LARGE DATA SCATTERING FOR NLKG ON WAVEGUIDE $\mathbb{R}^d \times \mathbb{T}$.

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Abstract. We consider the pure-power defocusing nonlinear Klein-Gordon equation, in the H^1–subcritical case, posed on the product space $\mathbb{R}^d \times \mathbb{T}$, where $\mathbb{T}$ is the one-dimensional flat torus. In this framework, we prove that scattering holds for any initial data belonging to the energy space $H^1 \times L^2$ for $1 \leq d \leq 4$. The strategy consists in proving a suitable profile decomposition theorem on the whole manifold to pursue a concentration-compactness and rigidity method along with the proofs of (global in time) Strichartz estimates.

1. Introduction

We consider the following Cauchy problem for the pure-power defocusing nonlinear Klein-Gordon equation posed on the waveguide $\mathbb{R}^d \times \mathbb{T}$, with $1 \leq d \leq 4$

$$\begin{cases} \partial_{tt} u - \Delta_{x,y} u + u = -|u|^\alpha u, \quad (t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \\ u(0,x,y) = f(x,y) \in H^1(\mathbb{R}^d \times \mathbb{T}) \\ \partial_t u(0,x,y) = g(x,y) \in L^2(\mathbb{R}^d \times \mathbb{T}) \end{cases}, \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \to \mathbb{R}$, $\mathbb{T}$ is the one-dimensional flat torus and $\Delta_{x,y} = \Delta_x + \Delta_y$ is the usual Laplace operator $\sum_{i=1}^d \partial^2_{x_i} + \partial^2_y$.
We consider nonlinearities that are energy subcritical on $\mathbb{R}^{d+1}$ and mass supercritical on $\mathbb{R}^d$, namely we restrict our attention to $\frac{4}{d} < \alpha < \frac{4}{d-1}$ for $2 \leq d \leq 4$ while $\alpha > 4$ for $d = 1$. For some particular choices of nonlinearities, aside from the natural question of existence of solutions, it is of interest to try to relate the long-time behaviour of nonlinear solutions to linear solutions in appropriate functional spaces. We wish to investigate the so-called energy scattering property for (1.1).

We briefly recall (assuming that a global well-posedness theory has been already established concerning the Cauchy problem (1.1)) what it is meant as scattering property: we investigate the completeness of the wave operator by showing that, a global solution $u(t,x,y)$ to (1.1) behaves as time $t \to \pm\infty$ – in the $H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$ topology – like a solution to the following linear equation

$$\begin{cases} \partial_{tt} v - \Delta_{x,y} v + v = 0, \quad (t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \\ v(0,x,y) = f^\pm \in H^1(\mathbb{R}^d \times \mathbb{T}) \\ \partial_t v(0,x,y) = g^\pm \in L^2(\mathbb{R}^d \times \mathbb{T}) \end{cases} \tag{1.2}$$

for some initial data $(f^\pm, g^\pm) \in H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$, respectively.

Our purpose is to carry on with the investigation of the second author and Visciglia started in [19]. In that paper the authors proved scattering for small energy data for the pure-power nonlinear energy-critical Klein-Gordon equation posed on $\mathbb{R}^d \times \mathcal{M}^2$, in both defocusing and focusing regimes (the latter corresponding to an opposite sign in front of the nonlinear term in (1.1)) and where $\mathcal{M}^2$ is a bidimensional compact

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manifold (in that case, the second order operator in (1.1) must be replaced by the the sum of the classical Laplacian $\sum_{i=1}^{d} \partial_{x_i}^2$ on $\mathbb{R}^d$ and the Laplace-Beltrami operator Δ_y on the compact manifold). For small initial data, once Strichartz estimates have been proved to hold globally in time, the global well-posedness and scattering can be proved by a perturbative argument.

Our aim is therefore to treat the large data theory for the class of Cauchy problems in (1.1). To the best of our knowledge, this is the first paper addressing to the problem of scattering about NLKG equation on a product space, since this research topic has started with the work of Tzvetkov and Visciglia for NLS on a mixed geometry setting, see [44].

The main result of this paper is stated as follows.

Theorem 1.1. *Assume that $d = 1$ and $\alpha > 4$ or $2 \le d \le 4$ and $\frac{4}{d} < \alpha < \frac{4}{d-1}$. Let*

$$u \in \mathcal{C}(\mathbb{R}; H^1(\mathbb{R}^d \times \mathbb{T})) \cap \mathcal{C}^1(\mathbb{R}; L^2(\mathbb{R}^d \times \mathbb{T})) \tag{1.3}$$

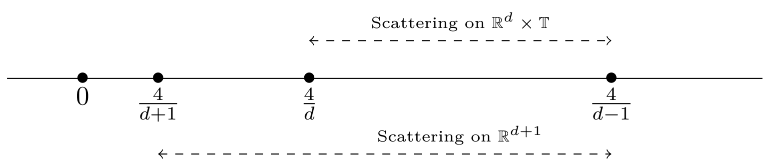
be the unique global solution to (1.1): then for $t \to +\infty$ (respectively $t \to -\infty$) there exists $(f^+, g^+) \in H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$ (respectively $(f^-, g^-) \in H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$) such that

$$\lim_{t\to+\infty} \left\| u(t,x) - u^+(t,x) \right\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \left\| \partial_t u(t,x) - \partial_t u^+(t,x) \right\|_{L^2(\mathbb{R}^d\times\mathbb{T})} = 0, \tag{1.4}$$

$$\left(\textit{respectively} \quad \lim_{t\to-\infty} \left\| u(t,x) - u^-(t,x) \right\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \left\| \partial_t u(t,x) - \partial_t u^-(t,x) \right\|_{L^2(\mathbb{R}^d\times\mathbb{T})} = 0 \right),$$

where $u^+(t,x,y), u^-(t,x,y) \in H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$ are the corresponding solutions to (1.2) with initial data (f^+, g^+) and (f^-, g^-).

Remark 1.2. The scattering property can be proved (in both small and large data cases) for α lying between the L^2–critical exponent on $\mathbb{R}^d$ and the H^1–critical one on $\mathbb{R}^d \times \mathbb{T}$. In fact, considering data which are constant in their compact variable, it is straightforward to see that for $\alpha < \frac{4}{d}$, the analysis is reduced to the L^2–subcritical case on $\mathbb{R}^d$, for which, at the best of our knowledge, no scattering result in energy space is known.
We quickly sum-up the interval where α should lie to fulfil our assumptions:



Picture for α such that Theorem 1.1 holds, with $1 \le d \le 4$ compared to the pure euclidean set-up.

Remark 1.3. Let us also notice that in the small data context, one can deal with both critical exponents $\frac{4}{d}$ and $\frac{4}{d-1}$, either in focusing and defocusing cases, due to the possibility to use a perturbative argument. Therefore, for small data, it is possible to add $d = 5, \alpha = 1$, which is energy critical (see page 5 for details).

In the next paragraphs, we give motivations on the study of such a models and we explain our main achievements.

1.1. **From the euclidean spaces or compact manifolds to the mixed geometry.** About the pure euclidean framework $\mathbb{R}^d$, there is a huge mathematical literature, not only for the Klein-Gordon equation but in general for other dispersive PDEs such as the nonlinear Schrödinger equations (NLS) and the nonlinear wave equations (NLW). We recall that, for such kind of equations, Strichartz estimates play an essential role for the local well-posedness and for the large time analysis of the solutions – once Strichartz estimates have been proved to hold globally in time. The nonlinear Klein-Gordon (NLKG) equation has been deeply studied in the euclidean context, producing a wide literature. We only give here some references amongst others about the scattering results, which is the issue investigated in this paper: in high dimension cases $d \geq 3$, we mention the early works [3, 4, 13–15, 31, 32] on the defocusing energy sub-critical cases, while for the low dimensional case $\mathbb{R}^d$ with $d = 1, 2$, the question of scattering has been solved in [35]. The focusing case have been investigated in [22, 23] both in the energy subcritical and critical cases. For a more complete picture of the known results, we refer the reader to the references contained in the previously cited papers.

For existence results for NLKG, valid on more general manifolds, we refer the reader to the early work by Kapitanskii [25] and Delort and Szeftel [10, 11]. Unlike the full euclidean setting, the compact one does not exhibit the same phenomena. This is due to the presence of periodic solutions inducing a lack of (global in time) summability on them. Basically, the main difference between the equation posed on the euclidean space with respect to a compact manifold is the lack of the dispersive nature of the linear flow in the latter situation. Therefore the question of "mixing" both configurations, to understand the competition of induced phenomena is natural, and the question is whether the dispersive property coming from $\mathbb{R}^d$ is still enough to deduce global feature of the solutions (in our context, their linear behaviour asymptotically in time).

In this paper we are interested in opening the treatment of larga data theory for the nonlinear Klein-Gordon equation (1.1) posed on $\mathbb{R}^d \times \mathbb{T}$, and with the aim of proving Theorem 1.1 above, our first main contribution in this paper is to show the validity of some Strichartz estimates on the whole product space for suitable ranges of exponents; these a priori estimates lead to local existence of solutions to (1.1) and due to conservation of the energy along the nonlinear Klein-Gordon flow and its positive definite character (a consequence of the defocusing nature of the nonlinearities of the equations (1.1)), these solutions can be globally extended on time by time-stepping (as stated in (1.3)). We recall that the energy is defined by

$$E(t) = E(u(t), \partial_t u(t)) := \frac{1}{2}\left(\|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + \frac{2}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} \right) \tag{1.5}$$

and the fact that it is a conserved quantity means that $E(t) = E(0)$ for any t belonging to the maximal interval of existence of the solutions. Furthermore, with these a priori estimates at hand, small data scattering can be shown by a perturbative argument. The small data scattering result is also the first step to carry on a strategy *à la Kenig & Merle*, whose second blueprint is the so-called profile decomposition theorem which we prove for the Klein-Gordon flow posed on the mixed geometry. Once a profile decomposition theorem is proved, we are able to construct a minimal (with respect to the energy) solution which is global in time but it does not enjoy the scattering property: namely, this solution does not satisfy a uniform bound in any Strichartz norm, since this would lead to the scattering property. Let us note the once Strichartz estimates are established, i.e. with a small data scattering theory at hand, the minimal energy non-scattering solution (also said soliton-like solution) is strictly positive. This soliton-like solution also satisfies, again as a byproduct of the profile decomposition, some compactness property which will be crucial to conclude the Kenig & Merle road map. This last step, referred as rigidity, is a Liouville-type theorem which aims to exclude such a global solution with unbounded Strichartz norms. In order to prove ridigity, beside the fundamental precompactness property of the minimal non-scattering solution, it is worth mentioning that in our paper we use only one dimensional Nakanishi/Morawetz estimates, even if the equation we are dealing with are posed on a multidimensional manifold. It is worth mentioning that this non-standard method of using a 1D tool was already used by the first author and Visciglia in [12] for NLS equations perturbed with a not decaying time-independent linear potential.

1.2. **NLS on product spaces.** As mentioned above, the study of qualitative properties of dispersive PDEs posed on a mixed geometry goes back to the work by Tzvetkov and Visciglia in [43], where a small data theory on $\mathbb{R}^d \times \mathcal{M}^k$ for NLS ($\mathcal{M}^k$ being a k-dimensional compact Riemannian manifold) is developed in term of some anisotropic function spaces, followed by a large data theory result by the same authors in [44], where they investigated energy scattering for the defocusing subcritical NLS posed on $\mathbb{R}^d \times \mathbb{T}$. Since these two works, several results have been appeared in the mathematical literature about NLS posed on product spaces. In is worth mentioning, keeping in mind the discussion in subsection 1.1 on the motivations to study dispersive PDEs on product spaces, that NLS on compact manifold have been investigated since the pioneering works [2] and [7]. It is rather impracticable to give a extend list of works on NLS on product spaces (as well on $\mathbb{R}^d$ or on compact manifolds), so we just mention a few of them: [21] and [38] about the issue of global well-posedness; [8], [17], [24] and [39] about scattering property of the solutions; [16] and [18] on modified scattering property of the solutions. In particular, we refer to [18, section 1.1] also for a more exhaustive list of references, and on motivations and backgrounds for NLS.

1.3. **Outline of the paper.**

We outline how the proof of Theorem 1.1 has been organized.
In Section 2, we give a proof of suitable global in time Strichartz estimates on the whole product space for some ranges of exponents. We then deduce global existence of the solution to the Cauchy problem (1.1) before concluding the section with small data scattering results, as it is the first step of the concentration-compactness and rigidity scheme. Section 3 presents the proof of a profile decomposition theorem, which in turn exhibits the existence of a non-trivial minimal energy soliton-like solution to (1.1) in Section 4. The latter one is a global non-scattering solution enjoying some compactness property. After the construction of this minimal element, we finally prove in Section 5, by means of a priori uniform bounds, that such a solution cannot exist.

1.4. **Notations.**

Along the paper, the space variable x refers to the euclidean component of the product space $\mathbb{R}^d \times \mathbb{T}$, while y belongs to the compact part: therefore $(x, y) \in \mathbb{R}^d \times \mathbb{T}$. Consequently the notation $\Delta_{\mathbb{R}^d}$ and $\Delta_{\mathbb{T}}$ is used when we consider the restrictions of Δ on $\mathbb{R}^d$ and $\mathbb{T}$, respectively. By analogous meaning, ∇ stands for the $(d + 1)$–components vector $\nabla = (\nabla_x, \partial_y)$.

