, \bar{x}). \tag{1.9}$$

Moreover we assume $V(x)$ periodic w.r.t. the variables $\bar{x} = (x_2, \dots, x_d)$. Namely we assume the existence of $d-1$ linear independent vectors $P_2, \dots, P_d \in \mathbb{R}^{d-1}$ such that for any fixed $x_1 \in \mathbb{R}$, the following holds:

$$\begin{gathered} V(x_1, \bar{x}) = V(x_1, \bar{x} + k_2 P_2 + \cdots + k_d P_d), \\ \forall\, \bar{x} = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}, \quad \forall\, (k_2, \dots, k_d) \in \mathbb{Z}^{d-1}. \end{gathered} \tag{1.10}$$

Some comments about this choice of assumptions on $V(x)$ are given in Remark 1.6.

Exactly as in $1D$ case mentioned above, we assume as a black-box the dispersive estimate

$$\|e^{it(\Delta - V)} f\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2} \|f\|_{L^1(\mathbb{R}^d)}, \quad \forall\, t \neq 0 \quad \forall\, f \in L^1(\mathbb{R}^d). \tag{1.11}$$

Next we recall the classical definition of scattering from nonlinear to linear solutions in a general setting. We recall that by classical arguments we have that once (1.11) is granted, then the local (and also the global, since the equation is defocusing) existence and uniqueness of solutions to (1.8) follows by standard arguments.

Definition 1.4. Let $u_0 \in H^1(\mathbb{R}^d)$ be given and $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ be the unique global solution to (1.8). Then we say that $u(t,x)$ *scatters to a linear solution* provided that

$$\lim_{t \to \pm\infty} \|u(t,x) - e^{it(\Delta - V)} \psi^\pm\|_{H^1(\mathbb{R}^d)} = 0$$

for suitable $\psi^\pm \in H^1(\mathbb{R}^d)$.

In the sequel we will also use the following auxiliary Cauchy problems that roughly speaking represent the Cauchy problems (1.8) in the regions $x_1 << 0$ and $x_1 >> 0$ (provide that we assume $a_- = 0$ and $a_+ = 1$ in (1.9)):

$$\begin{cases} i\partial_t u + \Delta u = |u|^\alpha u & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0) = \psi \in H^1(\mathbb{R}^d) \end{cases}, \tag{1.12}$$

and

$$\begin{cases} i\partial_t u + (\Delta - 1) u = |u|^\alpha u & (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u(0) = \psi \in H^1(\mathbb{R}^d) \end{cases}. \tag{1.13}$$

Notice that those problems are respectively the analogue of (1.3) and (1.4) in higher dimensional setting.

We can now state our main result about scattering from nonlinear to linear solutions in general dimension $d \geq 1$.

Theorem 1.5. *Let $V \in \mathcal{C}^1(\mathbb{R}^d;\mathbb{R})$ be a bounded, nonnegative potential which satisfies (1.9) with $a_- = 0$, $a_+ = 1$, (1.10) and assume moreover:*

- $|\nabla V(x_1, \bar{x})| \overset{|x_1|\to\infty}{\longrightarrow} 0$ *uniformly in* $\bar{x} \in \mathbb{R}^{d-1}$;
- *the decay estimate (1.11) is satisfied;*
- $x_1 \cdot \partial_{x_1} V(x) \leq 0$ *for any* $x \in \mathbb{R}^d$.

Then for every $u_0 \in H^1(\mathbb{R}^d)$ the unique corresponding global solution $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d))$ to (1.8) scatters.

Remark 1.6. Next we comment about the assumptions done on the potential $V(x)$ along Theorem 1.5. Roughly speaking we assume that the potential $V(x_1, \ldots, x_d)$ is steplike and repulsive w.r.t. x_1 and it is periodic w.r.t. $(x_2, \ldots, x_d)$. The main motivation of this choice is that this situation is reminiscent, according with [DS], of the higher dimensional version of the $1D$ double scattering channels mentioned above. Moreover we highlight the fact that the repulsivity of the potential in one unique direction is sufficient to get scattering, despite to other situations considered in the literature where repulsivity is assumed w.r.t. the full set of variables $(x_1, \ldots, x_d)$. Another point is that along the proof of Theorem 1.5 we show how to deal with a partially periodic potential $V(x)$, despite to the fact that, at the best of our knowledge, the large data scattering for potentials periodic w.r.t. the full set of variables has not been established elsewhere, either in the $1D$ case (see Remark 1.3).

Remark 1.7. Next we comment about the repulsivity assumption on $V(x)$. As pointed out in [H], this assumption on the potential plays the same role of the convexity assumption for the obstacle problem studied by Killip, Visan and Zhang in [KVZ]. The author highlights the fact that both strict convexity of the obstacle and the repulsivity of the potential prevent wave packets to refocus once they are reflected by the obstacle or by the potential. From a technical point of view the repulsivity assumption is done in order to control the right sign in the virial identities, and hence to conclude the rigidity part of the Kenig and Merle argument. In this paper, since we assume repulsivity only in one direction we use a suitable version of the Nakanishi-Morawetz time-dependent estimates in order to get the rigidity part in the Kenig and Merle road map. Of course it is a challenging mathematical question to understand whether or not the repulsivity assumption (partial or global) on $V(x)$ is a necessary condition in order to get scattering.

When we specialize in $1D$, we are able to complete the theory of double scattering channels in the energy space. Therefore how to concern the linear part of our work, we give the following result, that in conjunction with Theorem 1.5 where we fix $d = 1$, provides the proof of Theorem 1.2.

Theorem 1.8. *Assume that $V(x) \in \mathcal{C}(\mathbb{R};\mathbb{R})$ satisfies the following space decay rate:*

$$\lim_{x\to+\infty} |x|^{1+\varepsilon}|V(x) - 1| = \lim_{x\to-\infty} |x|^{1+\varepsilon}|V(x)| = 0 \quad \text{for some} \quad \varepsilon > 0. \tag{1.14}$$

Then for every $\psi \in H^1(\mathbb{R})$ we have

$$\lim_{t\to\pm\infty} \|e^{it(\partial_x^2 - V)}\psi - e^{it\partial_x^2}\eta_\pm - e^{it(\partial_x^2-1)}\gamma_\pm\|_{H^1(\mathbb{R})} = 0$$

for suitable $\eta_\pm, \gamma_\pm \in H^1(\mathbb{R})$.

Notice that Theorem 1.8 is a purely linear statement. The main point (compared with other results in the literature) is that the asymptotic convergence is stated with respect to the H^1 topology and not with respect to the weaker L^2 topology. Indeed we point out that the content of Theorem 1.8 is well-known and has been proved in [DS] in the L^2 setting. However, it seems natural to us to understand, in view of Theorem 1.5, whether or not the result can be extended in the H^1 setting. In fact according with Theorem 1.5 the asymptotic convergence of the nonlinear dynamic to linear dynamic occurs in the energy space and not only in L^2. As far as we know the issue of H^1 linear scattering has not been previously discussed in the literature, not even in the case of a potential which decays in both directions $\pm\infty$.

For this reason we have decided to state Theorem 1.8 as an independent result.

1.1. **Notations.**

The spaces $L^p_I L^q = L^p_t(I; L^q_x(\mathbb{R}^d))$ are the usual time-space Lebesgue mixed spaces endowed with norm defined by

$$\|u\|_{L^p_t(I;L^q_x(\mathbb{R}^d))} = \left(\int_I \left|\int_{\mathbb{R}^d} |u(t,x)|^q\,dx\right|^{p/q} dt\right)^{1/p}$$

and by the context it will be clear which interval $I \subseteq \mathbb{R}$, bounded or unbounded, is considered. If $I = \mathbb{R}$ we will lighten the notation by writing L^pL^q. The operator τ_z will denote the translation operator $\tau_z f(x) := f(x-z)$. If $z \in \mathbb{C}$, $\Re z$ and $\Im z$ are the common notations for the real and imaginary parts of a complex number and $\bar{z}$ is its complex conjugate.

In what follows, when dealing with a dimension $d \geq 2$, we write $\mathbb{R}^d \ni x := (x_1, \bar{x})$ with $\bar{x} \in \mathbb{R}^{d-1}$. For $x \in \mathbb{R}^d$ the quantity $|x|$ will denote the usual norm in $\mathbb{R}^d$.

With standard notation, the Hilbert spaces $L^2(\mathbb{R}^d), H^1(\mathbb{R}^d), H^2(\mathbb{R}^d)$ will be denoted simply by L^2, H^1, H^2 and likely for all the Lebesgue $L^p(\mathbb{R}^d)$ spaces. By $(\cdot,\cdot)_{L^2}$ we means the usual L^2-inner product, i.e. $(f,g)_{L^2} = \int_{\mathbb{R}^d} f\bar{g}\,dx$, $\forall\, f,g \in L^2$, while the energy norm $\mathcal{H}$ is the one induced by the inner product $(f,g)_{\mathcal{H}} := (f,g)_{\dot{H}^1} + (Vf,g)_{L^2}$.

Finally, if $d \geq 3$, $2^* = \frac{2d}{d-2}$ is the Sobolev conjugate of 2 (2^* being $+\infty$ in dimension $d \leq 2$), while if $1 \leq p \leq \infty$ then p' is the conjugate exponent given by $p' = \frac{p}{p-1}$.

2. Strichartz Estimates

The well known Strichartz estimates are a basic tool in the studying of the nonlinear Schrödinger equation and we will assume the validity of them in our context. Roughly speaking, we can say that these essential space-time estimates arise from the so-called dispersive estimate for the Schrödinger propagator

$$\|e^{it(\Delta-V)}f\|_{L^\infty} \lesssim |t|^{-d/2}\|f\|_{L^1}, \quad \forall\, t \neq 0 \quad \forall\, f \in L^1, \tag{2.1}$$

which is proved in $1D$ in [D'AS], under suitable assumptions on the steplike potential $V(x)$, and we take for granted by hypothesis.

As a first consequence we get the following Strichartz estimates

$$\|e^{it(\Delta-V)}f\|_{L^aL^b} \lesssim \|f\|_{L^2}$$

where $a, b \in [1, \infty]$ are assumed to be Strichartz admissible, namely

$$\frac{2}{a} = d\left(\frac{1}{2} - \frac{1}{b}\right). \tag{2.2}$$

We recall, as already mentioned in the introduction, that along our paper we are assuming the validity of the dispersive estimate (2.1) also in higher dimensional setting.

We fix from now on the following Lebesgue exponents

$$r = \alpha + 2, \qquad p = \frac{2\alpha(\alpha+2)}{4-(d-2)\alpha}, \qquad q = \frac{2\alpha(\alpha+2)}{d\alpha^2-(d-2)\alpha-4}.$$

(where α is given by the nonlinearity in (1.8)). Next, we give the linear estimates that will be fundamental in our study:

$$\|e^{it(\Delta-V)}f\|_{L^{\frac{4(\alpha+2)}{d\alpha}}L^r} \lesssim \|f\|_{H^1}, \tag{2.3}$$

$$\|e^{it(\Delta-V)}f\|_{L^{\frac{2(d+2)}{d}}L^{\frac{2(d+2)}{d}}} \lesssim \|f\|_{H^1}, \tag{2.4}$$

$$\|e^{it(\Delta-V)}f\|_{L^pL^r} \lesssim \|f\|_{H^1}. \tag{2.5}$$

The last estimate that we need is (some in between) the so-called inhomogeneous Strichartz estimate for non-admissible pairs:

$$\left\|\int_0^t e^{i(t-s)(\Delta-V)}g(s)\,ds\right\|_{L^pL^r} \lesssim \|g\|_{L^{q'}L^{r'}}, \tag{2.6}$$

whose proof is contained in [CW].

Remark 2.1. In the unperturbed framework, i.e. in the absence of the potential, and for general dimensions, we refer to [FXC] for comments and references about Strichartz estimates (2.3), (2.4), (2.5) and (2.6).

3. Perturbative nonlinear results

The results in this section are quite standard and hence we skip the complete proofs which can be found for instance in [BV, CT, FXC]. In fact the arguments involved are a compound of dispersive properties of the linear propagator and a standard perturbation argument.

Along this section we assume that the estimate (1.11) is satisfied by the propagator associated with the potential $V(x)$. We do not need for the moment to assume the other assumptions done on $V(x)$.

We also specify that in the sequel the couple (p, r) is the one given in Section 2.

Lemma 3.1. *Let $u_0 \in H^1$ and assume that the corresponding solution to (1.8) satisfies $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1) \cap L^pL^r$. Then $u(t,x)$* scatters *to a linear solution in H^1.*

Proof. It is a standard consequence of Strichartz estimates. □

Lemma 3.2. *There exists $\varepsilon_0 > 0$ such that for any $u_0 \in H^1$ with $\|u_0\|_{H^1} \leq \varepsilon_0$, the solution $u(t,x)$ to the Cauchy problem (1.8)* scatters *to a linear solution in H^1.*

Proof. It is a simple consequence of Strichartz estimates. □

Lemma 3.3. *For every $M > 0$ there exist $\varepsilon = \varepsilon(M) > 0$ and $C = C(M) > 0$ such that: if $u(t,x) \in \mathcal{C}(\mathbb{R}; H^1)$ is the unique global solution to (1.8) and $w \in \mathcal{C}(\mathbb{R}; H^1) \cap L^pL^r$ is a global solution to the perturbed problem*

$$\begin{cases} i\partial_t w + \Delta w - Vw = |w|^\alpha w + e(t,x) \\ w(0,x) = w_0 \in H^1 \end{cases}$$

satisfying the conditions $\|w\|_{L^pL^r} \leq M$, $\|\int_0^t e^{i(t-s)(\Delta - V)} e(s)\, ds\|_{L^pL^r} \leq \varepsilon$ and $\|e^{it(\Delta-V)}(u_0 - w_0)\|_{L^pL^r} \leq \varepsilon$, then $u \in L^pL^r$ and $\|u - w\|_{L^pL^r} \leq C\varepsilon$.

Proof. The proof is contained in [FXC], see Proposition 4.7, and it relies on (2.6). □

4. Profile decomposition

The main content of this section is the following profile decomposition theorem.

Theorem 4.1. *Let $V(x) \in L^\infty$ satisfies: $V \geq 0$, (1.10), (1.9) with $a_- = 0$ and $a_+ = 1$, the dispersive relation (1.11) and suppose that $|\nabla V(x_1, \bar{x})| \to 0$ as $|x_1| \to \infty$ uniformly in $\bar{x} \in \mathbb{R}^{d-1}$. Given a bounded sequence $\{v_n\}_{n\in\mathbb{N}} \subset H^1$, $\forall\, J \in \mathbb{N}$ and $\forall\, 1 \leq j \leq J$ there exist two sequences $\{t_n^j\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{x_n^j\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ and $\psi^j \in H^1$ such that, up to subsequences,*

$$v_n = \sum_{1\leq j\leq J} e^{it_n^j(\Delta - V)} \tau_{x_n^j} \psi^j + R_n^J$$

with the following properties:

- *for any fixed j we have the following dichotomy for the time parameters t_n^j:*
$$\text{either} \quad t_n^j = 0 \quad \forall\, n \in \mathbb{N} \quad \text{or} \quad t_n^j \overset{n\to\infty}{\longrightarrow} \pm\infty;$$
- *for any fixed j we have the following scenarios for the space parameters $x_n^j = (x_{n,1}^j, \bar{x}_n^j) \in \mathbb{R} \times \mathbb{R}^{d-1}$:*
$$\text{either} \quad x_n^j = 0 \quad \forall\, n \in \mathbb{N}$$
$$\text{or} \quad |x_{n,1}^j| \overset{n\to\infty}{\longrightarrow} \infty$$
$$\text{or} \quad x_{n,1}^j = 0, \quad \bar{x}_n^j = \sum_{l=2}^d k_{n,l}^j P_l \quad \text{with} \quad k_{n,l}^j \in \mathbb{Z} \quad \text{and} \quad \sum_{l=2}^d |k_{n,l}^j| \overset{n\to\infty}{\longrightarrow} \infty,$$
where P_l are given in (1.10);
- *(orthogonality condition) for any $j \neq k$*
$$|x_n^j - x_n^k| + |t_n^j - t_n^k| \overset{n\to\infty}{\longrightarrow} \infty;$$
- *(smallness of the remainder) $\forall\, \varepsilon > 0 \quad \exists\, J = J(\varepsilon)$ such that*
$$\limsup_{n\to\infty} \|e^{it(\Delta - V)} R_n^J\|_{L^pL^r} \leq \varepsilon;$$
- *by defining $\|v\|_{\mathcal{H}}^2 = \int (|\nabla v|^2 + V|v|^2) dx$ we have, as $n \to \infty$,*
$$\|v_n\|_{L^2}^2 = \sum_{1\leq j\leq J} \|\psi^j\|_{L^2}^2 + \|R_n^J\|_{L^2}^2 + o(1), \quad \forall\, J \in \mathbb{N},$$
$$\|v_n\|_{\mathcal{H}}^2 = \sum_{1\leq j\leq J} \|\tau_{x_n^j} \psi^j\|_{\mathcal{H}}^2 + \|R_n^J\|_{\mathcal{H}}^2 + o(1), \quad \forall\, J \in \mathbb{N};$$

- $\forall\, J \in \mathbb{N}$ *and* $\forall\ 2 < q < 2^*$ *we have, as* $n \to \infty$,

$$\|v_n\|_{L^q}^q = \sum_{1\le j\le J} \|e^{it_n^j(\Delta - V)}\tau_{x_n^j}\psi^j\|_{L^q}^q + \|R_n^J\|_{L^q}^q + o(1);$$

- *with* $E(v) = \frac{1}{2}\int \left(|\nabla v|^2 + V|v|^2 + \frac{2}{\alpha+2}|v|^{\alpha+2}\right)dx$, *we have, as* $n \to \infty$,

$$E(v_n) = \sum_{1\le j\le J} E(e^{it_n^j(\Delta - V)}\tau_{x_n^j}\psi^j) + E(R_n^j) + o(1), \quad \forall\, J \in \mathbb{N}. \tag{4.1}$$

First we prove the following lemma.