With $L^p := L^p(\mathbb{R}^d \times \mathbb{T})$ we mean the usual Lebesgue spaces and L^p_x and L^p_y stand for $L^p(\mathbb{R}^d)$ and $L^p(\mathbb{T})$ respectively. The same holds for the Hilbert space $H^s := H^s(\mathbb{R}^d \times \mathbb{T})$ with compact notations $H^s_x := H^s(\mathbb{R}^d)$ and $H^s_y := H^s(\mathbb{T})$.

The Bochner space $L^p(I; X)$ is classically defined as the space of functions $f : I \subseteq \mathbb{R} \to X$ having finite $L^p(I; X)$ norm, where

$$\|f\|_{L^p(I;X)} := \left(\int_I \|f\|^p_X(t)\, dt\right)^{1/p}.$$

If $I = \mathbb{R}$ we simply write L^pX. For any real $p \geq 1$, we denote with p' its conjugate given by $p' = \frac{p}{p-1}$. For a vector (f, g) we write $(f, g)^T = \begin{pmatrix} f \\ g \end{pmatrix}$ when convenient.
We indicate by $\mathcal{F}$ and $\mathcal{F}^{-1}$ the Fourier transform and its inverse, respectively, with respect to the x variable.

The expressions $A \lesssim B$ or $A \gtrsim B$ mean that there exists a universal constant $C > 0$ such that $A \leq CB$ or $A \geq CB$, respectively, while $A \sim B$ means that both previous relations hold true.

2. Strichartz estimates on waveguide

In this section we prove some Strichartz estimates on the whole product space, and deduce global existence of the solution to the Cauchy problem (1.1) in our setting, before handling small data scattering results. These results are the first step to perform a concentration-compactness method in the subsequent sections.
In our context, energy conservation is not enough to handle the global existence problem and we need global in time Strichartz estimates, for which we do not have any restriction on the euclidean dimension d. These are stated in the following theorem.

Theorem 2.1 (Strichartz estimates). *Let $d \in \mathbb{N}$ and $1 \le q, r \le \infty$ such that (q,r) satisfies*

$$\begin{cases} \dfrac{2q}{q-4} \le r, & q \ge 4, \quad \text{if } d = 1 \\ \\ \dfrac{2dq}{dq-4} \le r \le \dfrac{2q(d+1)}{q(d-1)-2}, & q > 2, \quad \text{if } d = 2 \quad q \ge 2, \ \text{if } d \ge 3 \end{cases}. \tag{2.1}$$

Let $w \in \mathcal{C}(\mathbb{R}; H^1) \cap \mathcal{C}^1(\mathbb{R}; L^2)$ be the unique solution to the following nonlinear problem:

$$\begin{cases} \partial_{tt} u - \Delta u + u = F, & (t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \\ u(0,x,y) = f \in H^1 \\ \partial_t u(0,x,y) = g \in L^2 \end{cases}, \tag{2.2}$$

where $F = F(t,x,y) \in L^1 L^2$. Then the estimate below holds:

$$\|w\|_{L^q L^r} \le C \left(\|f\|_{H^1} + \|g\|_{L^2} + \|F\|_{L^1 L^2} \right).$$

Remark 2.2. The method we apply to obtain Strichartz estimates on the whole product space is divided into the following steps:

(1) we state the estimates on $\mathbb{R}^d$, involving Besov spaces;
(2) we use embedding theorems to deduce some estimates that hold in Lebesgue spaces posed on $\mathbb{R}^d$;
(3) we use a scaling argument to handle masses different from one;
(4) we write (2.2) in the basis of eigenfunctions of $\mathbb{T}$ and prove Theorem 2.1 properly summing on the coefficients.

This approach has been used by the second author and Visciglia in [19], dealing with energy-critical nonlinearities in order to treat the small data theory. There, only critical embeddings were needed to prove small data scattering. In our subcritical setting, one has to consider a wider range of Strichartz estimates to prove such results, obtained with "subcritical" embeddings. Deeper discussions about these estimates will be made along the proof of Theorem 2.1.

Remark 2.3. Global existence of the solution as in Theorem 1.1 then classically follows: a standard contraction principle performed on a small time interval $T = T(\|f\|_{H^1} + \|g\|_{L^2})$ implies local well-posedness on suitable Banach spaces. Energy conservation (1.5) gives therefore global existence by time-stepping, since we are in the defocusing case.
We do not write the details of the proof of global existence in this paper since it does not require any tricky computation.

We are now able to state the small data scattering result.

Theorem 2.4 (Small data scattering). *Let $d = 1$ and $\alpha \ge 4$ or $2 \le d \le 5$ and α be such that $\frac{4}{d} \le \alpha \le \frac{4}{d-1}$. Then there exists $\varepsilon > 0$ such that for all $(f,g) \in H^1 \times L^2$ satisfying $\|f\|_{H^1} + \|g\|_{L^2} < \varepsilon$, the solution $u(t,x,y)$ to the Cauchy problem (1.1) is global and scatters in the sense of (1.4).*

Remark 2.5. It is worth mentioning that the analysis for small initial data can be stated without any further restriction in the focusing case, namely replacing in (1.1) the sign in front of the nonlinear term with a plus sign. Furthermore, observe that the result of the theorem above is valid also in the critical cases. The main restriction on α is carried by the fact that $(\alpha+1, 2\alpha+2)$ should satisfy (2.1). It is immediate to check that $d = 5, \alpha = 1$ is the only case that can be handled for $d > 4$ and it is critical.

In order to prove Theorem 2.1 we recall the definition of the Besov spaces. Given a cut-off function χ_0 such that

$$C_c^\infty(\mathbb{R}^d; \mathbb{R}) \ni \chi_0(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \le 1 \\ 0 & \text{if} \quad |\xi| > 2 \end{cases}$$

then are defined the following dyadic functions

$$\varphi_j(\xi) = \chi_0(2^{-j}\xi) - \chi_0(2^{-j+1}\xi),$$

yielding to the partition of the unity

$$\chi_0(\xi) + \sum_{j>0} \varphi_j(\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^d.$$

By denoting with $\mathcal{S}(\mathbb{R}^d)$ the set of all tempered distributions on $\mathbb{R}^d$, let us introduce the operators P_j, $j \in \mathbb{N} \cup \{0\}$, acting on $\mathcal{S}(\mathbb{R}^d)$ and defined as follows:

$$\begin{aligned} P_0 f &:= \mathcal{F}^{-1}\left(\chi_0 \mathcal{F}(f)\right), \\ P_j f &:= \mathcal{F}^{-1}\left(\varphi_j \mathcal{F}(f)\right), \quad \forall j \in \mathbb{N}. \end{aligned}$$

Let $s \in \mathbb{R}$. Then, for $0 < q \leq \infty$, the Besov space $B^s_{q,2}$ is defined by

$$B^s_{q,2}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}(\mathbb{R}^d) \middle| \left\{ 2^{js} \|P_j f\|_{L^q(\mathbb{R}^d)} \right\}_{j \in \mathbb{N}\cup\{0\}} \in l^2 \right\},$$

where l^2 is the classical space of square-summable sequences.

2.1. **Proof of Theorem 2.1.** We rigorously prove the steps listed in Remark 2.2.

Step 1. We begin with the following proposition which is given in a pure euclidean context.

Proposition 2.6 (Strichartz estimates for the euclidean case (from [33]))**.** *Let $d \geq 1$ and $2 \leq q, \rho \leq \infty$ such that*

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{\rho}\right) \quad (\textit{with the restriction } q > 2 \textit{ if } d = 2,\, q \geq 4 \textit{ if } d = 1). \tag{2.3}$$

Consider $w = w(t,x)$ satisfying

$$\begin{cases} \partial_{tt} w - \Delta_{\mathbb{R}^d} w + w = F, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ w(0,x) = f \in H^1(\mathbb{R}^d) \\ \partial_t w(0,x) = g \in L^2(\mathbb{R}^d) \end{cases}, \tag{2.4}$$

where $F = F(t,x) \in L^1(\mathbb{R}; L^2(\mathbb{R}^d))$. Then

$$\|w\|_{L^q(\mathbb{R}; B^s_{\rho,2}(\mathbb{R}^d))} \leq C\left(\|f\|_{H^1(\mathbb{R}^d)} + \|g\|_{L^2(\mathbb{R}^d)} + \|F\|_{L^1(\mathbb{R}; L^2(\mathbb{R}^d))} \right), \tag{2.5}$$

where $C > 0$ depends only on the choice of the pair (q, r) and on the dimension d and $s \in [0,1]$ is defined by

$$s = 1 - \frac{1}{2}\left(\frac{d}{2} + 1\right)\left(\frac{1}{\rho'} - \frac{1}{\rho}\right) = 1 - \frac{1}{2}\left(\frac{d}{2} + 1\right)\left(1 - \frac{2}{\rho}\right). \tag{2.6}$$

Proof. The proof is detailed in [33] for the non-endpoint cases, and by using the same dispersive estimates contained therein, a Keel and Tao argument, see [26], for $d \geq 3$ gives the endpoint cases. It is worth mentioning that in the former work, the estimates above are stated for more general settings (for the space in which the source term lies, the data space and the range of admissible pairs). We choose to give here the version fixing the setting that is suitable for our framework and especially for the use of the scaling argument with homogenous spaces in which Besov spaces are embedded (see the next step of the proof of Theorem 2.1 below). Indeed, the source term is estimated in [33, Lemma 2.1] by means of the $L^{\tilde{q}'}(\mathbb{R}; B^{s-1}_{\tilde{\rho}',2}(\mathbb{R}^d))$ norm of F, for $(\tilde{q}, \tilde{\rho})$ an admissible pair. So our required estimate (2.5) follows by selecting $(\tilde{q}, \tilde{\rho}) = (\infty, 2)$ therefore by noticing that $s - 1 = 0$ and using the fact that $B^0_{2,2}(\mathbb{R}^d)$ is equivalent to $L^2(\mathbb{R}^d)$. □

In conclusion of this first step through the proof of Strichartz estimates on product spaces, it is worth mentioning, besides the already mentioned ones, the early works [3, 4, 13–15, 37] about the Strichartz estimates for the Klein-Gordon equation in a euclidean framework, as well as the more recent papers [29, 30] for the endpoint cases when $d \geq 3$.

Step 2. We state the following embedding theorem contained in [41, 42] and references therein.

Theorem 2.7 (Embedding theorems)**.** *Let $d \geq 1$, $s > 0$, and $1 < r, \rho < \infty$. Consider the Besov space $B^s_{\rho,2}(\mathbb{R}^d)$ and the Lebesgue space $L^r(\mathbb{R}^d)$. Then the embedding relations below hold:*

(1) $B^s_{\rho,2}(\mathbb{R}^d) \hookrightarrow L^\rho(\mathbb{R}^d)$ ($\rho = 1, \infty$ allowed);

(2) for $\rho^* := \dfrac{d\rho}{d - s\rho}$, when $d > s\rho$, then $B^s_{\rho,2}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ for $\rho \leq r \leq \rho^*$;

(3) if $d \leq s\rho$, then $B^s_{\rho,2}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ for $\rho \leq r < +\infty$.

Remark 2.8. Observe that in the statement of the embedding theorem, s is not assumed to be the same of (2.6). We kept the same station since in the sequel they will be identified.

Performing quick computations, we notice that for s satisfying (2.6) and (q, ρ) as in (2.3), the conditions of the theorem yield

$$\begin{aligned} d - s\rho &< 0 \qquad \text{if} \quad d = 1, \\ d - s\rho &= 0 \qquad \text{if} \quad d = 2, \\ d - s\rho &> 0 \qquad \text{if} \quad d \geq 3. \end{aligned}$$

By [40], we have in our setting $B^s_{\rho,2}(\mathbb{R}^d) \hookrightarrow W^{s,\rho}(\mathbb{R}^d)$. Thus, by using the Sobolev embedding $W^{s,\rho}(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$ valid for

$$\begin{aligned} &r \in [\rho, \infty] &&\text{if} \quad d = 1, \\ &r \in [\rho, \infty) &&\text{if} \quad d = 2, \\ &r \in [\rho, \rho^*] &&\text{if} \quad d \geq 3, \end{aligned}$$

and computing ρ, ρ^* in terms of q, we obtain

$$\frac{2dq}{dq - 4} \leq r \qquad \text{if} \quad d = 1 \text{ or } 2, \tag{2.7}$$

$$\frac{2dq}{dq - 4} \leq r \leq \frac{2d^2 q}{d^2 q - 2d - 2dq + 4} \qquad \text{if} \quad d \geq 3. \tag{2.8}$$

Strichartz estimates involving Lebesgue spaces instead of Besov spaces follow immediately by applying the previous embedding theorem to Proposition 2.6, with r satisfying (2.7) or (2.8).

Step 3. By defining $w_\lambda := w\left(\sqrt{\lambda} t, \sqrt{\lambda} x\right)$ with w as in (2.4) and noticing that it satisfies

$$\begin{cases} \partial_{tt} w_\lambda - \Delta_{\mathbb{R}^d} w_\lambda + \lambda w_\lambda = F_\lambda, & (t, x) \in \mathbb{R} \times \mathbb{R}^d \\ w_\lambda(0, \cdot) = f_\lambda & \\ \partial_t w_\lambda(0, \cdot) = g_\lambda & \end{cases}, \tag{2.9}$$

where

$$f_\lambda(x) = f\left(\sqrt{\lambda} x\right), \quad g_\lambda(x) = \sqrt{\lambda} g\left(\sqrt{\lambda} x\right), \quad F_\lambda(t, x) = \lambda F\left(\sqrt{\lambda} t, \sqrt{\lambda} x\right),$$

we can claim the next result.