Lemma 4.2. *Given a bounded sequence* $\{v_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^d)$ *we define*

$$\Lambda = \left\{ w \in L^2 \quad | \quad \exists \{x_k\}_{k\in\mathbb{N}} \quad \text{and} \quad \{n_k\}_{k\in\mathbb{N}} \quad \text{s. t.} \quad \tau_{x_k} v_{n_k} \overset{L^2}{\rightharpoonup} w \right\}$$

and

$$\lambda = \sup\{\|w\|_{L^2}, \quad w \in \Lambda\}.$$

Then for every $q \in (2, 2^*)$ *there exists a constant* $M = M(\sup_n \|v_n\|_{H^1}) > 0$ *and an exponent* $e = e(d,q) > 0$ *such that*

$$\limsup_{n\to\infty} \|v_n\|_{L^q} \le M\lambda^e.$$

Proof. We consider a Fourier multiplier ζ where ζ is defined as

$$C_c^\infty(\mathbb{R}^d;\mathbb{R}) \ni \zeta(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le 1 \\ 0 & \text{if} \quad |\xi| > 2 \end{cases}.$$

By setting $\zeta_R(\xi) = \zeta(\xi/R)$, we define the pseudo-differential operator with symbol ζ_R, classically given by $\zeta_R(|D|)f = \mathcal{F}^{-1}(\zeta_R \mathcal{F} f)(x)$ and similarly we define the operator $\tilde{\zeta}_R(|D|)$ with the associated symbol given by $\tilde{\zeta}_R(\xi) = 1 - \zeta_R(\xi)$. Here by $\mathcal{F}, \mathcal{F}^{-1}$ we mean the Fourier transform operator and its inverse, respectively. For any $q \in (2, 2^*)$ there exists a $\epsilon \in (0,1)$ such that $H^\epsilon \hookrightarrow L^{\frac{2d}{d-2\epsilon}} =: L^q$. Then

$$\begin{aligned} \|\tilde{\zeta}_R(|D|)v_n\|_{L^q} &\lesssim \|\langle\xi\rangle^\epsilon \tilde{\zeta}_R(\xi)\hat{v}_n\|_{L^2_\xi} \\ &= \|\langle\xi\rangle^{\epsilon-1}\langle\xi\rangle\tilde{\zeta}_R(\xi)\hat{v}_n\|_{L^2_\xi} \\ &\lesssim R^{-(1-\epsilon)} \end{aligned}$$

where we have used the boundedness of $\{v_n\}_{n\in\mathbb{N}}$ in H^1 at the last step. For the localized part we consider instead a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that

$$\|\zeta_R(|D|)v_n\|_{L^\infty} \le 2|\zeta_R(|D|)v_n(y_n)|$$

and we have that up to subsequences, by using the well-known properties $\mathcal{F}^{-1}(fg) = \mathcal{F}^{-1}f * \mathcal{F}^{-1}g$ and $\mathcal{F}^{-1}\left(f\left(\frac{\cdot}{r}\right)\right) = r^d(\mathcal{F}^{-1}f)(r\cdot)$,

$$\limsup_{n\to\infty} |\zeta_R(|D|)v_n(y_n)| = R^d \limsup_{n\to\infty} \left| \int \eta(Rx)v_n(x - y_n)\,dx \right| \lesssim R^{d/2}\lambda$$

where we denoted $\eta = \mathcal{F}^{-1}\zeta$ and we used Cauchy-Schwartz inequality. Given $\theta \in (0,1)$ such that $\frac{1}{q} = \frac{1-\theta}{2}$, by interpolation follows that

$$\|\zeta_R(|D|)v_n\|_{L^q} \le \|\zeta_R(|D|)v_n\|_{L^\infty}^\theta \|\zeta_R(|D|)v_n\|_{L^2}^{1-\theta} \lesssim R^{\frac{d\theta}{2}}\lambda^\theta$$

$$\limsup_{n\to\infty} \|v_n\|_{L^q} \lesssim \left(R^{\frac{d\theta}{2}}\lambda^\theta + R^{-1+\epsilon}\right)$$

and the proof is complete provided we select as radius $R = \lambda^{-\beta}$ with $0 < \beta = \theta\left(1 - \epsilon + \frac{d\theta}{2}\right)^{-1}$ and so $e = \theta(1-\epsilon)\left(1 - \epsilon + \frac{d\theta}{2}\right)^{-1}$. □

Based on the previous lemma we can prove the following result.

Lemma 4.3. *Let $\{v_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R}^d)$. There exists, up to subsequences, a function $\psi \in H^1$ and two sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ such that*

$$\tau_{-x_n} e^{it_n(\Delta - V)} v_n = \psi + W_n, \tag{4.2}$$

where the following conditions are satisfied:

$$W_n \overset{H^1}{\rightharpoonup} 0,$$

$$\limsup_{n\to\infty} \|e^{it(\Delta-V)} v_n\|_{L^\infty L^q} \le C\left(\sup_n \|v_n\|_{H^1}\right) \|\psi\|_{L^2}^e$$

with the exponent $e > 0$ given in Lemma 4.2. Furthermore, as $n \to \infty$, v_n fulfills the Pythagorean expansions below:

$$\|v_n\|_{L^2}^2 = \|\psi\|_{L^2}^2 + \|W_n\|_{L^2}^2 + o(1); \tag{4.3}$$

$$\|v_n\|_{\mathcal{H}}^2 = \|\tau_{x_n}\psi\|_{\mathcal{H}}^2 + \|\tau_{x_n} W_n\|_{\mathcal{H}}^2 + o(1); \tag{4.4}$$

$$\|v_n\|_{L^q}^q = \|e^{it_n(\Delta-V)}\tau_{x_n}\psi\|_{L^q}^q + \|e^{it_n(\Delta-V)}\tau_{x_n} W_n\|_{L^q}^q + o(1), \qquad q \in (2, 2^*). \tag{4.5}$$

Moreover we have the following dichotomy for the time parameters t_n:

$$\text{either} \quad t_n = 0 \quad \forall\, n \in \mathbb{N} \quad \text{or} \quad t_n \overset{n\to\infty}{\longrightarrow} \pm\infty. \tag{4.6}$$

Concerning the space parameters $x_n = (x_{n,1}, \bar{x}_n) \in \mathbb{R} \times \mathbb{R}^{d-1}$ we have the following scenarios:

$$\begin{aligned} &\text{either} \quad x_n = 0 \quad \forall\, n \in \mathbb{N} \\ &\text{or} \quad |x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty \end{aligned} \tag{4.7}$$

$$\text{or} \quad x_{n,1} = 0, \quad \bar{x}_n^j = \sum_{l=2}^d k_{n,l} P_l \quad \text{with} \quad k_{n,l} \in \mathbb{Z} \quad \text{and} \quad \sum_{l=2}^d |k_{n,l}| \overset{n\to\infty}{\longrightarrow} \infty.$$

Proof. We choose a sequences of times $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\|e^{it_n(\Delta-V)} v_n\|_{L^q} > \frac{1}{2} \|e^{it(\Delta-V)} v_n\|_{L^\infty L^q}. \tag{4.8}$$

According to Lemma 4.2 we can choose a sequence of space translations such that

$$\tau_{-x_n}(e^{it_n(\Delta-V)} v_n) \overset{H^1}{\rightharpoonup} \psi,$$

obtaining (4.2). Let us remark that the choice of the time sequence in (4.8) is possible since the norms H^1 and $\mathcal{H}$ are equivalent. Then

$$\limsup_{n\to\infty} \|e^{it_n(\Delta-V)} v_n\|_{L^q} \lesssim \|\psi\|_{L^2}^e,$$

which in turn implies by (4.8) that

$$\limsup_{n\to\infty} \|e^{it(\Delta-V)} v_n\|_{L^\infty L^q} \lesssim \|\psi\|_{L^2}^e,$$

where the exponent is the one given in Lemma 4.2. From the definition of ψ we can write

$$\tau_{-x_n} e^{it_n(\Delta - V)} v_n = \psi + W_n, \qquad W_n \overset{H^1}{\rightharpoonup} 0 \tag{4.9}$$

and the Hilbert structure of L^2 gives (4.3).
Next we prove (4.4). We have

$$v_n = e^{-it_n(\Delta-V)}\tau_{x_n}\psi + e^{-it_n(\Delta-V)}\tau_{x_n}W_n, \qquad W_n \overset{H^1}{\rightharpoonup} 0$$

and we conclude provided that we show

$$(e^{-it_n(\Delta-V)}\tau_{x_n}\psi, e^{-it_n(\Delta-V)}\tau_{x_n}W_n)_{\mathcal{H}} \overset{n\to\infty}{\longrightarrow} 0. \tag{4.10}$$

Since we have

$$\begin{aligned}&(e^{-it_n(\Delta-V)}\tau_{x_n}\psi, e^{-it_n(\Delta-V)}\tau_{x_n}W_n)_{\mathcal{H}}\\ &= (\psi, W_n)_{\dot H^1} + \int V(x+x_n)\psi(x)\bar W_n(x)\,dx\end{aligned}$$

and $W_n \overset{H^1}{\rightharpoonup} 0$, it is sufficient to show that

$$\int V(x+x_n)\psi(x)\bar W_n(x)\,dx \overset{n\to\infty}{\longrightarrow} 0. \tag{4.11}$$

If (up to subsequence) $x_n \overset{n\to\infty}{\longrightarrow} x^* \in \mathbb{R}^d$ or $|x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty$, where we have splitted $x_n = (x_{n,1}, \bar x_n) \in \mathbb{R}\times\mathbb{R}^{d-1}$, then we have that the sequence $\tau_{-x_n}V(x) = V(x+x_n)$ pointwise converges to the function $\tilde V(x) \in L^\infty$ defined by

$$\tilde V(x) = \begin{cases} 1 & if \quad x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty \\ V(x+x^*) & if \quad x_n \overset{n\to\infty}{\longrightarrow} x^* \in \mathbb{R}^d \\ 0 & if \quad x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty \end{cases}$$

and hence

$$\begin{aligned}\int V(x+x_n)\psi(x)\bar W_n(x)\,dx &= \int [V(x+x_n) - \tilde V(x)]\psi(x)\bar W_n(x)\,dx\\ &\quad + \int \tilde V(x)\psi(x)\bar W_n(x)\,dx.\end{aligned}$$

The function $\tilde V(x)\psi(x)$ belongs to L^2 since $\tilde V$ is bounded and $\psi \in H^1$, and since $W_n \rightharpoonup 0$ in H^1 (and then in L^2) we have that

$$\int \tilde V(x)\psi(x)\bar W_n(x)\,dx \overset{n\to\infty}{\longrightarrow} 0.$$

Moreover by using Cauchy-Schwartz inequality

$$\left|\int [V(x+x_n) - \tilde V(x)]\psi(x)\bar W_n(x)\,dx\right| \le \sup_n \|W_n\|_{L^2}\|[V(\cdot + x_n) - \tilde V(\cdot)]\psi(\cdot)\|_{L^2};$$

since $\left|[V(\cdot+x_n) - \tilde V(\cdot)]\psi(\cdot)\right|^2 \lesssim |\psi(\cdot)|^2 \in L^1$ we have by dominated convergence theorem that also

$$\int [V(x+x_n) - \tilde V(x)]\psi(x)\bar W_n(x)\,dx \overset{n\to\infty}{\longrightarrow} 0,$$

and we conclude (4.11) and hence (4.10). It remains to prove (4.10) in the case when, up to subsequences, $x_{n,1} \overset{n\to\infty}{\longrightarrow} x_1^*$ and $|\bar x_n| \overset{n\to\infty}{\longrightarrow} \infty$. Up to subsequences we can assume therefore that $\bar x_n = \bar x^* + \sum_{l=2}^d k_{n,l}P_l + o(1)$ with $\bar x^* \in \mathbb{R}^{d-1}$, $k_{n,l} \in \mathbb{Z}$

and $\sum_{l=2}^{d}|k_{n,l}| \overset{n\to\infty}{\longrightarrow} \infty$. Then by using the periodicity of the potential V w.r.t. the $(x_2,\ldots,x_d)$ variables we get:

$$\begin{aligned}&(e^{-it_n(\Delta-V)}\tau_{x_n}\psi, e^{-it_n(\Delta-V)}\tau_{x_n}W_n)_{\mathcal{H}} =\\&(e^{-it_n(\Delta-V)}\tau_{(x_1^*,\bar{x}_n)}\psi, e^{-it_n(\Delta-V)}\tau_{(x_1^*,\bar{x}_n)}W_n)_{\mathcal{H}} + o(1) =\\&(\tau_{(x_1^*,\bar{x}^*)}\psi, \tau_{(x_1^*,\bar{x}^*)}W_n)_{\mathcal{H}} + o(1) =\\&(\psi, W_n)_{\dot{H}^1} + \int V(x+(x_1^*,\bar{x}^*))\psi(x)\bar{W}_n\,dx = o(1)\end{aligned}$$

where we have used the fact that $W_n \overset{H^1}{\rightharpoonup} 0$.
We turn now our attention to the orthogonality of the non quadratic term of the energy, namely (4.5). The proof is almost the same of the one carried out in [BV], with some modification.

Case 1. Suppose $|t_n| \overset{n\to\infty}{\longrightarrow} \infty$. By (2.1) we have $\|e^{it(\Delta-V)}\|_{L^1\to L^\infty} \lesssim |t|^{-d/2}$ for any $t\neq 0$. We recall that for the evolution operator $e^{it(\Delta-V)}$ the L^2 norm is conserved, so the estimate $\|e^{it(\Delta-V)}\|_{L^{p\prime}\to L^p} \lesssim |t|^{-d\left(\frac12-\frac1p\right)}$ holds from Riesz-Thorin interpolation theorem, thus we have the conclusion provided that $\psi\in L^1\cap L^2$. If this is not the case we can conclude by a straightforward approximation argument. This implies that if $|t_n|\to\infty$ as $n\to\infty$ then for any $p\in(2,2^*)$ and for any $\psi\in H^1$

$$\|e^{it_n(\Delta-V)}\tau_{x_n}\psi\|_{L^p} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus we conclude by (4.9).

Case 2. Suppose now that $t_n \overset{n\to\infty}{\longrightarrow} t^*\in\mathbb{R}$ and $x_n \overset{n\to\infty}{\longrightarrow} x^*\in\mathbb{R}^d$. In this case the proof relies on a combination of the Rellich-Kondrachov theorem and the Brezis-Lieb Lemma contained in [BL], provided that

$$\|e^{it_n(\Delta-V)}(\tau_{x_n}\psi) - e^{it^*(\Delta-V)}(\tau_{x^*}\psi)\|_{H^1} \overset{n\to\infty}{\longrightarrow} 0, \qquad \forall\,\psi\in H^1.$$

But this is a straightforward consequence of the continuity of the linear propagator (see [BV] for more details).

Case 3. It remains to consider $t_n \overset{n\to\infty}{\longrightarrow} t^*\in\mathbb{R}$ and $|x_n| \overset{n\to\infty}{\longrightarrow} \infty$. Also here we can proceed as in [BV] provided that for any $\psi\in H^1$ there exists a $\psi^*\in H^1$ such that

$$\|\tau_{-x_n}(e^{it_n(\Delta-V)}(\tau_{x_n}\psi)) - \psi^*\|_{H^1} \overset{n\to\infty}{\longrightarrow} 0.$$

Since translations are isometries in H^1, it suffices to show that for some $\psi^*\in H^1$

$$\|e^{it_n(\Delta-V)}\tau_{x_n}\psi - \tau_{x_n}\psi^*\|_{H^1} \overset{n\to\infty}{\longrightarrow} 0.$$

We decompose $x_n=(x_{n,1},\bar{x}_n)\in\mathbb{R}\times\mathbb{R}^{d-1}$ and we consider the two scenarios: $|x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty$ and $\sup_n|x_{n,1}|<\infty$.
If $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$, by continuity in H^1 of the flow, it is enough to prove that

$$\|e^{it^*(\Delta-V)}\tau_{x_n}\psi - e^{it^*\Delta}\tau_{x_n}\psi\|_{H^1} \overset{n\to\infty}{\longrightarrow} 0.$$

We observe that

$$e^{it^*(\Delta-V)}\tau_{x_n}\psi - e^{it^*\Delta}\tau_{x_n}\psi = \int_0^{t^*} e^{i(t^*-s)(\Delta-V)}(Ve^{-is\Delta}\tau_{x_n}\psi)(s)\,ds$$

and hence,

$$\|e^{it^*(\Delta-V)}\tau_{x_n}\psi - e^{it^*\Delta}\tau_{x_n}\psi\|_{H^1} \leq \int_0^{t^*} \|(\tau_{-x_n}V)e^{is\Delta}\psi\|_{H^1} ds.$$

We will show that

$$\int_0^{t^*} \|(\tau_{-x_n}V)e^{is\Delta}\psi\|_{H^1} ds \overset{n\to\infty}{\longrightarrow} 0. \tag{4.12}$$

Since we are assuming $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$, for fixed $x \in \mathbb{R}^d$ we get $V(x+x_n) \overset{n\to\infty}{\longrightarrow} 0$, namely $(\tau_{-x_n}V)(x) \overset{n\to\infty}{\longrightarrow} 0$ pointwise; since $V \in L^\infty$, $|\tau_{-x_n}V|^2|e^{it\Delta}\psi|^2 \leq \|V\|_{L^\infty}^2|e^{it\Delta}\psi|^2$ and $|e^{it\Delta}\psi|^2 \in L^1$, by the dominated convergence theorem we get that

$$\|(\tau_{-x_n}V)e^{it\Delta}\psi\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

Analogously, since $|x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty$ implies $|\nabla\tau_{-x_n}V(x)| \overset{n\to\infty}{\longrightarrow} 0$, we get

$$\|\nabla(\tau_{-x_n}Ve^{it\Delta}\psi)\|_{L^2} \leq \|(e^{it\Delta}\psi)\nabla\tau_{-x_n}V\|_{L^2} + \|(\tau_{-x_n}V)\nabla(e^{it\Delta}\psi)\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

We conclude (4.12) by using the dominated convergence theorem w.r.t the measure ds.

For the case $x_{n,1} \overset{n\to\infty}{\longrightarrow} \infty$ we proceed similarly.

If $\sup_{n\in\mathbb{N}}|x_{n,1}| < \infty$, then up to subsequence $x_{n,1} \overset{n\to\infty}{\longrightarrow} x_1^* \in \mathbb{R}$. The thesis follows by choosing $\psi^* = e^{it^*(\Delta-V)}\tau_{(x_1^*,\bar{x}^*)}\psi$, with $\bar{x}^* \in \mathbb{R}^{d-1}$ defined as follows (see above the proof of (4.4)): $\bar{x}_n = \bar{x}^* + \sum_{l=2}^d k_{n,l}P_l + o(1)$ with $k_{n,l} \in \mathbb{Z}$ and $\sum_{l=2}^d |k_{n,l}| \overset{n\to\infty}{\longrightarrow} \infty$.
Finally, it is straightforward from [BV] that the conditions on the parameters (4.6) and (4.7) hold.