Proposition 2.9. *Consider a pair (q, ρ) as in (2.3), s given by (2.6) and r as in Theorem 2.7. Consider w given by (2.4) for which Proposition 2.6 holds. Then one has for (2.9)*

$$\lambda^{\frac{1}{2}\left(\frac{d}{r} + \frac{1}{q} - \frac{d}{2} + 1\right)} \|w_\lambda\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^d))} \leq C\left(\sqrt{\lambda} \|f_\lambda\|_{L^2(\mathbb{R}^d)} + \|f_\lambda\|_{\dot{H}^1(\mathbb{R}^d)} + \|g_\lambda\|_{L^2(\mathbb{R}^d)} + \|F_\lambda\|_{L^1(\mathbb{R}; L^2(\mathbb{R}^d))}\right). \tag{2.10}$$

Proof. The proof follows by straightforward algebraic calculations. ∎

Step 4. Once we can rely on the ingredients of the previous steps, we finally use the strategy from [19,43] to conclude with the desired result. We write $\{\lambda_j\}_{j\geq 0}$ for the set of eigenvalues of $-\Delta_{\mathbb{T}}$, sorted in ascending order and taking in account their multiplicities; we also introduce $\{\Phi_j(y)\}_{j\geq 0}$, the corresponding eigenfunctions associated to λ_j, i.e.

$$-\Delta_{\mathbb{T}}\Phi_j = \lambda_j\Phi_j, \quad \lambda_j \geq 0, \quad j \in \mathbb{N}\cup\{0\}. \tag{2.11}$$

The latter sequence $\{\Phi_j(y)\}_{j\geq 0}$ provides an orthonormal basis of $L^2(\mathbb{T})$. We now consider the solution to (2.2) and we write the functions in terms of (2.11):

$$\begin{aligned} w(t,x,y) &= \sum_{j=0}^{\infty} w_j(t,x)\Phi_j(y), \\ F(t,x,y) &= \sum_{j=0}^{\infty} F_j(t,x)\Phi_j(y), \\ f(x,y) &= \sum_{j=0}^{\infty} f_j(x)\Phi_j(y), \\ g(x,y) &= \sum_{j=0}^{\infty} g_j(x)\Phi_j(y), \end{aligned} \tag{2.12}$$

with $w_j = w_j(t,x)$ satisfying

$$\partial_{tt}w_j - \Delta_{\mathbb{R}^d}w_j + w_j + \lambda_j w_j = F_j, \quad w_j(0,x) = f_j(x), \quad \partial_t w_j(0,x) = g_j(x).$$

Taking $\lambda = 1+\lambda_j$ in (2.10) it follows that

$$\begin{aligned} (\lambda_j+1)^{\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right)}\|w_j\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))} \leq C\Big((\lambda_j+1)^{1/2}\|f_j\|_{L^2(\mathbb{R}^d)} + \|f_j\|_{\dot H^1(\mathbb{R}^d)} + \|g_j\|_{L^2(\mathbb{R}^d)} \\ +\|F_j\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d))}\Big). \end{aligned}$$

Then, summing in j the squares one obtains

$$\begin{aligned} \left\|(\lambda_j+1)^{\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right)}w_j\right\|_{l^2_jL^q(\mathbb{R};L^r(\mathbb{R}^d))} \leq C\left(\left\|(\lambda_j+1)^{1/2}f_j\right\|_{l^2_jL^2(\mathbb{R}^d)} + \|f_j\|_{l^2_j\dot H^1(\mathbb{R}^d)} + \|g_j\|_{l^2_jL^2(\mathbb{R}^d)} \\ +\|F_j\|_{l^2_jL^1(\mathbb{R};L^2(\mathbb{R}^d))}\right). \end{aligned}$$

Since $\max\{1,2\} = 2 \leq \min\{q,\rho\}$, the Minkowski inequality can be applied

$$\begin{aligned} \left\|(\lambda_j+1)^{\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right)}w_j\right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d))l^2_j} \leq& C\left(\left\|(\lambda_j+1)^{1/2}f_j\right\|_{L^2(\mathbb{R}^d)l^2_j} + \right. \\ & \left. \|g_j\|_{L^2(\mathbb{R}^d)l^2_j} + \|F_j\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d))l^2_j}\right). \end{aligned}$$

Hence, by means of the Plancherel identity, one is able to handle the y variable to obtain

$$\left\|(1-\Delta_y)^{\frac{1}{2}\left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right)}w\right\|_{L^q(\mathbb{R};L^r(\mathbb{R}^d)L^2(\mathbb{T}))} \leq C\left(\|f\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \|g\|_{L^2(\mathbb{R}^d\times\mathbb{T})} + \|F\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d\times\mathbb{T}))}\right)$$

which in turn implies

$$\|w\|_{L^q(\mathbb{R};L^r_x(\mathbb{R}^d)H^\gamma_y(\mathbb{T}))} \leq C\left(\|f\|_{H^1(\mathbb{R}^d\times\mathbb{T})} + \|g\|_{L^2(\mathbb{R}^d\times\mathbb{T})} + \|F\|_{L^1(\mathbb{R};L^2(\mathbb{R}^d)L^2(\mathbb{T}))}\right),$$

where

$$\gamma = \left(\frac{d}{r}+\frac{1}{q}+1-\frac{d}{2}\right).$$

We easily see that $\gamma \geq 0$ when $d = 1, 2$, and by computing the condition $\gamma \geq 0$ for $d \geq 3$, we have

$$\begin{aligned}\gamma \geq 0 \iff & \frac{2dq + 2r + 2qr - dqr}{2qr} \geq 0 \\ \iff & r \leq \frac{2dq}{dq - 2q - 2}.\end{aligned}$$

Since $\dfrac{2dq}{dq-2q-2} \geq \rho^* > r$, we can establish that under (2.8), γ is always nonnegative. The proof is then completed by using a Sobolev embedding available for $\gamma \geq 0$

$$H^\gamma(\mathbb{T}) \hookrightarrow L^r(\mathbb{T}) \tag{2.13}$$

which holds (at least) under one of the following conditions:

$$\begin{aligned} &\bullet\quad 2\gamma < 1, \frac{2}{1-2\gamma} \geq r, \quad \text{which is the “usual” condition to have Sobolev embedding,} \\ &\bullet\quad 2\gamma \geq 1, \quad r \geq 2, \qquad \text{which ensures (2.13) with } H^\gamma(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T}) \text{ allowing to control any} \\ &\qquad\qquad\qquad\qquad\qquad\quad L^r \text{ norm with the } H^\gamma \text{ norm since } \mathbb{T} \text{ is of finite volume.}\end{aligned} \tag{2.14}$$

Then by gluing together all conditions (2.7),(2.8),(2.14) in terms of q, we exhibit the exponent r for which the Strichartz estimates can be proved: for $d = 1$, since $\gamma > 1/2$, we have $H^\gamma(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{T})$ and so

$$\frac{2q}{q-4} \leq r.$$

For $d \geq 2$

$$\frac{2dq}{dq-4} \leq r \leq \min\left\{\frac{2d^2q}{d^2q - 2d - 2dq + 4}, \frac{2q(d+1)}{dq - q - 2}\right\},$$

that is

$$\frac{2dq}{dq-4} \leq r \leq \frac{2q(d+1)}{dq-q-2},$$

which concludes the proof of Theorem 2.1.

As already introduced before, the tool given by the Strichartz estimates implies the small data scattering, which holds also in the critical cases.

Proof of Theorem 2.4. We recall that in the statement of Theorem 2.4, we consider $\frac{4}{d} \leq \alpha \leq \frac{4}{d-1}$ for $2 \leq d \leq 5$ and $\alpha \geq 4$ if $d = 1$. It is standard to prove that the smallness assumption of Theorem 2.4 implies that the solution belongs to $\mathcal{C}(\mathbb{R}; H^1) \cap \mathcal{C}^1(\mathbb{R}; L^2) \cap L^{\alpha+1}L^{2(\alpha+1)}$, therefore let us give, for sake of completeness, a proof of the scattering as consequence of such space-time regularity of the solution. We handle both focusing and defocusing nonlinearities as pointed out in Remark 1.3.

We rewrite (1.1) in the vector form. More precisely if u is a solution to (1.1) then the vector $(u, \partial_t u)^T$ satisfies

$$\partial_t \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Delta + 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} + \begin{pmatrix} 0 \\ \pm |u|^\alpha u \end{pmatrix}.$$

We have that the following exponential matrix operator

$$e^{tH} = \begin{pmatrix} \cos\left(t \cdot \sqrt{1-\Delta}\right) & \dfrac{\sin\left(t \cdot \sqrt{1-\Delta}\right)}{\sqrt{1-\Delta}} \\ -\sin\left(t \cdot \sqrt{1-\Delta}\right) \cdot \left(\sqrt{1-\Delta}\right) & \cos\left(t \cdot \sqrt{1-\Delta}\right) \end{pmatrix} \tag{2.15}$$

is unitary on the energy space $H^1 \times L^2$ (see [36]). Moreover

$$\begin{pmatrix} u \\ \partial_t u \end{pmatrix} = e^{tH} \begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t e^{(t-s)H} \begin{pmatrix} 0 \\ |u|^\alpha u \end{pmatrix} (s) ds$$

and then, since e^{tH} is unitary

$$e^{-tH} \begin{pmatrix} u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} + \int_0^t e^{-sH} \begin{pmatrix} 0 \\ |u|^\alpha u \end{pmatrix} (s) ds.$$

We now write $\vec{V}(t) = e^{-tH} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}$, and consider $0 < \tau < t$. Then

$$\|\vec{V}(t) - \vec{V}(\tau)\|_{H^1 \times L^2} \leq C \int_\tau^t \||u|^\alpha u(s)\|_{L^2} ds \leq C \|u\|^{\alpha+1}_{L^{\alpha+1}([\tau,t],L^{2(\alpha+1)})},$$

and since $\|u\|^{\alpha+1}_{L^{\alpha+1}([\tau,t],L^{2(\alpha+1)})}$ tends to zero as t, τ tends to infinity (since the solution belongs to $L^{\alpha+1}\left(\mathbb{R}; L^{2(\alpha+1)}\right))$ we can claim that there exist $(f^\pm, g^\pm) \in H^1 \times L^2$ such that $\vec{V}(t) \to \begin{pmatrix} f^\pm \\ g^\pm \end{pmatrix}$ in $H^1 \times L^2$ as $t \to \pm\infty$. Using again the unitary property of the linear flow e^{tH} the proof is concluded. □

We conclude this section on the Strichartz estimates with the following comments.

Remark 2.10. We emphasize that the restricted range of Strichartz estimates cannot allow a simple use of Morawetz estimates to deduce scattering in the classical manner, i.e. as in $\mathbb{R}^d$. This would require a investigation on the possible generalization of the Strichartz estimates on the mixed geometry, which falls out of our goal. Furthermore, we think that the use of a concentration/compactness method how to concern the proof of large data scattering is more elegant (and general), and allows us to use only homogeneous function spaces.

Remark 2.11. The proof of Theorem 2.4 shows as a uniform in time control of the Strichartz norm yields to scattering of global solutions. The concentration/compactness and rigidity scheme therefore will be implemented in order to show that every global solution to (1.1) has a bounded global in time $L^{\alpha+1}L^{2(\alpha+1)}$ Strichartz norm, independently of the size of the initial datum.

3. PROFILE DECOMPOSITION THEOREM

In this section we provide a profile decomposition theorem which is the main ingredient in the proof of scattering properties in the whole energy space.
We start with the following preliminary lemma. We use the following convention

$$2^* = \begin{cases} \frac{2(d+1)}{d-1}, & \text{if} \quad d \geq 2 \\ +\infty, & \text{if} \quad d = 1 \end{cases}. \tag{3.1}$$

Lemma 3.1. *Let $\{v_n(x,y)\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R}^d \times \mathbb{T})$, with $1 \leq d \leq 4$, be a bounded sequence. Define the set*

$$\begin{aligned} \Lambda := \Lambda(\{v_n(x,y)\}_{n\in\mathbb{N}}) = \{w(x,y) \in L^2 \,\big|\, \exists \{(x_n,y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^d \times \mathbb{T} \quad &\textit{such that,} \\ \textit{up to subsequence,} \quad v_n(x-x_n, y-y_n) \overset{L^2}{\rightharpoonup} w(x,y)\}& \end{aligned} \tag{3.2}$$

and let

$$\lambda := \lambda(\{v_n(x,y)\}_{n\in\mathbb{N}}) = \sup\{\|w\|_{L^2}, w \in \Lambda\}. \tag{3.3}$$

Then, for any q such that $2q \in (2, 2^*)$ we have

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim \lambda^e$$

where

$$e = e(q, d) = \frac{q-1}{3-5q}\left(\frac{q(d-1)-(d+1)}{q}\right) > 0.$$

Proof. By the Sobolev embedding theorem, see [20], the energy space embeds continuously in the Lebesgue space L^{2^*}. In particular $H^1(\mathbb{R}^d \times \mathbb{T}) \hookrightarrow L^{2q}(\mathbb{R}^d \times \mathbb{T})$ for any $q \in [1, 2^*/2]$ if $d \geq 2$ while $q \geq 1$ if $d = 1$, where 2^* is defined in (3.1). Consider, as Fourier multiplier, a cut-off function in the frequencies space $\mathbb{R}^d_\xi$ where the cut-off is given by

$$C_c^\infty(\mathbb{R}^d; \mathbb{R}) \ni \chi(\xi) = \begin{cases} 1 & \text{if} \quad |\xi| \leq 1 \\ 0 & \text{if} \quad |\xi| > 2 \end{cases}.$$

By setting $\chi_R(\xi) = \chi(\xi/R)$, $R > 0$, we define the pseudo-differential operator with symbol χ_R. It is given by $\chi_R(|D|)f = \mathcal{F}^{-1}(\chi_R \mathcal{F}f)(x)$ and similarly we define the operator $\tilde{\chi}_R(|D|)$ with the associated symbol given by $\tilde{\chi}_R(\xi) = 1 - \chi_R(\xi)$. Later on we will also use the well-known properties

$$\mathcal{F}(fg) = \mathcal{F}(f) * \mathcal{F}(g)$$
$$\mathcal{F}(f(\sigma \cdot)) = \sigma^{-d}\mathcal{F}f(\cdot/\sigma)$$

which hold for any smooth functions $f, g : \mathbb{R}^d \to \mathbb{R}$ and any positive real number σ. In order to apply the Hausdorff-Young inequality $\mathcal{F} : L^p \to L^{p'}$ for any $p \in [1, 2]$, we set $2q = p'$ and $p = \frac{p'}{p'-1} = \frac{2q}{2q-1} \in (1, 2)$. We then use the Hölder inequality with $\frac{1}{p} = \frac{1}{2} + \frac{1}{r}$ and by exploiting the precise structure of $\mathbb{T}$ we can write, for every $n \in \mathbb{N}$,

$$v_n(x, y) = \sum_{k\in\mathbb{Z}} v_n^k(x)e^{iky}, \tag{3.4}$$

where the functions v_n^k are the Fourier coefficients, and similarly

$$\tilde{\chi}_R(|D|)v_n(x, y) = \sum_{k\in\mathbb{Z}} \tilde{\chi}_R(|D|)v_n^k(x)e^{iky}.$$

We first notice the embedding $H^{\frac{1}{2}-\frac{1}{2q}}(\mathbb{T}) \hookrightarrow L^{2q}(\mathbb{T})$, which enables to write

$$\begin{aligned}
\|\tilde{\chi}_R(|D|)v_n\|^2_{L^{2q}} \lesssim \|\tilde{\chi}_R(|D|)v_n\|^2_{L^{2q}_x H_y^{\frac{1}{2}-\frac{1}{2q}}} &= \left\| \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} |\tilde{\chi}_R(|D|)v_n^k|^2 \right\|_{L^q_x} \\
&\lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \|\tilde{\chi}_R(|D|)v_n^k\|^2_{L^{2q}_x} \lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \|\mathcal{F}^{-1}(\tilde{\chi}_R(|\xi|)\hat{v}_n^k)(x)\|^2_{L^{2q}_x} \\
&\lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \|\tilde{\chi}_R(|\xi|)\hat{v}_n^k(\xi)\|^2_{L_\xi^{2q/(2q-1)}} \\
&\lesssim \sum_{k\in\mathbb{Z}} \langle k\rangle^{1-\frac{1}{q}} \|\langle\xi\rangle^{\frac{1}{2}+\frac{1}{2q}}\hat{v}_n^k(\xi)\|^2_{L^2_\xi} \|\tilde{\chi}_R(|\xi|)\langle\xi\rangle^{-\frac{1}{2}-\frac{1}{2q}}\|^2_{L_\xi^{2q/(q-1)}}
\end{aligned} \tag{3.5}$$

where an Hölder inequality was used in the last step. We notice that the last factor in the r.h.s. term is controlled as follows:

$$
\begin{aligned}
\|\tilde\chi_R(|\xi|)\langle\xi\rangle^{-\frac12-\frac{1}{2q}}\|^2_{L_\xi^{2q/(q-1)}} &\lesssim \left(\int_{|\xi|\ge R}\frac{d\xi}{(1+|\xi|^2)^{\left(\frac14+\frac{1}{4q}\right)\left(\frac{2q}{q-1}\right)}}\right)^{\frac{q-1}{q}}\\
&\lesssim \left(\int_{|\xi|\ge R}\frac{d\xi}{|\xi|^{\frac{q+1}{q-1}}}\right)^{\frac{q-1}{q}}\\
&\lesssim \left(\int_R^\infty \frac{1}{\rho^{\frac{q+1}{q-1}-d+1}}\,d\rho\right)^{\frac{q-1}{q}}\\
&\lesssim \left(R^{d-\frac{q+1}{q-1}}\right)^{\frac{q-1}{q}} = R^{\frac{d(q-1)}{q}-\frac{q+1}{q}}
\end{aligned}
$$

where the integrability of the term has been checked and $\frac{d(q-1)}{q}-(1+\frac1q)<0$. Thus, by the Plancherel identity, estimate (3.5) can be concluded as

$$
\begin{aligned}
\|\tilde\chi_R(|D|)v_n\|^2_{L^{2q}} &\lesssim R^{\frac{d(q-1)}{q}-(1+\frac1q)}\sum_{k\in\mathbb{Z}}\langle k\rangle^{1-\frac1q}\int\langle\xi\rangle^{1+\frac1q}|\hat v_n^k(\xi)|^2\,d\xi\\
&\lesssim R^{\frac{d(q-1)}{q}-(1+\frac1q)}\|v_n\|^2_{H^1}.
\end{aligned}
$$

Recalling that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in H^1, we summarize the above estimate with

$$
\|\tilde\chi_R(|D|)v_n\|_{L^{2q}} \lesssim R^{\frac{d(q-1)}{2q}-\frac{q+1}{2q}} = R^{\frac{q(d-1)-(d+1)}{2q}}.
$$

We now use (3.4) and we define the localized part of v_n as

$$
\chi_R(|D|)v_n(x,y) = \sum_{|k|\le M}\chi_R(|D|)v_n^k(x)e^{iky} + \sum_{|k|>M}\chi_R(|D|)v_n^k(x)e^{iky} := \chi_R^{\le M}(|D|)v_n + \chi_R^{>M}(|D|)v_n.
$$

We estimate the tail $\chi_R^{>M}(|D|)v_n$ as follows. By means of the Minkowski inequality and the Cauchy-Schwarz inequality we get

$$
\begin{aligned}
\|\chi_R^{>M}(|D|)v_n\|_{L^2} &\le C(Vol(\mathbb{T}))\sum_{|k|>M}\|\chi_R(|D|)v_n^k(x)\|_{L_x^2}\\
&\lesssim \left(\sum_{|k|>M}\frac{1}{k^2}\right)^{1/2}\left(\sum_{|k|>M}k^2\|\chi_R(|\xi|)\hat v_n^k(\xi)\|^2_{L_\xi^2}\right)^{1/2}\\
&\lesssim \left(\sum_{|k|>M}\frac{1}{k^2}\right)^{1/2}\left(\sum_{|k|>M}k^2\|\hat v_n^k(\xi)\|^2_{L_\xi^2}\right)^{1/2}\\
&\lesssim \left(\sum_{|k|>M}\frac{1}{k^2}\right)^{1/2}\|v_n\|_{L_x^2H_y^1} \lesssim \left(\sum_{|k|>M}\frac{1}{k^2}\right)^{1/2}.
\end{aligned}
$$

Since

$$
\sum_{k=M+1}^{\infty}a_k \le \int_M^\infty f(x)\,dx
$$

where $f : [1, +\infty) \to \mathbb{R}^+$ is a decreasing function such that $a_k = f(k)$, (it is assumed here that $f(\zeta) = \zeta^{-2}$) then

$$\left(\sum_{|k|>M+1} k^{-2} \right)^{1/2} \lesssim M^{-1/2}$$

and so

$$\|\chi_R^{>M}(|D|)v_n\|_{L^2} \lesssim M^{-1/2}.$$

A straightforward and classical application of the Hölder inequality yields that if $f \in L^{p_1} \cap L^{p_2}$ and $\theta \in [0, 1]$, for p defined as $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ we have the interpolation estimate

$$\|f\|_{L^p} \leq \|f\|_{L^{p_1}}^{\theta} \|f\|_{L^{p_2}}^{1-\theta}. \tag{3.6}$$

Therefore by (3.6) we get

$$\begin{aligned} \|\chi_R^{>M}(|D|)v_n\|_{L^{2q}_{x,y}} &\leq \|\chi_R^{>M}(|D|)v_n\|_{L^2}^{\theta} \|\chi_R^{>M}(|D|)v_n\|_{L^{2^*}}^{1-\theta} \\ &\lesssim \|\chi_R^{>M}(|D|)v_n\|_{L^2}^{\frac{(d+1)}{2q} - \frac{d-1}{2}} \lesssim M^{-\frac{1}{2}\left(\frac{(d+1)}{2q} - \frac{d-1}{2}\right)}. \end{aligned} \tag{3.7}$$

It remains to estimate the term $\sum_{|k|\leq M} \chi_R(|D|)v_n^k(x)e^{iky}$. Denoting by D_M the Dirichlet Kernel

$$D_M(y) = \sum_{k=-M}^{M} e^{iky},$$

we can write

$$\chi_R^{\leq M}(|D|)v_n(x, y) = \sum_{|k|\leq M} \chi_R(|D|)v_n^k(x)e^{iky} = \int_{\mathbb{T}} \chi_R(|D|)v_n(x, z)D_M(y - z)\, dz,$$

and we choose a sequence $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}^d \times \mathbb{T}$ such that

$$\begin{aligned} \|\chi_R^{\leq M}(|D|)v_n\|_{L^\infty} &\leq 2 \left|\chi_R^{\leq M}(|D|)v_n(x_n, y_n)\right| \\ &= 2R^d \left| \int_{\mathbb{R}^d\times\mathbb{T}} \eta(Rx)D_M(y)v_n(x - x_n, y - y_n)\, dy\, dx \right|, \end{aligned}$$

where $R^d\eta(Rx) = \mathcal{F}^{-1}(\chi_R(|\xi|))$. Observe that $\eta(Rx)D_M(y)$ is a function in $L^2_{x,y}$ and that $\|\eta(Rx)D_M(y)\|_{L^2} \lesssim R^{-d/2}M\|\eta\|_{L^2}$. Up to subsequences, from (3.2) and (3.3) we get

$$\begin{aligned} \limsup_{n\to\infty} \|\chi_R^{\leq M}(|D|)v_n\|_{L^\infty} &\leq \limsup_{n\to\infty} 2R^d \left| \int_{\mathbb{R}^d\times\mathbb{T}} \eta(Rx)D_M(y)v_n(x - x_n, y - y_n)\, dy\, dx \right| \\ &= 2R^d \left| \int_{\mathbb{R}^d\times\mathbb{T}} \eta(Rx)D_M(y)w(x, y)\, dy\, dx \right| \\ &\leq 2R^{d/2}M\lambda\|\eta\|_{L^2} \lesssim R^{d/2}M\lambda \end{aligned}$$

thus, again by interpolation, we infer that

$$\|\chi_R^{\leq M}(|D|)v_n\|_{L^{2q}} \lesssim \|\chi_R^{\leq M}(|D|)v_n\|_{L^\infty}^{1-1/q} \|\chi_R^{\leq M}(|D|)v_n\|_{L^2}^{1/q} \lesssim \|\chi_R^{\leq M}(|D|)v_n\|_{L^\infty}^{1-1/q},$$

and then

$$\limsup_{n\to\infty} \|\chi_R^{\leq M}(|D|)v_n\|_{L^{2q}_{x,y}} \lesssim R^{\frac{d}{2}\left(\frac{q-1}{q}\right)} M^{\frac{q-1}{q}} \lambda^{\frac{q-1}{q}}. \tag{3.8}$$

Combining (3.5),(3.7) and (3.8), we obtain

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim R^{\frac{q(d-1)-(d+1)}{2q}} + M^{\frac{1}{2}\left(\frac{q(d-1)-(d+1)}{2q}\right)} + R^{\frac{d}{2}\left(\frac{q-1}{q}\right)} M^{\frac{q-1}{q}} \lambda^{\frac{q-1}{q}},$$

and by choosing $M \sim R^2$ we end up with

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim R^{\frac{q(d-1)-(d+1)}{2q}} + \left(R^{\frac{d+4}{2}}\lambda\right)^{\frac{q-1}{q}}.$$

We now consider as radius $R = \lambda^\beta$, and so

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim \lambda^{\beta\left(\frac{q(d-1)-(d+1)}{2q}\right)} + \left(\lambda^{\beta\left(\frac{d+4}{2}\right)+1}\right)^{\frac{q-1}{q}}.$$

Defining now β in this way:

$$\begin{aligned}
&\beta\left(\frac{q(d-1)-(d+1)}{2q}\right) = \frac{q-1}{q}\left(\beta\left(\frac{d+4}{2}\right)+1\right)\\
\iff &\beta\left(\frac{q(d-1)-(d+1)}{2q} - \frac{q-1}{q}\frac{d+4}{2}\right) = \frac{q-1}{q}\\
\iff &\beta(q(d-1)-(d+1)-(q-1)(d+4)) = 2(q-1)\\
\iff &\beta(3-5q) = 2(q-1) \iff \beta = \frac{2(q-1)}{3-5q},
\end{aligned}$$

we observe that $\beta = \frac{2(q-1)}{3-5q} < 0$, and we conclude with

$$\limsup_{n\to\infty} \|v_n\|_{L^{2q}} \lesssim \lambda^{\frac{q-1}{3-5q}\left(\frac{q(d-1)-(d+1)}{q}\right)}.$$

□

Remark 3.2. We notice that w actually belongs to $H^1_{x,y}$ since the weak limit clearly enjoys this regularity.