□

Proof of Theorem 4.1. The proof of the profile decomposition theorem can be carried out as in [BV] iterating the previous lemma. □

5. Nonlinear profiles

The results of this section will be crucial along the construction of the minimal element. We recall that the couple (p,r) is the one given in Section 2. Moreover for every sequence $x_n \in \mathbb{R}^d$ we use the notation $x_n = (x_{n,1}, \bar{x}_n) \in \mathbb{R} \times \mathbb{R}^{d-1}$.

Lemma 5.1. *Let $\psi \in H^1$ and $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be such that $|x_{n,1}| \overset{n\to\infty}{\longrightarrow} \infty$. Up to subsequences we have the following estimates:*

$$x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty \implies \|e^{it\Delta}\psi_n - e^{it(\Delta-V)}\psi_n\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0, \tag{5.1}$$

$$x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty \implies \|e^{it(\Delta-1)}\psi_n - e^{it(\Delta-V)}\psi_n\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0, \tag{5.2}$$

where $\psi_n := \tau_{x_n}\psi$.

Proof. Assume $x_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$ (the case $x_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$ can be treated similarly). We first prove that

$$\sup_{n\in\mathbb{N}} \|e^{it(\Delta-V)}\psi_n\|_{L^p_{(T,\infty)}L^r} \overset{T\to\infty}{\longrightarrow} 0. \tag{5.3}$$

Let $\varepsilon>0$. By density there exists $\tilde\psi\in C_c^\infty$ such that $\|\tilde\psi-\psi\|_{H^1}\leq\varepsilon$, then by the estimate (2.5)

$$\|e^{it(\Delta-V)}(\tilde\psi_n-\psi_n)\|_{L^pL^r}\lesssim\|\tilde\psi_n-\psi_n\|_{H^1}=\|\tilde\psi-\psi\|_{H^1}\lesssim\varepsilon.$$

Since $\tilde\psi\in L^{r'}$, by interpolation between the dispersive estimate (2.1) and the conservation of the mass along the linear flow, we have

$$\|e^{it(\Delta-V)}\tilde\psi_n\|_{L^r}\lesssim|t|^{-d\left(\frac12-\frac1r\right)}\|\tilde\psi\|_{L^{r'}},$$

and since $f(t)=|t|^{-d\left(\frac12-\frac1r\right)}\in L^p(|t|>1)$, there exists $T>0$ such that

$$\sup_n\|e^{it(\Delta-V)}\tilde\psi_n\|_{L^p_{|t|\geq T}L^r}\leq\varepsilon,$$

hence we get (5.3). Now to obtain (5.1), we are reduced to show that for a fixed $T>0$

$$\|e^{it\Delta}\psi_n-e^{it(\Delta-V)}\psi_n\|_{L^p_{(0,T)}L^r}\overset{n\to\infty}{\longrightarrow}0.$$

Since $w_n=e^{it\Delta}\psi_n-e^{it(\Delta-V)}\psi_n$ is the solution of the following linear Schrödinger equation

$$\begin{cases} i\partial_t w_n+\Delta w_n-Vw_n=-Ve^{it\Delta}\psi_n\\ w_n(0)=0\end{cases},$$

by combining (2.5) with the Duhamel formula we get

$$\|e^{it\Delta}\psi_n-e^{it(\Delta-V)}\psi_n\|_{L^p_{(0,T)}L^r}\lesssim\|(\tau_{-x_n}V)e^{it\Delta}\psi\|_{L^1_{(0,T)}H^1}.$$

The thesis follows from the dominated convergence theorem. □

Lemma 5.2. *Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a sequence such that $x_{n,1}\overset{n\to\infty}{\longrightarrow}-\infty$, (resp. $x_{n,1}\overset{n\to\infty}{\longrightarrow}+\infty$) and $v\in\mathcal{C}(\mathbb{R};H^1)$ be the unique solution to (1.12) (resp. (1.13)). Define $v_n(t,x):=v(t,x-x_n)$. Then, up to a subsequence, the followings hold:*

$$\left\|\int_0^t[e^{i(t-s)\Delta}\left(|v_n|^\alpha v_n\right)-e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)]ds\right\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0;\tag{5.4}$$

$$\left(\textit{resp.}\left\|\int_0^t[e^{i(t-s)(\Delta-1)}\left(|v_n|^\alpha v_n\right)-e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)]ds\right\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0\right).\tag{5.5}$$

Proof. Assume $x_{n,1}\overset{n\to\infty}{\longrightarrow}-\infty$ (the case $x_{n,1}\overset{n\to\infty}{\longrightarrow}+\infty$ can be treated similarly). We start by showing that

$$\lim_{T\to\infty}\left(\sup_n\left\|\int_0^t e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)ds\right\|_{L^p_{(T,\infty)}L^r}\right)=0.\tag{5.6}$$

By Minkowski inequality and the interpolation of the dispersive estimate (2.1) with the conservation of the mass, we have

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)ds\right\|_{L^r_x}&\lesssim\int_0^t|t-s|^{-d\left(\frac12-\frac1r\right)}\||v_n|^\alpha v_n\|_{L^{r'}_x}ds\\ &\lesssim\int_{\mathbb{R}}|t-s|^{-d\left(\frac12-\frac1r\right)}\||v|^\alpha v\|_{L^{r'}_x}ds=|t|^{-d\left(\frac12-\frac1r\right)}*g\end{aligned}$$

with $g(s)=\||v|^\alpha v(s)\|_{L_x^{r'}}$. We conclude (5.6) provided that we show $|t|^{-d\left(\frac12-\frac1r\right)}*g(t)\in L_t^p$. By using the Hardy-Littlewood-Sobolev inequality (see for instance Stein's book [ST], p. 119) we get:

$$\left\||t|^{-1+\frac{(2-d)\alpha+4}{2(\alpha+2)}}*g(t)\right\|_{L_t^p}\lesssim\||v|^\alpha v\|_{L^{\frac{2\alpha(\alpha+2)}{((2-d)\alpha+4)(\alpha+1)}}L^{r'}}=\|v\|_{L^pL^r}^{\alpha+1}.$$

Since v scatters, then it belongs to L^pL^r, and so we can claim the validity of (5.6).

Consider now T fixed: we have to show that

$$\left\|\int_0^t[e^{i(t-s)\Delta}\left(|v_n|^\alpha v_n\right)-e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)]ds\right\|_{L^p_{(0,T)}L^r}\overset{n\to\infty}{\longrightarrow}0.$$

As usual we observe that

$$w_n(t,x)=\int_0^t e^{i(t-s)\Delta}\left(|v_n|^\alpha v_n\right)ds-\int_0^t e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)ds$$

is the solution of the following linear Schrödinger equation

$$\begin{cases} i\partial_t w_n+\Delta w_n-Vw_n=-V\displaystyle\int_0^t e^{i(t-s)\Delta}\left(|v_n|^\alpha v_n\right)ds\\ w_n(0)=0\end{cases}$$

and likely for Lemma 5.1 we estimate

$$\begin{aligned}\left\|\int_0^t e^{i(t-s)\Delta}\left(|v_n|^\alpha v_n\right)ds-\int_0^t e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)ds\right\|_{L^p_{(0,T)}L^r}\\ \lesssim\|(\tau_{-x_n}V)|v|^\alpha v\|_{L^1_{(0,T)}H^1}.\end{aligned}$$

By using the dominated convergence theorem we conclude the proof. □

The previous results imply the following useful corollaries.

Corollary 5.3. *Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a sequence such that $x_{n,1}\overset{n\to\infty}{\longrightarrow}-\infty$, and let $v\in\mathcal{C}(\mathbb{R};H^1)$ be the unique solution to (1.12) with initial datum $v_0\in H^1$. Then*

$$v_n(t,x)=e^{it(\Delta-V)}v_{0,n}-i\int_0^t e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)ds+e_n(t,x)$$

where $v_{0,n}(x):=\tau_{x_n}v_0(x)$, $v_n(t,x):=v(t,x-x_n)$ and $\|e_n\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0$.

Proof. It is a consequence of (5.1) and (5.4). □

Corollary 5.4. *Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a sequence such that $x_{n,1}\overset{n\to\infty}{\longrightarrow}+\infty$, and let $v\in\mathcal{C}(\mathbb{R};H^1)$ be the unique solution to (1.13) with initial datum $v_0\in H^1$. Then*

$$v_n(t,x)=e^{it(\Delta-V)}v_{0,n}-i\int_0^t e^{i(t-s)(\Delta-V)}\left(|v_n|^\alpha v_n\right)ds+e_n(t,x)$$

where $v_{0,n}(x):=\tau_{x_n}v_0(x)$, $v_n(t,x):=v(t,x-x_n)$ and $\|e_n\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0$.