We now fix some notations used in the following part. We define with $v(t,x,y)$ or simply $v(t)$ the free evolution with respect to the linear Klein-Gordon equation, with Cauchy datum $\vec{v}^0 = (v_0, v_1)$ and we define by $\vec{v}(t) = e^{tH}\vec{v}^0 = (v(t), \partial_t v(t))^T$, where e^{tH} has been introduced in (2.15). Then, we give the following decomposition for a time-independent bounded sequence in $H^1 \times L^2$. We first introduce the following lemma which will be useful in the sequel. To shorten the notation, we write from now on $\mathcal{H} = H^1 \times L^2$, and for a two dimensional vector $\vec{\psi}$ we denote its components by $(\psi, \partial\psi)$.

Lemma 3.3. *Let $\vec{f}_n \rightharpoonup 0$ in $\mathcal{H}$. Then we have:*

- $t_n \to \bar{t} \in \mathbb{R} \implies e^{t_n H}\vec{f}_n(x,y) \rightharpoonup 0$ *in* $\mathcal{H}$,
- $e^{(t_n^2 - t_n^1)H}\vec{f}_n(x-(x_n^1-x_n^2), y) \rightharpoonup \vec{g} \neq 0 \implies |t_n^2 - t_n^1| + |x_n^2 - x_n^1| \to +\infty.$

Proof. For the first point, we make use of the continuity property of the propagator: by denoting by $(\cdot,\cdot)_{\mathcal{H}}$ the scalar product in $\mathcal{H}$, for any $\vec{\psi} \in \mathcal{H}$ we have

$$\begin{aligned}
(e^{t_nH}\vec{f}_n, \vec{\psi})_{\mathcal{H}} &= (e^{t_nH}\vec{f}_n - e^{\bar{t}H}\vec{f}_n, \vec{\psi}) + (\vec{f}_n, e^{-\bar{t}H}\vec{\psi})_{\mathcal{H}}\\
&= (\vec{f}_n, e^{-t_nH}\vec{\psi} - e^{\bar{t}H}\vec{\psi})_{\mathcal{H}} + (\vec{f}_n(x,y), e^{-\bar{t}H}\vec{\psi})_{\mathcal{H}}\\
&= (\vec{f}_n, e^{-t_nH}\vec{\psi} - e^{\bar{t}H}\vec{\psi})_{\mathcal{H}} + o(1).
\end{aligned}$$

The conclusion follows, up to subsequences, since it holds in the $L^2 \times L^2$ topology by exploiting the continuity of the flow.

The second point is proved in its contrapositive form. Suppose that $s_n := (t_n^2 - t_n^1)$ and $\xi_n := (x_n^2 - x_n^1)$ are bounded. Then, up to subsequences, $s_n \to s \in \mathbb{R}$ and $\xi_n \to \bar{\xi} \in \mathbb{R}^d$. We prove that $e^{s_nH}\vec{f}_n(x-\xi_n, y) \rightharpoonup 0$ in $\mathcal{H}$. But as before

$$\begin{aligned}
(e^{s_nH}\vec{f}_n(x-\xi_n,y), \vec{\psi})_{\mathcal{H}} &= (e^{sH}\vec{f}_n(x,y), \vec{\psi}(x+\xi,y))_{\mathcal{H}} + o(1)\\
&= (\vec{f}_n(x,y), e^{-sH}\vec{\psi}(x+\xi,y))_{\mathcal{H}} + o(1) \rightharpoonup 0.
\end{aligned}$$

$\square$

We can now state the following result, whose iteration will give the profile decomposition theorem.

Proposition 3.4. *Let $\{\vec{v}_n^0\}_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{H}$ and $1 \le d \le 4$. Then, for suitable sequences $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}, \{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$, possibly after extraction of subsequences (still denoted with the subscript n), we can write, for every $n \in \mathbb{N}$*

$$\vec{v}_n(-t_n, x - x_n, y) = \vec{\psi}(x, y) + \vec{W}_n(x, y)$$

with $\vec{v}_n(t, x, y) = e^{tH}\vec{v}_n^0$. Moreover, the following properties hold:

$$\vec{W}_n \overset{n\to\infty}{\rightharpoonup} 0 \quad in \quad \mathcal{H},$$

$$\limsup_{n\to\infty} \|v_n(t, x, y)\|_{L^\infty L^q} \lesssim \|\psi\|_{L^2}^e \quad for\ any \quad q \in (2, 2^*), \tag{3.9}$$

where $e > 0$ is given in Lemma 3.1 and as $n \to \infty$

$$\|\vec{v}_n^0\|_{\mathcal{H}}^2 = \|\vec{\psi}\|_{\mathcal{H}}^2 + \|\vec{W}_n\|_{\mathcal{H}}^2 + o(1). \tag{3.10}$$

Similarly, for the $L^{\alpha+2}$ norm, as $n \to \infty$, we have

$$\|v_n^0\|_{L^{\alpha+2}}^{\alpha+2} = \|\psi\|_{L^{\alpha+2}}^{\alpha+2} + \|W_n\|_{L^{\alpha+2}}^{\alpha+2} + o(1). \tag{3.11}$$

Furthermore, the translation sequences $\{t_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ satisfy the dichotomies below:

$$\begin{aligned} &either \quad t_n = 0 \quad \forall\, n \in \mathbb{N}, \qquad or \quad t_n \overset{n\to+\infty}{\longrightarrow} \pm\infty; \\ &either \quad x_n = 0 \quad \forall\, n \in \mathbb{N}, \qquad or \quad |x_n| \overset{n\to+\infty}{\longrightarrow} \infty. \end{aligned} \tag{3.12}$$

Proof. Define $\vec{v}_n(t, x, y) := e^{tH}\vec{v}_n^0$, namely $\vec{v}_n(t)$ is the linear evolution of $\vec{v}_n^0$ by the linear Klein-Gordon flow. Since the energy is preserved along the flow, the sequence $\vec{v}_n(t)$ is bounded in $L_t^\infty\mathcal{H}$ and by Sobolev embedding the sequence $\{v_n(t)\}_{n\in\mathbb{N}}$ is bounded in $L^\infty L^q$ norm, for any $q \in (2, 2^*)$. Thus, let us now choose a sequence of times $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\|v_n(-t_n)\|_{L^q} > \frac{1}{2}\|v_n(\cdot)\|_{L^\infty L^q}. \tag{3.13}$$

In the spirit and with the notation of the previous lemma, we consider $\Lambda = \Lambda\left(\{v_n(-t_n, x, y)\}_{n\in\mathbb{N}}\right)$ and $\lambda = \lambda\left(\{v_n(-t_n, x, y)\}_{n\in\mathbb{N}}\right)$. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^d \times \mathbb{T}$ and $\vec{\psi}(x, y) = (\psi, \partial\psi)(x, y) \in \mathcal{H}$ be such that, up to subsequences,

$$\vec{v}_n(-t_n, x - x_n, y - y_n) \rightharpoonup \vec{\psi}$$

in $\mathcal{H}$ as $n \to \infty$. Then we get

$$\vec{v}_n(-t_n, x - x_n, y - y_n) = \vec{\psi} + \vec{W}_n, \qquad \vec{W}_n \rightharpoonup 0 \text{ as } n \to \infty, \tag{3.14}$$

the latter weak convergence occurring in $\mathcal{H}$, and in addition

$$\lambda \lesssim \|\psi\|_{L^2}. \tag{3.15}$$

The relation (3.15) along with Lemma 3.1 implies that

$$\limsup_{n\to\infty} \|v_n(-t_n)\|_{L^q} \lesssim \|\psi\|_{L^2}^e \quad \text{for} \quad \text{any} \quad q \in (2, 2^*),$$

and then (3.9) follows by (3.13).

By definition, from (3.14) we can write

$$\vec{v}_n^0(x, y) = e^{t_nH}\vec{\psi}(x + x_n, y + y_n) + e^{t_nH}\vec{W}_n(x + x_n, y + y_n), \tag{3.16}$$

and since e^{tH} is an isometry on $\mathcal{H}$ and its adjoint is given by e^{-tH}, together with the fact that $\vec{W}_n \rightharpoonup 0$ as $n \to \infty$, we get

$$\|\vec{v}_n^0\|_{\mathcal{H}}^2 = \|\vec{\psi}\|_{\mathcal{H}}^2 + \|\vec{W}_n\|_{\mathcal{H}}^2 + o_n(1).$$

We pursue the proof by showing the orthogonality property of the potential energy, by distinguishing three cases. In the following, the Lebesgue exponent $\alpha+2$, is given by the same α appearing in the nonlinearity of (1.1).

Case 1: $|t_n|\to\infty$. From (3.16) we see that (3.11) holds, observing that W_n is uniformly bounded and using the dispersive estimate (B.1) and a density argument; hence the orthogonality in $L^{\alpha+2}$.

Since $\{y_n\}_{n\in\mathbb{N}}\subset\mathbb{T}$ which is compact, in the next two cases we can assume that up to subsequence $y_n\to\bar y\in\mathbb{T}$.

Case 2: $t_n\to\bar t$ *&* $x_n\to\bar x$. We claim the following:

$$\vec v_n^0(x,y)-e^{t_nH}\vec\psi(x+x_n,y+y_n)=e^{t_nH}\vec W_n(x+x_n,y+y_n)\to 0,$$

for almost every $(x,y)\in\mathbb{R}^d\times\mathbb{T}$. In fact

$$(e^{t_nH}\vec W_n(x+x_n,y+y_n),\vec\psi)_{\mathcal{H}}=(\vec W_n,e^{-\bar tH}\vec\psi(x-\bar x,y-\bar y))_{\mathcal{H}}+o_n(1)=o_n(1),$$

if we localize in the euclidean part, i.e. if we consider the restriction of $e^{t_nH}\vec W_n(x+x_n,y+y_n)$ on a compact set $K\subset\mathbb{R}^d$. The compactness of $K\times\mathbb{T}$ gives by the Rellich-Kondrakhov theorem, that $W_n(t_n,x+x_n,y+y_n)$ strongly converges towards zero in $L^p(K\times\mathbb{T})$ for any $p\in(2,2^*)$, see [20]. Therefore we have (x,y)–almost everywhere convergence towards zero of $W_n(t_n,x+x_n,y+y_n)$. We recall that the Brezis-Lieb Lemma (see [5]) holds on a general measured space, therefore the same argument given in [1] yields to the $L^{\alpha+2}$ orthogonality in the case $t_n\to\bar t$ and $x_n\to\bar x$.

Case 3: $t_n\to\bar t$ *&* $|x_n|\to\infty$. Similar arguments apply to the remaining situation $t_n\to\bar t$ and $|x_n|\to\infty$.

It remains to prove that we can rearrange the sequences of translation parameters $\{t_n\}_{n\in\mathbb{N}}$, $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$. Namely, we wish to have that for any $n\in\mathbb{N}$, $t_n=0$ or $t_n\to\pm\infty$, and similarly for $\{x_n\}_{n\in\mathbb{N}}$, while y_n can be assumed to be trivial. In the following, by $t_n\to\bar t$ and $x_n\to\bar x$ we will implicitly assume that this possibly holds after extraction of subsequences from bounded sequences.

Case I: $t_n\to\bar t$ *&* $|x_n|\to\infty$. By continuity of the linear flow

$$e^{t_nH}\vec\psi\xrightarrow{\mathcal{H}}\vec\phi,\qquad \vec\phi(x,y):=e^{\bar tH}\vec\psi(x,y).$$

We rewrite $\vec v_n^0$ as

$$\vec v_n^0(x-x_n,y-y_n)=\vec\phi(x,y)+e^{t_nH}\vec W_n(x,y)+\vec r_n(x,y)=\vec\phi(x,y)+\vec\rho_n(x,y),$$

where $\vec r_n\to 0$ strongly in $\mathcal{H}$ and $\vec\rho_n=e^{t_nH}\vec W_n(x,y)+\vec r_n(x,y)$. From Lemma 3.3 it follows that if $\vec h_n\rightharpoonup 0$ in $\mathcal{H}$ and $t_n\to\bar t$ then $e^{t_nH}\vec h_n\rightharpoonup 0$. Therefore $\vec\rho_n\rightharpoonup 0$ in $\mathcal{H}$. Its is true whether $x_n\to x_0\in\mathbb{R}^d$ or $|x_n|\to\infty$. Translating the profiles by $\bar y$, namely by choosing $\vec\phi(x,y):=e^{\bar tH}\vec\psi(x,y-\bar y)$ we can also assume that $y_n=0$.
Case II: $t_n\to\bar t$ *&* $x_n\to\bar x$. If $t_n\to\bar t\in\mathbb{R}$ and also $x_n\to\bar x\in\mathbb{R}^d$ we proceed similarly by adding a space translation: namely as before but considering $\vec\phi:=e^{\bar tH}\vec\psi(x-\bar x,y-\bar y)$.

Case III: $t_n\to\pm\infty$ *&* $x_n\to\bar x$. If $t_n\to\pm\infty$ and $x_n\to\bar x\in\mathbb{R}$ then we change the function by translating in the space variables only, i.e. we consider $\vec\phi:=\vec\psi(x-\bar x,y-\bar y)$.

Case IV: $|t_n|\to\infty$ *&* $|x_n|\to\infty$. By extracting subsequences we have the desired property, again by translating the profiles in the y variable only. □

We can now state the linear profile decomposition for a bounded sequence of linear solutions in the energy space.