Proof. It is a consequence of (5.2) and (5.5). □

Lemma 5.5. *Let $v(t,x)\in\mathcal{C}(\mathbb{R};H^1)$ be a solution to (1.12) (resp. (1.13)) and let $\psi_\pm\in H^1$ (resp. $\varphi_\pm\in H^1$) be such that*

$$\|v(t,x)-e^{it\Delta}\psi_\pm\|_{H^1}\overset{t\to\pm\infty}{\longrightarrow}0$$
$$\left(\textit{resp. } \|v(t,x)-e^{it(\Delta-1)}\varphi_\pm\|_{H^1}\overset{t\to\pm\infty}{\longrightarrow}0\right).$$

Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^d$, $\{t_n\}_{n\in\mathbb{N}}\subset\mathbb{R}$ be two sequences such that $x_{n,1}\overset{n\to\infty}{\longrightarrow}-\infty$ (resp. $x_{n,1}\overset{n\to\infty}{\longrightarrow}+\infty$) and $|t_n|\overset{n\to\infty}{\longrightarrow}\infty$. Let us define moreover $v_n(t,x):=v(t-t_n,x-x_n)$ and $\psi_n^\pm(x):=\tau_{x_n}\psi_\pm(x)$ (resp. $\varphi_n^\pm(x)=\tau_{x_n}\varphi_\pm(x)$). Then, up to subsequence, we get

$$\begin{aligned}(5.7)\qquad & t_n\to\pm\infty\Longrightarrow\|e^{i(t-t_n)\Delta}\psi_n^\pm-e^{i(t-t_n)(\Delta-V)}\psi_n^\pm\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0\quad\text{and}\\ & \left\|\int_0^t[e^{i(t-s)\Delta}\big(|v_n|^\alpha v_n\big)ds-e^{i(t-s)(\Delta-V)}\big(|v_n|^\alpha v_n\big)]ds\right\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0\end{aligned}$$

$$\Bigg(\textit{resp. } t_n\to\pm\infty\Longrightarrow\|e^{i(t-t_n)(\Delta-1)}\varphi_n^\pm-e^{i(t-t_n)(\Delta-V)}\varphi_n^\pm\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0\quad\text{and}$$
$$\left\|\int_0^t[e^{i(t-s)(\Delta-1)}\big(|v_n|^\alpha v_n\big)ds-e^{i(t-s)(\Delta-V)}\big(|v_n|^\alpha v_n\big)]ds\right\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0\Bigg).$$

Proof. It is a multidimensional suitable version of Proposition 3.6 in [BV]. Nevertheless, since in [BV] the details of the proof are not given, we expose below the proof of the most delicate estimate, namely the second estimate in (5.7). After a change of variable in time, proving (5.7) is clearly equivalent to prove

$$\left\|\int_{-t_n}^t e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)ds-\int_{-t_n}^t e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)ds\right\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0.$$

We can focus on the case $t_n\to\infty$ and $x_{n,1}\overset{n\to\infty}{\longrightarrow}+\infty$, being the other cases similar. The idea of the proof is to split the estimate above in three different regions, i.e. $(-\infty,-T)\times\mathbb{R}^d$, $(-T,T)\times\mathbb{R}^d$, $(T,\infty)\times\mathbb{R}^d$ for some fixed T which will be chosen in an appropriate way below. The strategy is to use translation in the space variable to gain smallness in the strip $(-T,T)\times\mathbb{R}^d$ while we use smallness of Strichartz estimate in half spaces $(-T,T)^c\times\mathbb{R}^d$. Actually in (T,∞) the situation is more delicate and we will also use the dispersive relation.

Let us define $g(t)=\|v(t)\|^{\alpha+1}_{L^{(\alpha+1)r'}}$ and for fixed $\varepsilon>0$ let us consider $T=T(\varepsilon)>0$ such that:

$$(5.8)\qquad\begin{cases}\||v|^\alpha v\|_{L^{q'}_{(-\infty,-T)}L^{r'}}<\varepsilon\\ \||v|^\alpha v\|_{L^{q'}_{(T,+\infty)}L^{r'}}<\varepsilon\\ \||v|^\alpha v\|_{L^1_{(-\infty,-T)}H^1}<\varepsilon\\ \left\||t|^{-d\left(\frac12-\frac1r\right)}*g(t)\right\|_{L^p_{(T,+\infty)}}<\varepsilon\end{cases}.$$

The existence of such a T is guaranteed by the integrability properties of v and its decay at infinity (in time). We can assume without loss of generality that $|t_n|>T$.

We split the term to be estimated as follows:

$$\begin{aligned}&\int_{-t_n}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)ds-\int_{-t_n}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)ds\\
=\,&e^{it\Delta}\int_{-t_n}^{-T} e^{-is\Delta}\tau_{x_n}(|v|^\alpha v)(s)ds-e^{it(\Delta-V)}\int_{-t_n}^{-T} e^{-is(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)ds\\
&+\int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)ds-\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)ds.\end{aligned}$$

By Strichartz estimate (2.5) and the third one of (5.8), we have, uniformly in n,

$$\begin{aligned}\left\|e^{it\Delta}\int_{-t_n}^{-T} e^{-is\Delta}\tau_{x_n}(|v|^\alpha v)(s)ds\right\|_{L^pL^r}&\lesssim\varepsilon,\\
\left\|e^{it(\Delta-V)}\int_{-t_n}^{-T} e^{-is(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)ds\right\|_{L^pL^r}&\lesssim\varepsilon.\end{aligned}$$

Thus, it remains to prove

$$\left\|\int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)ds-\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)ds\right\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0.$$

and we split it by estimating it in the regions mentioned above. By using (2.6) and the first one of (5.8) we get uniformly in n the following estimates:

$$\begin{aligned}\left\|\int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)ds\right\|_{L^p_{(-\infty,-T)}L^r}&\lesssim\||v|^\alpha v\|_{L^{q'}_{(-\infty,-T)}L^{r'}}\lesssim\varepsilon,\\
\left\|\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)ds\right\|_{L^p_{(-\infty,-T)}L^r}&\lesssim\||v|^\alpha v\|_{L^{q'}_{(-\infty,-T)}L^{r'}}\lesssim\varepsilon.\end{aligned}$$

The difference $w_n=\int_{-T}^t e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)ds-\int_{-T}^t e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)(s)ds$ satisfies the following Cauchy problem

$$\begin{cases}i\partial_t w_n+(\Delta-V)w_n=-V\displaystyle\int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)(s)\,ds\\ w_n(-T)=0\end{cases},$$

then w_n satisfies the integral equation

$$w_n(t)=\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\Big(-V\int_{-T}^{s} e^{i(s-\sigma)\Delta}\tau_{x_n}(|v|^\alpha v)(\sigma)\,d\sigma\Big)\,ds$$

which we estimate in the region $(-T,T)\times\mathbb{R}^d$. By Sobolev embedding $H^1\hookrightarrow L^r$, Hölder and Minkowski inequalities we have therefore

$$\begin{aligned}\left\|\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\Big(-V\int_{-T}^{s} e^{i(s-\sigma)\Delta}\tau_{x_n}(|v|^\alpha v)(\sigma)\,d\sigma\Big)\,ds\right\|_{L^p_{(-T,T)}L^r}&\lesssim\\
\lesssim T^{1/p}\int_{-T}^{T}\left\|(\tau_{-x_n}V)\int_{-T}^{s} e^{i(s-\sigma)\Delta}|v|^\alpha v(\sigma)\,d\sigma\right\|_{H^1}ds&\lesssim\varepsilon\end{aligned}$$

by means of Lebesgue's theorem.
It remains to estimate in the region $(T,\infty)\times\mathbb{R}^d$ the terms

$$\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)\,ds \qquad \text{and} \qquad \int_{-T}^{t} e^{i(t-s)\Delta}\tau_{x_n}(|v|^\alpha v)\,ds.$$

We consider only one term being the same for the other. Let us split the estimate as follows:

$$\begin{aligned}\left\|\int_{-T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)\,ds\right\|_{L^p_{(T,\infty)}L^r} &\leq \\ \leq \left\|\int_{-T}^{T} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)\,ds\right\|_{L^p_{(T,\infty)}L^r} \\ + \left\|\int_{T}^{t} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)\,ds\right\|_{L^p_{(T,\infty)}L^r}.\end{aligned}$$

The second term is controlled by Strichartz estimates, and it is $\lesssim\varepsilon$ since we are integrating in the region where $\||v|^\alpha v\|_{L^{q'}((T,\infty);L^{r'})}<\varepsilon$ (by using the second of (5.8)), while the first term is estimated by using the dispersive relation. More precisely

$$\begin{aligned}\left\|\int_{-T}^{T} e^{i(t-s)(\Delta-V)}\tau_{x_n}(|v|^\alpha v)\,ds\right\|_{L^p_{(T,\infty)}L^r} &\lesssim \\ \lesssim \left\|\int_{-T}^{T} |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}\|v\|^{\alpha+1}_{L^{(\alpha+1)r'}}\,ds\right\|_{L^p_{(T,\infty)}} \\ \lesssim \left\|\int_{\mathbb{R}} |t-s|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}\|v\|^{\alpha+1}_{L^{(\alpha+1)r'}}\,ds\right\|_{L^p_{(T,\infty)}} \lesssim \varepsilon\end{aligned}$$

where in the last step we used Hardy-Sobolev-Littlewood inequality and the fourth of (5.8).

□

As consequences of the previous lemma we obtain the following corollaries.

Corollary 5.6. *Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a sequence such that $x_{n,1}\overset{n\to\infty}{\longrightarrow}-\infty$ and let $v\in\mathcal{C}(\mathbb{R};H^1)$ be a solution to (1.12) with initial datum $\psi\in H^1$. Then for a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $|t_n|\overset{n\to\infty}{\longrightarrow}\infty$*

$$v_n(t,x)=e^{it(\Delta-V)}\psi_n-i\int_0^t e^{i(t-s)(\Delta-V)}\big(|v_n|^\alpha v_n\big)ds+e_n(t,x)$$

where $\psi_n:=e^{-it_n(\Delta-V)}\tau_{x_n}\psi$, $v_n:=v(t-t_n,x-x_n)$ and $\|e_n\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0$.