Theorem 3.5 (Linear Profile Decomposition)**.** *Let $\{\vec{u}_n(t,x,y)\}_{n\in\mathbb{N}}$ be a sequence of solutions to the linear Klein-Gordon equation, bounded in $H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$ for $1 \le d \le 4$. Recall that $\|\vec{u}_n(t,x,y)\|_{\mathcal{H}} = \|\vec{u}_n(0,x,y)\|_{\mathcal{H}}$, thus we are assuming that $\sup_n \|\vec{u}_n(0)\|_{\mathcal{H}} < \infty$. For any integer $k \ge 1$ the decomposition below holds:*

$$\vec{u}_n(t,x,y) = \sum_{1\le j<k} \vec{v}^j(t-t_n^j, x-x_n^j, y) + \vec{R}_n^k(t,x,y),$$

where $\vec{v}^j$ are solutions to linear Klein-Gordon with suitable initial data and the translation sequences satisfy

$$\lim_{n\to\infty} \left(|t_n^k - t_n^j| + |x_n^k - x_n^j|\right) = \infty, \qquad \forall\, j \ne k,$$

along with the same dichotomy property of (3.12). Moreover, for $q \in (2, 2^*)$

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|R_n^k\|_{L^\infty L^q} = 0$$

which in turn implies that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|R_n^k\|_{L^{\alpha+1} L^{2(\alpha+1)}} = 0.$$

Furthermore as $n \to \infty$,

$$\|\vec{u}_n(0,x,y)\|_{\mathcal{H}}^2 = \sum_{1\le j<k} \|\vec{v}_n^j\|_{\mathcal{H}}^2 + \|\vec{R}_n^k\|_{\mathcal{H}}^2 + o_n(1),$$

and

$$\|u_n(0,x,y)\|_{L^{\alpha+2}}^{\alpha+2} = \sum_{1\le j<k} \|v_n^j\|_{L^{\alpha+2}}^{\alpha+2} + \|R_n^k\|_{L^{\alpha+2}}^{\alpha+2} + o_n(1).$$

Proof. We iterate several times the result of Proposition 3.4. We consider $\{\vec{v}_n\}_{n\in\mathbb{N}}$ as the sequence of initial data of the linear solution $\{\vec{u}_n(t,x,y)\}_{n\in\mathbb{N}}$; namely we consider the sequence $\{\vec{u}_n(0,x,y)\}_{n\in\mathbb{N}}$ as a bounded sequence in $\mathcal{H}$. Let $\{t_n^1\}_{n\in\mathbb{N}}$ be the sequence given in the proposition above and $\{x_n^1\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$ be such that, up to subsequences,

$$\vec{u}_n(-t_n^1, x - x_n^1, y) \rightharpoonup \vec{\psi}^1(x,y)$$

in $\mathcal{H}$. Then

$$\vec{u}_n(-t_n^1, x - x_n^1, y) = \vec{\psi}^1(x,y) + \vec{W}_n^1(x,y),$$

with $\vec{W}_n^1 \rightharpoonup 0$ in $\mathcal{H}$. It follows, as $n \to \infty$, that

$$\vec{u}_n(0,x,y) = e^{t_n^1 H}\vec{\psi}^1(x + x_n^1, y) + e^{t_n^1 H}\vec{W}_n^1(x + x_n^1, y) := e^{t_n^1 H}\vec{\psi}^1(x + x_n^1, y) + \vec{R}_n^1(x,y),$$

where

$$e^{-t_n^1 H}\vec{R}_n^1(x - x_n^1, y) = \vec{W}_n^1(x,y) \rightharpoonup 0$$

in $\mathcal{H}$, and that

$$\|\vec{u}_n(0)\|_{\mathcal{H}}^2 = \|\vec{\psi}^1\|_{\mathcal{H}}^2 + \|\vec{R}_n^1\|_{\mathcal{H}}^2 + o_n(1) = \|\vec{\psi}^1\|_{\mathcal{H}}^2 + \|\vec{W}_n^1\|_{\mathcal{H}}^2 + o_n(1).$$

A similar claim can be proved for the $L^{\alpha+2}$ norm (potential energy). We now consider the functions $\vec{R}_n^1(x,y) = e^{t_n^1 H}\vec{W}_n^1(x + x_n^1, y)$ as bounded sequence in $\mathcal{H}$. As before, we can write

$$\vec{R}_n^1(x,y) = e^{t_n^2 H}\vec{\psi}^2(x + x_n^2, y) + e^{t_n^2 H}\vec{W}_n^2(x + x_n^2, y) := e^{t_n^2 H}\vec{\psi}^2(x + x_n^2, y) + \vec{R}_n^2(x,y),$$

where $\vec{W}_n^2 \rightharpoonup 0$ in $\mathcal{H}$ and

$$\|\vec{R}_n^1\|_{\mathcal{H}}^2 = \|\vec{\psi}^2\|_{\mathcal{H}}^2 + \|\vec{R}_n^2\|_{\mathcal{H}}^2 + o_n(1) = \|\vec{\psi}^2\|_{\mathcal{H}}^2 + \|\vec{W}_n^2\|_{\mathcal{H}}^2 + o_n(1).$$

It implies that at the second step we have

$$\vec{u}_n(0,x,y) = e^{t_n^1 H}\vec{\psi}^1(x + x_n^1, y) + e^{t_n^2 H}\vec{\psi}^2(x + x_n^2, y) + \vec{R}_n^2(x,y),$$

and by "applying" the linear propagator on both sides we get

$$\vec{u}_n(t,x,y) = e^{(t+t_n^1)H}\vec{\psi}^1(x + x_n^1, y) + e^{(t+t_n^2)H}\vec{\psi}^2(x + x_n^2, y) + e^{tH}\vec{R}_n^2(x,y).$$

Moreover, as $n \to \infty$,

$$\|\vec{u}(t,x,y)_n\|_{\mathcal{H}}^2 = \|\vec{\psi}^1\|_{\mathcal{H}}^2 + \|\vec{\psi}^2\|_{\mathcal{H}}^2 + \|\vec{R}_n^2\|_{\mathcal{H}}^2 + o_n(1) = \|\vec{\psi}^1\|_{\mathcal{H}}^2 + \|\vec{\psi}^2\|_{\mathcal{H}}^2 + \|\vec{W}_n^2\|_{\mathcal{H}}^2 + o_n(1),$$

and the orthogonality for the $L^{\alpha+2}-$norm can be proved similarly. Recall that

$$e^{t_n^1 H}\vec{W}_n^1(x+x_n^1,y) = \vec{R}_n^1(x,y) = e^{t_n^2 H}\vec{\psi}^2(x+x_n^2,y) + e^{t_n^2 H}\vec{W}_n^2(x+x_n^2,y),$$

and so

$$e^{(t_n^1-t_n^2)H}\vec{W}_n^1(x+(x_n^1-x_n^2),y) = \vec{\psi}^2(x,y) + \vec{W}_n^2(x,y),$$

with $\vec{W}_n^2 \rightharpoonup 0$ in $\mathcal{H}$, and this implies the weak convergence in $\mathcal{H}$

$$e^{(t_n^1-t_n^2)H}\vec{W}_n^1(x+(x_n^1-x_n^2),y) \rightharpoonup \vec{\psi}^2(x,y).$$

Lemma 3.3, which is the equivalent of [1, Lemma 2.1] in our context, allows us to conclude with the orthogonality condition

$$|t_n^1 - t_n^2| + |x_n^1 - x_n^2| \to \infty.$$

Iterating this construction we end up, at the k^{th} step, with

$$\vec{u}_n(t,x,y) = e^{(t+t_n^1)H}\vec{\psi}^1(x+x_n^1,y) + \cdots + e^{(t+t_n^{k-1})H}\vec{\psi}^{k-1}(x+x_n^{k-1},y) + e^{tH}\vec{R}_n^k(x,y),$$

where

$$\vec{R}_n^k(x,y) = e^{t_n^k H}\vec{W}_n^k(x+x_n^k,y), \qquad \vec{W}_n^k \rightharpoonup 0 \text{ in } \mathcal{H}.$$

Moreover the free energy orthogonality holds:

$$\|\vec{u}_n(t,x,y)\|_{\mathcal{H}}^2 = \|\vec{\psi}^1\|_{\mathcal{H}}^2 + \cdots + \|\vec{\psi}^{k-1}\|_{\mathcal{H}}^2 + \|\vec{R}_n^k\|_{\mathcal{H}}^2,$$

and by the fact that the l.h.s. is uniformly bounded in $L_t^\infty\mathcal{H}$ we get

$$\lim_{k\to\infty}\|\psi^k\|_{L^2} \le \lim_{k\to\infty}\|\psi^k\|_{H^1} \le \lim_{k\to\infty}\|\vec{\psi}^k\|_{\mathcal{H}} = 0.$$

Using (3.15) we obtain the smallness of the remainders in the sense of

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\|R_n^k\|_{L^\infty L^q} = 0.$$

The proof of the smallness in the Strichartz norm

$$\lim_{k\to\infty}\limsup_{n\to\infty}\|R_n^k\|_{L^{\alpha+1}L^{2(\alpha+1)}} = 0,$$

is done by interpolation (see Lemma A.1 for the detailed computations). The proof of Theorem 3.5 is complete. □

4. Existence of a critical element

Theorem 3.5 is the key tool for the construction of a minimal (with respect to the energy) non-scattering solution to (1.1) with some compactness property. We define the following critical energy:

$$\begin{aligned} E_c = \sup\{E > 0 \,|\, &\text{ for any } (f,g) \in \mathcal{H} \quad \text{with} \quad E(f,g) < E \\ &\implies u_{(f,g)}(t) \in L^{\alpha+1}L^{2(\alpha+1)} < \infty\} \end{aligned}$$

where $u_{(f,g)}(t)$ denotes the global solution to (1.1) with Cauchy data (f,g). Our final aim, at the end of the paper, is to exclude that E_c is finite.

The result stated in Theorem 2.4 ensures that $E_c > 0$. The strategy consists in a contradiction argument. If we suppose that E_c is finite, we will show that there exists a critical solution u_c to (1.1), with energy E_c, such that it does not belong to the Strichartz space $L^{\alpha+1}L^{2(\alpha+1)}$. It will moreover enjoy some compactness properties. The latter will imply that such critical solution must be the trivial one, hence a contradiction.

We first proceed with the construction of the critical solution, based on the profile decomposition theorem Theorem 3.5. We remark that it is a linear statement. Since we are dealing with a nonlinear equation, we give the following perturbation lemma which will enable us to absorb the nonlinear terms in the remainders of the profile decomposition theorem, in a proper way.

Lemma 4.1. *For any $M > 0$ there exist $\varepsilon = \varepsilon(M) > 0$ (possibly very small) and $c = c(M) > 0$ (possibly very large) such that the following fact holds. Fix $t_0 \in \mathbb{R}$ and suppose that*

$$\|v\|_{L^{\alpha+1}L^{2(\alpha+1)}} \leq M$$

$$\|e_u\|_{L^1L^2} + \|e_v\|_{L^1L^2} + \|w_0\|_{L^{\alpha+1}L^{2(\alpha+1)}} \leq \varepsilon' \leq \varepsilon(M)$$

where $u, v \in \bigcap_{h\in\{0,1\}} \mathcal{C}^h(\mathbb{R}; H^{1-h})$, $e_z = \partial_{tt}z - \Delta_{x,y}z + z + |z|^\alpha z$ and $\vec{w}_0(t) = e^{(t-t_0)H}(\vec{u}(t_0) - \vec{v}(t_0))$. Then

$$\|u\|_{L^{\alpha+1}L^{2(\alpha+1)}} < \infty,$$

$$\|\vec{u} - \vec{v} - \vec{w}_0\|_{L^\infty\mathcal{H}} + \|u - v\|_{L^{\alpha+1}L^{2(\alpha+1)}} \leq c(M)\varepsilon'.$$

Proof. The proof is a straightforward modification of the one contained in [36] (where the authors consider the cubic focusing NLKG on $\mathbb{R}^3$) by means of following inequality to estimate the nonlinear part:

$$||u+v|^\alpha(u+v) - |u|^\alpha u| \leq C(|u|^\alpha + |v|^\alpha)|v| = C(|u|^\alpha|v| + |v|^{\alpha+1}).$$

□

Once every ingredient is given, we continue with the extraction of the critical solution. We therefore assume that $E_c < \infty$. Let $\{(f_n, g_n)\}_{n\in\mathbb{N}} \in \mathcal{H}$ be a sequence of Cauchy data such that $E(f_n, g_n) \to E_c$ as $n \to +\infty$ and let $u_n(t) := u_{(f_n,g_n)}(t)$ be the corresponding solutions to (1.1) which exist globally in time but do not belong to $L^{\alpha+1}L^{2(\alpha+1)}$, i.e. $\|u_n\|_{L^{\alpha+1}L^{2(\alpha+1)}} = \infty$. The last condition means that we are considering a maximising sequence $\{(f_n, g_n)\}_{n\in\mathbb{N}} \in \mathcal{H}$ whose corresponding solutions do not satisfy the scattering property.