Corollary 5.7. *Let $\{x_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^d$ be a sequence such that $x_{n,1}\overset{n\to\infty}{\longrightarrow}+\infty$ and let $v\in\mathcal{C}(\mathbb{R};H^1)$ be a solution to (1.13) with initial datum $\psi\in H^1$. Then for a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $|t_n|\overset{n\to\infty}{\longrightarrow}\infty$*

$$v_n(t,x)=e^{it(\Delta-V)}\psi_n-i\int_0^t e^{i(t-s)(\Delta-V)}\big(|v_n|^\alpha v_n\big)ds+e_n(t,x)$$

where $\psi_n:=e^{-it_n(\Delta-V)}\tau_{x_n}\psi$, $v_n:=v(t-t_n,x-x_n)$ and $\|e_n\|_{L^pL^r}\overset{n\to\infty}{\longrightarrow}0$.

We shall also need the following results, for whose proof we refer to [BV].

Proposition 5.8. *Let $\psi \in H^1$. There exists $\hat{U}_\pm \in \mathcal{C}(\mathbb{R}_\pm; H^1) \cap L^p_{\mathbb{R}_\pm} L^r$ solution to* (1.8) *such that*

$$\|\hat{U}_\pm(t,\cdot) - e^{-it(\Delta - V)}\psi\|_{H^1} \overset{t\to\pm\infty}{\longrightarrow} 0.$$

Moreover, if $t_n \to \mp\infty$, then

$$\hat{U}_{\pm,n} = e^{it(\Delta-V)}\psi_n - i\int_0^t e^{i(t-s)(\Delta-V)}\big(|\hat{U}_{\pm,n}|^\alpha \hat{U}_{\pm,n}\big)ds + h_{\pm,n}(t,x)$$

where $\psi_n := e^{-it_n(\Delta-V)}\psi$, $\hat{U}_{\pm,n}(t,\cdot) =: \hat{U}_\pm(t-t_n,\cdot)$ and $\|h_{\pm,n}(t,x)\|_{L^pL^r} \overset{n\to\infty}{\longrightarrow} 0$.

6. Existence and extinction of the critical element

In view of the results stated in Section 3, we define the following quantity belonging to $(0,\infty]$:

$$E_c = \sup\Big\{E > 0 \textit{ such that if } \varphi \in H^1 \textit{ with } E(\varphi) < E$$
$$\textit{then the solution of } (1.8) \textit{ with initial data } \varphi \textit{ is in } L^pL^r\Big\}.$$

Our aim is to show that $E_c = \infty$ and hence we get the large data scattering.

6.1. **Existence of the Minimal Element.**

Proposition 6.1. *Suppose $E_c < \infty$. Then there exists $\varphi_c \in H^1$, $\varphi_c \not\equiv 0$, such that the corresponding global solution $u_c(t,x)$ to* (1.8) *does not scatter. Moreover, there exists $\bar{x}(t) \in \mathbb{R}^{d-1}$ such that $\{u_c(t, x_1, \bar{x} - \bar{x}(t))\}_{t\in\mathbb{R}^+}$ is a relatively compact subset in H^1.*

Proof. If $E_c < \infty$, there exists a sequence φ_n of elements of H^1 such that

$$E(\varphi_n) \overset{n\to\infty}{\longrightarrow} E_c,$$

and by denoting with $u_n \in \mathcal{C}(\mathbb{R}; H^1)$ the corresponding solution to (1.1) with initial datum φ_n then

$$u_n \notin L^pL^r.$$

We apply the profile decomposition to φ_n :

$$\varphi_n = \sum_{j=1}^J e^{-it_n^j(-\Delta+V)}\tau_{x_n^j}\psi^j + R_n^J. \tag{6.1}$$

Claim 6.2. *There exists only one non-trivial profile, that is $J = 1$.*

Assume $J > 1$. For $j \in \{1,\dots,J\}$ to each profile ψ^j we associate a nonlinear profile U_n^j. We can have one of the following situations, where we have reordered without loss of generality the cases in these way:

(1) $(t_n^j, x_n^j) = (0,0) \in \mathbb{R}\times\mathbb{R}^d$,
(2) $t_n^j = 0$ and $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} -\infty$,
(3) $t_n^j = 0$, and $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} +\infty$,
(4) $t_n^j = 0$, $x_{n,1}^j = 0$ and $|\bar{x}_n^j| \overset{n\to\infty}{\longrightarrow} \infty$,
(5) $x_n^j = \vec{0}$ and $t_n^j \overset{n\to\infty}{\longrightarrow} -\infty$,
(6) $x_n^j = \vec{0}$ and $t_n^j \overset{n\to\infty}{\longrightarrow} +\infty$,
(7) $x_{n,1}^j \overset{n\to\infty}{\longrightarrow} -\infty$ and $t_n^j \overset{n\to\infty}{\longrightarrow} -\infty$,

(8) $x^j_{n,1} \overset{n\to\infty}{\longrightarrow} -\infty$ and $t^j_n \overset{n\to\infty}{\longrightarrow} +\infty$,
(9) $x^j_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$ and $t^j_n \overset{n\to\infty}{\longrightarrow} -\infty$,
(10) $x^j_{n,1} \overset{n\to\infty}{\longrightarrow} +\infty$ and $t^j_n \overset{n\to\infty}{\longrightarrow} +\infty$,
(11) $x^j_{n,1} = 0$, $t^j_n \overset{n\to\infty}{\longrightarrow} -\infty$ and $|\bar{x}^j_n| \overset{n\to\infty}{\longrightarrow} \infty$,
(12) $x^j_{n,1} = 0$, $t^j_n \overset{n\to\infty}{\longrightarrow} +\infty$ and $|\bar{x}^j_n| \overset{n\to\infty}{\longrightarrow} \infty$.

Notice that despite to [BV] we have twelve cases to consider and not six (this is because we have to consider a different behavior of $V(x)$ as $|x| \to \infty$). Since the argument to deal with the cases above is similar to the ones considered in [BV] we skip the details. The main point is that for instance in dealing with the cases (2) and (3) above we have to use respectively Corollary 5.3 and Corollary 5.4. When instead $|\bar{x}^j_n| \overset{n\to\infty}{\longrightarrow} \infty$ and $x^j_{1,n} = 0$ we use the fact that this sequences can be assumed, according with the profile decomposition Theorem 4.1 to have components which are integer multiples of the periods, so the translations and the nonlinear equation commute and if $|t_n| \overset{n\to\infty}{\longrightarrow} \infty$ we use moreover Proposition 5.8. We skip the details. Once it is proved that $J = 1$ and

$$\varphi_n = e^{it_n(\Delta - V)}\psi + R_n$$

with $\psi \in H^1$ and $\limsup_{n\to\infty} \|e^{it(\Delta-V)}R_n\|_{L^pL^r} = 0$, then the existence of the critical element follows now by [FXC], ensuring that up to subsequence φ_n converges to ψ in H^1 and so $\varphi_c = \psi$. We define by u_c the solution to (1.8) with Cauchy datum φ_c, and we call it critical element. This is the minimal (with respect to the energy) non-scattering solution to (1.8). We can assume therefore with no loss of generality that $\|u_c\|_{L^p((0,+\infty);L^r)} = \infty$. The precompactenss of the trajectory up to translation by a path $\bar{x}(t)$ follows again by [FXC].

□

6.2. **Extinction of the Minimal Element.** Next we show that the unique solution that satisfies the compactness properties of the minimal element $u_c(t,x)$ (see Proposition 6.1) is the trivial solution. Hence we get a contradiction and we deduce that necessarily $E_c = \infty$.

The tool that we shall use is the following Nakanishi-Morawetz type estimate.

Lemma 6.3. *Let $u(t,x)$ be the solution to (1.8), where $V(x)$ satisfies $x_1 \cdot \partial_{x_1} V(x) \leq 0$ for any $x \in \mathbb{R}^d$, then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{t^2|u|^{\alpha+2}}{(t^2 + x_1^2)^{3/2}}\, dx_1\, d\bar{x}\, dt < \infty. \tag{6.2}$$

Proof. The proof follows the ideas of [N]; we shall recall it shortly, with the obvious modifications of our context. Let us introduce

$$m(u) = a\partial_{x_1}u + gu$$

with

$$a = -\frac{2x_1}{\lambda}, \quad g = -\frac{t^2}{\lambda^3} - \frac{it}{\lambda}, \quad \lambda = (t^2 + x_1^2)^{1/2}$$

and by using the equation solved by $u(t,x)$ we get

$$
\begin{aligned}
0 &= \Re\{(i\partial_t u + \Delta u - Vu - |u|^\alpha u)\bar{m})\} \\
&= \frac{1}{2}\partial_t\Big(-\frac{2x_1}{\lambda}\Im\{\bar{u}\partial_{x_1}u\} - \frac{t|u|^2}{\lambda}\Big) \\
&\quad + \partial_{x_1}\Re\{\partial_{x_1}u\bar{m} - al_V(u) - \partial_{x_1}g\frac{|u|^2}{2}\} \\
&\quad + \frac{t^2G(u)}{\lambda^3} + \frac{|u|^2}{2}\Re\{\partial^2_{x_1}g\} \\
&\quad + \frac{|2it\partial_{x_1}u + x_1u|^2}{2\lambda^3} - x_1\partial_{x_1}V\frac{|u|^2}{\lambda} \\
&\quad + div_{\bar{x}}\Re\{\bar{m}\nabla_{\bar{x}}u\}.
\end{aligned}
\tag{6.3}
$$

with $G(u) = \frac{\alpha}{\alpha+2}|u|^{\alpha+2}$, $l_V(u) = \frac{1}{2}\left(-\Re\{i\bar{u}\partial_t u\} + |\partial_{x_1}u|^2 + \frac{2|u|^{\alpha+2}}{\alpha+2} + V|u|^2\right)$ and $div_{\bar{x}}$ is the divergence operator w.r.t. the $(x_2,\dots,x_d)$ variables. Making use of the repulsivity assumption in the x_1 direction, we get (6.2) by integrating (6.3) on $\{1<|t|<T\}\times\mathbb{R}^d$, obtaining

$$\int_1^T\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\frac{t^2|u|^{\alpha+2}}{(t^2+x_1^2)^{3/2}}\,dx_1\,d\bar{x}\,dt \le C,$$

where $C = C(M,E)$ depends on mass and energy and then letting $T\to\infty$. □

Lemma 6.4. *Let $u(t,x)$ be a nontrivial solution to (1.8) such that for a suitable choice $\bar{x}(t)\in\mathbb{R}^{d-1}$ we have that $\{u(t,x_1,\bar{x}-\bar{x}(t))\}\subset H^1$ is a precompact set. If $\bar{u}\in H^1$ is one of its limit points, then $\bar{u}\neq 0$.*