Since $E(f_n, g_n) \to E_c$ and the energy is a conserved quantity, we can state that $\vec{u}_n^0 := (f_n, g_n)$ is uniformly bounded in $\mathcal{H}$. For the Klein-Gordon linear flow preserves the $\mathcal{H}$ norm, the sequence $e^{tH}\vec{u}_n^0$ is uniformly bounded in $L^\infty\mathcal{H}$. Thus we can apply the linear profile decomposition to this sequence of free solutions and we can write

$$e^{tH}\vec{u}_n^0 = \sum_{1\leq j<k} \vec{v}_n^j(t) + \vec{R}_n^k(t,x,y),$$

where $\vec{v}_n^j(t) = \vec{v}^j(t - t_n^j, x - x_n^j, y) = e^{(t-t_n^j)H}\vec{\psi}^j(x - x_n^j, y)$ for suitable $\vec{\psi}^j \in \mathcal{H}$. We recall that the profile decomposition theorem given above ensures the orthogonality of the translation sequences in the sense of

$$\lim_{n\to+\infty} \left(|t_n^h - t_n^j| + |x_n^h - x_n^j|\right) = +\infty, \tag{4.1}$$

for all $j \neq h$, the smallness of the remainders in the sense of

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|R_n^k(t)\|_{L^\infty L^q \cap L^{\alpha+1}L^{2(\alpha+1)}} = 0,$$

as well as the pythagorean expansions of the quadratic and super quadratic terms of the energy. More precisely, for $n \to \infty$,

$$\|(f_n, g_n)\|_{\mathcal{H}}^2 = \|\vec{u}_n(0,x,y)\|_{\mathcal{H}}^2 = \|\vec{u}_n(t,x,y)\|_{\mathcal{H}}^2 = \sum_{1\leq j<k} \|\vec{v}_n^j\|_{\mathcal{H}}^2 + \|\vec{R}_n^k\|_{\mathcal{H}}^2 + o_n(1), \tag{4.2}$$

and

$$\|u_n(0,x,y)\|_{L^{\alpha+2}}^{\alpha+2} = \sum_{1\leq j<k} \|v_n^j\|_{L^{\alpha+2}}^{\alpha+2} + \|R_n^k\|_{L^{\alpha+2}}^{\alpha+2} + o_n(1). \tag{4.3}$$

We suppose by the absurd that $k > 1$. Due to (4.1), we can have that at most in one case both the space and the time translation sequences are trivial, . Without loss of generality we can suppose that this case happens when $j = 1$, and since we are assuming $k > 1$ we have, by orthogonality of the energy expressed by summing up (4.2) and (4.3), that $\vec{\psi}^1$ is such that the corresponding solution $z^1 := u_{\vec{\psi}^1}$ to (1.1) scatters, as it belongs to $L^{\alpha+1}L^{2(\alpha+1)}$ by definition. In the other cases $j \geq 2$, we associate to a linear profile $\vec{\psi}^j$, a nonlinear profile in a proper way. We associate a nonlinear profile V^j to each linear profile v^j by exploiting to the following procedure: V^j is a nonlinear solution to (1.1) such that

$$\lim_{n\to\infty} \|\vec{v}^j(t_n^j) - \vec{V}^j(t_n^j)\|_{\mathcal{H}} = 0.$$

Recall that by the dichotomy property of the parameters, for every j, $\lim_{n\to\infty} t_n^j = 0$ or $\lim_{n\to\infty} |t_n^j| = \infty$. Then V^j is locally defined both in a neighbourhood of $t = 0$ or $|t| = \infty$: the first property follows by the local well-posedness theory, while the second one by the existence of the wave operators. Due to the defocusing nature of the equation, V^j is actually globally defined. Orthogonality of the energy given by (4.2) together with (4.3) implies that any nonlinear profile V^j has an energy less than the minimal one E_c. Let us define

$$V(t) = \sum_{j=1}^{k} V^j(t - t_n^j, x - x_n^j, y);$$

we use the perturbation lemma with V and u_n replacing v and u of Lemma 4.1. As in [36] this would imply that

$$\limsup_{n\to\infty} \|\sum_{j=1}^{k} V^j(t - t_n^j, x - x_n^j, y)\|_{L^{\alpha+1}L^{2(\alpha+1)}} \le C < \infty, \quad \text{uniformly in } \; k,$$

and Lemma 4.1 gives

$$\limsup_{n\to\infty} \|u_n\|_{L^{\alpha+1}L^{2(\alpha+1)}} \le C < \infty,$$

which is a contradiction. Therefore $k = 1$, and the precompactness of the trajectory up to a translation also follows by [36]. We can summarize the core result of this section in the following theorem.

Theorem 4.2. *There exists an initial datum $(f_c, g_c) \in H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$ such that the corresponding solution $u_c(t)$ to (1.1) is global and $\|u_c\|_{L^{\alpha+1}L^{2(\alpha+1)}} = \infty$. Moreover there exists a path $x(t) \in \mathbb{R}^d$ such that $\{u_c(t, x - x(t), y), \partial_t u_c(t, x - x(t), y), t \in \mathbb{R}^+\}$ is precompact in $H^1(\mathbb{R}^d \times \mathbb{T}) \times L^2(\mathbb{R}^d \times \mathbb{T})$.*

5. Rigidity

This section establishes that the minimal element built in the previous section cannot exist. The first step is to prove the validity of the finite propagation speed in our framework. It will be useful to control the growth of the translation path $x(t) \in \mathbb{R}^d$ given in Theorem 4.2. Let us first recall this result.

Lemma 5.1. *Let f be smooth and $B(x_0, r) \subset \mathbb{R}^d$ the ball centered in x_0 with radius r. The following equality holds:*

$$\frac{d}{dr} \int_{B(x_0,r)} f(x)\, dx = \int_{\partial B(x_0,r)} f(\sigma)\, d\sigma,$$

where $\partial B(x_0, r)$ is the boundary of $B(x_0, r)$ and $d\sigma$ is the surface measure on $\partial B(x_0, r)$.

Proof. The proof is straightforward once switched in radial coordinates. □

We then state the following, which is the finite time propagation speed mentioned above. The notation $B(x_0, r)^c$ stands for $\mathbb{R}^d \setminus B(x_0, r)$.

Proposition 5.2. *Let u be the solution to (1.1) with Cauchy datum (u_0, u_1) vanishing on $B(x_0, r)^c \times \mathbb{T}$, for some $r > 0$. Then $\vec{u}(t) = (u, \partial_t u)(t)$ vanishes on $K(x_0, r) := \{t \ge 0,\ x \in B(x_0, r + t)^c,\ y \in \mathbb{T}\}$.*

Proof. Fix $r > 0$, $x_0 \in \mathbb{R}^d$, consider the balls $B(x_0, r + t) := B(t + r)$ and define the local energy $E_r(t)$ as

$$E_r(t) = \frac{1}{2} \int_{\mathbb{T}} \int_{B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha + 2} |u|^{\alpha+2} \right)(t)\, dx\, dy.$$

Assume that $u(t,x,y)$ is smooth enough (by a classical regularization argument, the following calculations then extend to rougher solutions), and let us calculate the first time derivative of the local energy:

$$\begin{aligned}
\frac{d}{dt}E_r(t) &= \int_{\mathbb{T}}\int_{B(r+t)} \partial_t u \partial_{tt} u + \sum_{i\in\{1,\dots,d\}} \partial_{x_i} u\, \partial_{x_i}\partial_t u + \partial_y u\, \partial_y \partial_t u \; dx\, dy \\
&\quad + \int_{\mathbb{T}}\int_{B(r+t)} u\partial_t u + \frac{1}{\alpha+2}|u|^\alpha u \partial_t u \; dx\, dy \\
&\quad + \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\, d\sigma\, dy \\
&= \int_{\mathbb{T}}\int_{B(r+t)} \partial_t u \partial_{tt} u + div_x(\partial_t u \nabla_x u) - \partial_t u \Delta_x u + \partial_y(\partial_t u \partial_y u) - \partial_t u \partial_{yy} u \; dx\, dy \\
&\quad + \int_{\mathbb{T}}\int_{B(r+t)} u\partial_t u + \frac{1}{\alpha+2}|u|^\alpha u \partial_t u \, dx\, dy \\
&\quad + \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\, d\sigma\, dy \\
&= \int_{\mathbb{T}}\int_{B(r+t)} div_x(\partial_t u \nabla_x u)\, dx\, dy + \int_{B(r+t)}\int_{\mathbb{T}} \partial_y(\partial_t u \partial_y u)\, dy\, dx \\
&\quad + \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\, d\sigma\, dy \\
&= -\int_{\mathbb{T}}\int_{\partial B(r+t)} \partial_t u \nabla u \cdot n_i \, d\sigma\, dy \\
&\quad + \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\, d\sigma\, dy.
\end{aligned}$$

where $n_i = n_i(x)$, $x \in \partial B$, denotes the inner normal vector to the boundary of B. Recall that the energy on the whole space in conserved, and so by using the Cauchy-Schwarz inequality

$$\begin{aligned}
\frac{d}{dt}(E - E_r(t)) &= \frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{T}}\int_{B(r+t)^c} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\, dx\, dy\right\} \\
&= \int_{\mathbb{T}}\int_{\partial B(r+t)} \partial_t u \nabla u \cdot n_i \, d\sigma\, dy \\
&\quad - \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\, d\sigma\, dy \\
&\le \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)} |\partial_t u|^2 + |\nabla u|^2 \, d\sigma\, dy \\
&\quad - \frac{1}{2}\int_{\mathbb{T}}\int_{\partial B(r+t)} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\, d\sigma\, dy \le 0,
\end{aligned}$$

and we obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbb{T}}\int_{B(r+t)^c} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t)\, dx\, dy\right\} \le 0,$$

namely the energy on $B(x_0, r+t)^c \times \mathbb{T}$ is decreasing. The conclusion follows. $\square$

We now give an estimate from above away from zero of a portion of the potential energy. This will be essential in the last section dealing with the rigidity part in the Kenig & Merle scheme. We borrow from the ideas of Bulut in [6] (where the author deals with the energy supercritical NLW).

Lemma 5.3. *Let $u(t,x,y)$ be a solution to (1.1). If $\{\vec{u}(t)\}_{t\in\mathbb{R}} \subset \mathcal{H}$ is a relatively compact set and $\vec{u}^* \in \mathcal{H}$ is one of its limit points, then $\vec{u}^* \neq 0$.*

Proof. This property simply follows from the conservation of energy (1.5). □

At this point we can give the following lemma, essentially based on the well-posedness of (1.1), in particular its continuous dependence on the initial data.

Lemma 5.4. *Let $u(t)$ be a nontrivial solution to (1.1) such that $\{u(t,x-x(t),y), \partial_t u(t,x-x(t),y)\}_{t\in\mathbb{R}}$ is relatively compact in $\mathcal{H}$. Then for any $A>0$, there exists $C(A)>0$ such that for any $t\in\mathbb{R}$,*

$$\int_t^{t+A}\int_{\mathbb{T}}\int_{|x-x(t)|\leq R} |u|^{\alpha+2}(s,x,y)\,dx\,dy\,ds \geq C(A), \tag{5.1}$$

for $R=R(A)$ large enough.

Proof. We argue by contradiction, supposing that there exists a sequence of times $\{t_n\}_{n\in\mathbb{N}}$ such that

$$\int_{t_n}^{t_n+A}\int_{\mathbb{R}^d\times\mathbb{T}} |u|^{\alpha+2}(s,x,y)\,dx\,dy\,ds < \frac{1}{n}.$$

By compactness, up to subsequence still denoted with the subscript n,

$$(u(t_n,x-x(t_n),y), \partial_t u(t_n,x-x(t_n),y)) \to (f,g) \in \mathcal{H}.$$

Let $(w(0),\partial_t w(0)) = (f,g)$ be an initial datum and $w(t)$ be the corresponding solution to (1.1): then we have, by the fact that $u\neq 0$,

$$E(w,\partial_t w) = E(f,g) = \lim_{n\to\infty} E(u(t_n,x-x(t_n),y), \partial_t u(t_n,x-x(t_n),y)) = E(u_0,u_1) \neq 0. \tag{5.2}$$

Local well-posedness and Strichartz estimates imply

$$\begin{aligned}
0 &= \lim_{n\to\infty}\int_{t_n}^{t_n+A}\int_{\mathbb{R}^d\times\mathbb{T}} |u|^{\alpha+2}(s,x,y)\,dx\,dy\,ds\\
&= \lim_{n\to\infty}\int_0^A\int_{\mathbb{R}^d\times\mathbb{T}} |u|^{\alpha+2}(t_n+s,x,y)\,dx\,dy\,ds\\
&= \lim_{n\to\infty}\int_0^A\int_{\mathbb{R}^d\times\mathbb{T}} |u|^{\alpha+2}(t_n+s,x-x(t_n),y)\,dx\,dy\,ds\\
&= \int_0^A\int_{\mathbb{R}^d\times\mathbb{T}} |w|^{\alpha+2}(s,x,y)\,dx\,dy\,ds,
\end{aligned}$$

which in turn gives that $w(t)=0$ almost everywhere in $(0,A)$. This contradicts (5.2), then

$$\int_t^{t+A}\int_{\mathbb{T}\times\mathbb{R}^d} |u|^{\alpha+2}\,dx\,dy\,dt \geq C'(A).$$

By exploiting again the precompactness property of the solution

$$\begin{aligned}
\int_t^{t+A}\int_{\mathbb{T}}\int_{|x-x(t)|\leq R} |u|^{\alpha+2}\,dx\,dy\,dt &= \int_t^{t+A}\left\{\int_{\mathbb{T}\times\mathbb{R}^d}|u|^{\alpha+2}\,dx\,dy - \int_{\mathbb{T}}\int_{|x-x(t)|\geq R}|u|^{\alpha+2}\,dx\,dy\right\}dt,\\
&\geq C'(A) - \frac{C'(A)}{2} = \frac{C'(A)}{2} =: C(A).
\end{aligned}$$

□

Corollary 5.5. *By interpolation the same property can be claimed for the localized L^2 norm of u. More precisely, under the same assumption of Lemma 5.4 on u, for any $A > 0$ there exists $C(A) > 0$ such that for any $t \in \mathbb{R}$*

$$\int_t^{t+A} \int_{\mathbb{T}} \int_{|x-x(t)|\le R} |u|^2(s,x,y)\,dx\,dy\,ds \ge C(A) \tag{5.3}$$

for $R = R(A)$ large enough.