Proof. This property simply follows from the conservation of the energy. □

Lemma 6.5. *If $u(t,x)$ is an in Lemma 6.4 then for any $\varepsilon>0$ there exists $R>0$ such that*

$$\sup_{t\in\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{|x_1|>R}(|u|^2 + |\nabla_x u|^2 + |u|^{\alpha+2})\,d\bar{x}\,dx_1 < \varepsilon. \tag{6.4}$$

Proof. This is a well-known property implied by the precompactness of the sequence. □

Lemma 6.6. *If $u(t,x)$ is an in Lemma 6.4 then there exist $R_0>0$ and $\varepsilon_0>0$ such that*

$$\int_{\mathbb{R}^{d-1}}\int_{|x_1|<R_0}|u(t,x_1,\bar{x}-\bar{x}(t))|^{\alpha+2}\,d\bar{x}\,dx_1 > \varepsilon_0 \qquad \forall\, t\in\mathbb{R}^+. \tag{6.5}$$

Proof. It is sufficient to prove that $\inf_{t\in\mathbb{R}^+}\|u(t,x_1,\bar{x}-\bar{x}(t))\|_{L^{\alpha+2}}>0$, then the result follows by combining this fact with Lemma 6.5. If by the absurd it is not true then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^+$ such that $u(t_n,x_1,\bar{x}-\bar{x}(t_n))\overset{n\to\infty}{\longrightarrow} 0$ in $L^{\alpha+2}$. On the other hand by the compactness assumption, it implies that $u(t_n,x_1,\bar{x}-\bar{x}(t_n))\overset{n\to\infty}{\longrightarrow} 0$ in H^1, and it is in contradiction with Lemma 6.4. □

We now conclude the proof of scattering for large data, by showing the extinction of the minimal element. Let $R_0 > 0$ and $\varepsilon_0 > 0$ be given by Lemma 6.6, then

$$\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} \frac{|u|^{\alpha+2}t^2}{(t^2+x_1^2)^{3/2}}\,dx_1\,d\bar{x}\,dt \geq \int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}}\int_{|x_1|<R_0} \frac{t^2|u(t,x_1,\bar{x}-\bar{x}(t))|^{\alpha+2}}{(t^2+x_1^2)^{3/2}}\,dx_1\,d\bar{x}\,dt$$
$$\geq \varepsilon \int_1^T \frac{t^2}{(t^2+R_0^2)^{3/2}}\,dt \to \infty \qquad \text{if} \quad T\to\infty.$$

Hence we contradict (6.2) and we get that the critical element cannot exist.

7. Double scattering channels in $1D$

This last section is devoted to prove Theorem 1.8. Following [DS] (see *Example 1,* page 283) we have the following property:

$$\forall\,\psi\in L^2 \quad \exists\,\eta_\pm,\gamma_\pm\in L^2 \text{ such that}$$
$$\|e^{it(\partial_x^2-V)}\psi - e^{it\partial_x^2}\eta_\pm - e^{it(\partial_x^2-1)}\gamma_\pm\|_{L^2} \overset{t\to\pm\infty}{\longrightarrow} 0. \tag{7.1}$$

Our aim is now to show that (7.1) actually holds in H^1 provided that $\psi\in H^1$. We shall prove this property for $t\to+\infty$ (the case $t\to-\infty$ is similar).

7.1. Convergence (7.1) occurs in H^1 provided that $\psi\in H^2$.

In order to do that it is sufficient to show that

$$\psi\in H^2 \Longrightarrow \eta_+,\gamma_+\in H^2. \tag{7.2}$$

Once it is proved then we conclude the proof of this first step by using the following interpolation inequality

$$\|f\|_{H^1}\leq \|f\|_{L^2}^{1/2}\|f\|_{H^2}^{1/2}$$

in conjunction with (7.1) and with the bound

$$\sup_{t\in\mathbb{R}} \|e^{it(\partial_x^2-V)}\psi - e^{it\partial_x^2}\eta_+ - e^{it(\partial_x^2-1)}\gamma_+\|_{H^2}<\infty$$

(in fact this last property follows by the fact that $D(\partial_x^2-V(x))=H^2$ is preserved along the linear flow and by (7.2)). Thus we show (7.2). Notice that by (7.1) we get

$$\|e^{-it\partial_x^2}e^{it(\partial_x^2-V)}\psi - \eta_+ - e^{-it}\gamma_+\|_{L^2} \overset{t\to\infty}{\longrightarrow} 0,$$

and by choosing as subsequence $t_n=2\pi n$ we get

$$\|e^{-it_n\partial_x^2}e^{it_n(\partial_x^2-V)}\psi - \eta_+ - \gamma_+\|_{L^2} \overset{n\to\infty}{\longrightarrow} 0.$$

By combining this fact with the bound $\sup_n \|e^{-it_n\partial_x^2}e^{it_n(\partial_x^2-V)}\psi\|_{H^2}<\infty$ we get $\eta_+ + \gamma_+\in H^2$. Arguing as above but by choosing $t_n=(2n+1)\pi$ we also get $\eta_+-\gamma_+\in H^2$ and hence necessarily $\eta_+,\gamma_+\in H^2$.

7.2. The map $H^2\ni\psi\mapsto(\eta_+,\gamma_+)\in H^2\times H^2$ satisfies $\|\gamma_+\|_{H^1}+\|\eta_+\|_{H^1}\lesssim\|\psi\|_{H^1}$.

Once this step is proved then we conclude by a straightforward density argument. By a linear version of the conservation laws (1.5), (1.6) we get

$$\|e^{it(\partial_x^2-V)}\psi\|_{H^1_V} = \|\psi\|_{H^1_V} \tag{7.3}$$

where

$$\|w\|_{H^1_V}^2 = \int |\partial_x w|^2 dx + \int V|w|^2 dx + \int |w|^2 dx.$$

Notice that this norm is clearly equivalent to the usual norm of H^1.
Next notice that by using the conservation of the mass we get

$$\|\eta_+ + \gamma_+\|_{L^2}^2 = \|\eta_+ + e^{-2n\pi i}\gamma_+\|_{L^2}^2 = \|e^{i2\pi n\partial_x^2}\eta_+ + e^{i2\pi n(\partial_x^2-1)}\gamma_+\|_{L^2}^2$$

and by using (7.1) we get

$$\|\eta_+ + \gamma_+\|_{L^2}^2 = \lim_{t\to\infty} \|e^{it(\partial_x^2-V)}\psi\|_{L^2}^2 = \|\psi\|_{L^2}^2$$

Moreover we have

$$\begin{aligned}\|\partial_x(\eta_+ + \gamma_+)\|_{L^2}^2 &= \|\partial_x(\eta_+ + e^{-2n\pi i}\gamma_+)\|_{L^2}^2 = \|\partial_x(e^{i2\pi n\partial_x^2}(\eta_+ + e^{-i2\pi n}\gamma_+))\|_{L^2}^2\\ &= \|\partial_x(e^{i2\pi n\partial_x^2}\eta_+ + e^{i2\pi n(\partial_x^2-1)}\gamma_+)\|_{L^2}^2\end{aligned}$$

and by using the previous step and (7.3) we get

$$\begin{aligned}\|\partial_x(\eta_+ + \gamma_+)\|_{L^2}^2 &= \lim_{t\to+\infty} \|\partial_x(e^{it(\partial_x^2-V)}\psi)\|_{L^2}^2\\ &\leq \lim_{t\to\infty} \|e^{it(\partial_x^2-V)}\psi\|_{H^1_V}^2 = \|\psi\|_{H^1_V}^2 \lesssim \|\psi\|_{H^1}^2.\end{aligned}$$

Summarizing we get

$$\|\eta_+ + \gamma_+\|_{H^1} \lesssim \|\psi\|_{H^1}.$$

By a similar argument and by replacing the sequence $t_n = 2\pi n$ by $t_n = (2n+1)\pi$ we get

$$\|\eta_+ - \gamma_+\|_{H^1} \lesssim \|\psi\|_{H^1}.$$

The conclusion follows.

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Luigi Forcella
Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56126 Pisa, Italy
E-mail address: `luigi.forcella@sns.it`

Nicola Visciglia
Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo, 5, 56100 Pisa, Italy
E-mail address: `nicola.visciglia@unipi.it`