The last ingredient to derive a contradiction to the existence of a such precompact solution is an a priori bound for the super-quadratic term of the energy which is due to Nakanishi, see [35]. The latter is a remarkable extension the low dimensional cases $\mathbb{R}^m$ with $m = 1, 2$ of the well-known Morawetz estimate proved by Morawetz and Strauss, see [31, 32], in higher dimensions. Those a priori bounds led to the scattering in the energy space both for the nonlinear Klein-Gordon equation and the nonlinear Schrödinger equation posed in the euclidean spaces $\mathbb{R}$ and $\mathbb{R}^2$.

5.1. **Nakanishi/Morawetz-type estimate.** We begin this section by giving the analogue in our domain of the decay result due to Nakanishi, [35]. Our approach is to use a multiplier that does not consider all the variables: neither the compact factor of the product space we work on (the y variable), nor a set of $d-1$ euclidean variables ($x_2, \dots, x_d$ for instance) will be "seen" by the multiplier. Consequently, we will show how the Nakanishi/Morawetz type estimate in one dimension is enough for a contradiction argument which will exclude soliton-like solutions, i.e. the u_c built in Theorem 4.2. This strategy to use a 1D tool to exclude the existence of a soliton-like solution has been used by the first author and Visciglia in [12] in order to show energy scattering for defocusing NLS perturbed with a partially periodic not-decaying time-independent potentials.

We verbatim report, for sake of completeness, the proof contained in [35, Lemma 5.1, equation (5.1)], then we analyze the extra term given by the remaining part of the second order in space operator involved in the equation. First, Nakanishi introduces the following quantities with relative notations (recall that in the following we are in a pure euclidean space, with $x \in \mathbb{R}^m$ and $m = 1, 2$):

$$r = |x|, \quad \theta = \frac{x}{r}, \quad \lambda = \sqrt{t^2 + r^2}, \quad \Theta = \frac{(-t, x)}{\lambda}$$
$$u_r = \theta \cdot \nabla_x u, \quad u_\theta = \nabla_x u - \theta u_r$$
$$l(u) = \frac{1}{2}\left(-|\partial_t u|^2 + |\nabla_x u|^2 + |u|^2 + \frac{2}{\alpha+2}|u|^{\alpha+2}\right)$$
$$(\partial_0, \partial_1, \partial_2) = (-\partial^0, \partial^1, \partial^2) = (\partial_t, \nabla_x)$$
$$g = \frac{m-1}{2\lambda} + \frac{t^2 - r^2}{2\lambda^3}, \quad M = \Theta \cdot (\partial_t u, \nabla_x u) + ug$$
$$(\partial_t^2 - \Delta_x) g = -\frac{5}{2\lambda^3} + 3\frac{t^2 - r^2}{\lambda^5} + 15\frac{(t^2 - r^2)^2}{2\lambda^7}.$$

Then by multiplying the equation $\partial_t^2 u - \Delta_x u + u + |u|^\alpha u = 0$ by M, with $u = u(t,x)$, we obtain the relation

$$\begin{aligned} 0 = (\partial_t^2 u - \Delta_x u + u + |u|^\alpha u) M = \sum_{\beta=0}^{m} \partial_\beta \left(-M\partial^\beta u + l(u)\Theta_\beta + \frac{|u|^2}{2}\partial^\beta g\right) \\ + \frac{|u_\omega|^2}{\lambda} + \frac{|u|^2}{2}(\partial_t^2 - \Delta_x) g + \frac{\alpha}{\alpha+2}|u|^{\alpha+2} g, \end{aligned} \tag{5.4}$$

where u_ω is the projection of $(\partial_t u, \nabla_x u)$ on the tangent space of $t^2 - |x|^2 = c$, c being a constant.

We focus on $m = 1$ and we go back to (1.1). We introduce the compact notation

$$\mathbb{R}^{d-1} \times \mathbb{T} =: \mathcal{M} \ni z := (\bar{x}, y) = (x_2, \dots, x_d, y).$$

Then the analogous of (5.4) is the following:

$$
\begin{aligned}
0=(\partial_t^2u-\Delta u+u+|u|^\alpha u)M=\sum_{\beta\in\{0,1\}}&\partial_\beta\left(-M\partial^\beta u+l(u)\Theta_\beta+\frac{|u|^2}{2}\partial^\beta g\right)\\
&+\frac{|u_\omega|^2}{\lambda}+\frac{|u|^2}{2}(\partial_t^2-\Delta_x)g+\frac{\alpha}{\alpha+2}|u|^{\alpha+2}g\\
&-M\Delta_z u.
\end{aligned}
\tag{5.5}
$$

Observe that the term g is nonnegative only in the region where $r<t$. Then after integrating (5.5) (now $u=u(t,x_1,z)$) on $\mathcal{C}:=\{(t,x_1)\,|\,2<t<T,|x_1|=r<t\}\times\mathcal{M}$, using the divergence theorem, the last relation we obtain is:

$$
\begin{aligned}
\left\{\int_{\mathcal{M}}\int_{r<t}-\partial_t uM+l(u)\frac{t}{\lambda}+\frac{|u|^2}{2}\partial_t g\,dx_1dz\right\}\Bigg|_{t=2}^{t=T}&=\int_{\mathcal{C}}\frac{|u_\omega|^2}{\lambda}+\frac{|u|^2}{2}(\partial_t^2-\partial_{x_1}^2)g+\frac{\alpha}{\alpha+2}|u|^{\alpha+2}g\,dx_1dz\,dt\\
&\quad+\frac{\sqrt{2}}{2}\int_{\mathcal{M}}\int_{2<r=t<T}|u|^2+\frac{2}{\alpha+2}|u|^{\alpha+2}\,dx_1dz\\
&\quad-\int_{\mathcal{C}}M\Delta_z u\,dx_1dz\,dt,
\end{aligned}
$$

noticing that $|u_\theta|^2=0$ if $m=1$. The l.h.s. of the above identity is bounded by the energy – as well as the middle term in the first integral in the r.h.s. – thanks to the estimate

$$
\left|\int_{\mathcal{C}}\frac{|u|^2}{2}(\partial_t^2-\partial_{x_1}^2)g\right|\lesssim\int_2^T\int_{\mathbb{T}\times\mathbb{R}^d}\frac{|u|^2}{t^3}\,dx_1dzdt\lesssim E.
$$

The energy flux through the curved surface, i.e. the second integral in the r.h.s. is estimated by the energy. In fact we have the following:

Lemma 5.6. *Any smooth solution u to (1.1) satisfies:*

$$
\int_{\mathcal{M}}\int_{2<|x_1|=t<T}|\partial_t u-\theta\partial_{x_1}u|^2+|\nabla_z u|^2+|u|^2+\frac{2}{\alpha+2}|u|^{\alpha+2}\,d\sigma\,dz\lesssim E.
\tag{5.6}
$$

Proof. The proof repeats the same analysis performed to prove the finite propagation speed property. Define

$$
e(t):=\frac{1}{2}\int_{\mathcal{M}}\int_{|x_1|<t}\left(|\partial_t u|^2+|u|^2+|\partial_{x_1}u|^2+\frac{2}{\alpha+2}|u|^{\alpha+2}\right)(t,x_1,z)\,dx_1\,dz.
$$

Differentiating $e(t)$ with respect to t, we obtain

$$
\begin{aligned}
\frac{d}{dt}e(t)&=\int_{\mathcal{M}}\int_{|x_1|<t}\left(\partial_t u\partial_t^2u+\partial_{x_1}u\partial_{x_1}\partial_t u+\partial_t uu+|u|^\alpha u\partial_t u\right)\,dx_1\,dz\\
&\quad+\frac{1}{2}\int_{\mathcal{M}}\int_{|x_1|=t}\left(|\partial_t u|^2+|u|^2+|\partial_{x_1}u|^2+\frac{2}{\alpha+2}|u|^{\alpha+2}\right)\,d\sigma\,dz\\
&=\int_{\mathcal{Y}}\int_{|x_1|<t}\partial_t u\left(\partial_t^2u-\partial_{x_1}^2u+u+|u|^\alpha u\right)+\partial_{x_1}(\partial_{x_1}u\cdot\partial_t u)\,dx_1\,dz\\
&\quad+\frac{1}{2}\int_{\mathcal{M}}\int_{|x_1|=t}\left(|\partial_t u|^2+|u|^2+|\partial_{x_1}u|^2+\frac{2}{\alpha+2}|u|^{\alpha+2}\right)\,d\sigma\,dz\\
&=\int_{\mathcal{Y}}\int_{|x_1|<t}\partial_t u\left(\partial_t^2u-\Delta u+u+|u|^\alpha u\right)+\partial_t u\Delta_z u+\partial_{x_1}(\partial_{x_1}u\cdot\partial_t u)\,dx_1\,dz\\
&\quad+\frac{1}{2}\int_{\mathcal{M}}\int_{|x_1|=t}\left(|\partial_t u|^2+|u|^2+|\partial_{x_1}u|^2+\frac{2}{\alpha+2}|u|^{\alpha+2}\right)\,d\sigma\,dz
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathcal{Y}} \int_{|x_1|<t} \partial_t u \Delta_z u + \partial_{x_1}(\partial_{x_1} u \cdot \partial_t u)\, dx_1\, dz \\
&\quad + \frac{1}{2} \int_{\mathcal{M}} \int_{|x_1|=t} \left(|\partial_t u|^2 + |u|^2 + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2} |u|^{\alpha+2} \right) d\sigma\, dz \\
&= -\frac{1}{2} \int_{\mathcal{M}} \int_{|x_1|<t} \partial_t |\nabla_z u|^2\, dx_1\, dz \\
&\quad + \frac{1}{2} \int_{\mathcal{M}} \int_{|x_1|=t} \left(|\partial_t u|^2 + |u|^2 + |\partial_{x_1} u|^2 + \frac{2}{\alpha+2} |u|^{\alpha+2} - 2\theta \partial_{x_1} u \cdot \partial_t u \right) d\sigma\, dz \\
&= -\frac{1}{2} \frac{d}{dt} \int_{\mathcal{M}} \int_{|x_1|<t} |\nabla_z u|^2\, dx\, dz + \frac{1}{2} \int_{\mathcal{Y}} \int_{|x_1|=t} |\nabla_z u|^2\, d\sigma\, dz \\
&\quad + \frac{1}{2} \int_{\mathcal{M}} \int_{|x_1|=t} \left(|\partial_t u - \theta \partial_{x_1} u|^2 + |u|^2 + \frac{2}{\alpha+2} |u|^{\alpha+2} \right) d\sigma\, dz
\end{aligned}
$$

therefore, integrating with respect to the time from 2 to T we obtain (5.6). □

Moreover, the energy estimate on the surface of the light cone gives

$$
\sup_t \int_{\mathbb{R}^{d-1}\times\mathbb{T}} \int_{\mathbb{R}} |u(|x_1| + t, x_1, z)|^2\, dx_1 dz \lesssim E.
$$

We now analyze the term $-\int_{\mathcal{C}} M \Delta_z u\, dx_1 dz\, dt$ in (5.5). We rewrite the term to be integrated as

$$
-M\Delta_z u = -div_z \left(M \nabla_z u \right) + \nabla_z u \cdot \nabla_z M := \mathcal{A} + \mathcal{B}.
$$

The second term is explicitly given by

$$
\begin{aligned}
\mathcal{B} &= -\frac{t}{2\lambda} \partial_t |\nabla_z u|^2 + \frac{1}{2\lambda} x_1 \cdot \partial_{x_1} |\nabla_z u|^2 + g |\nabla_z u|^2 \\
&= -\frac{1}{2} \partial_t \left(\frac{t}{\lambda} |\nabla_z u|^2 \right) + \frac{1}{2} |\nabla_z u|^2 \partial_t \left(\frac{t}{\lambda} \right) + \frac{1}{2\lambda} \left(\partial_{x_1}(x_1 |\nabla_z u|^2) - |\nabla_z u|^2 \right) + g |\nabla_z u|^2 \\
&= -\frac{1}{2} \partial_t \left(\frac{t}{\lambda} |\nabla_z u|^2 \right) + \frac{|x_1|^2}{2\lambda^3} |\nabla_z u|^2 + \frac{1}{2\lambda} \partial_{x_1}(x_1 |\nabla_z u|^2) - \frac{|\nabla_z u|^2}{2\lambda} + g |\nabla_z u|^2 \\
&= -\frac{1}{2} \partial_t \left(\frac{t}{\lambda} |\nabla_z u|^2 \right) + \frac{|x_1|^2}{2\lambda^3} |\nabla_z u|^2 + \partial_{x_1} \left(\frac{x_1}{2\lambda} |\nabla_z u|^2 \right) - \partial_{x_1} \left(\frac{1}{2\lambda} \right) x_1 |\nabla_z u|^2 - \frac{|\nabla_z u|^2}{2\lambda} + g |\nabla_z u|^2 \\
&= -\frac{1}{2} \partial_t \left(\frac{t}{\lambda} |\nabla_z u|^2 \right) + \frac{|x_1|^2}{2\lambda^3} |\nabla_z u|^2 + \partial_{x_1} \left(\frac{x_1}{2\lambda} |\nabla_z u|^2 \right) + \frac{|x_1|^2}{2\lambda^3} |\nabla_z u|^2 - \frac{|\nabla_z u|^2}{2\lambda} + g |\nabla_z u|^2 \\
&= -\frac{1}{2} \partial_t \left(\frac{t}{\lambda} |\nabla_z u|^2 \right) + \partial_{x_1} \left(\frac{x_1}{2\lambda} |\nabla_z u|^2 \right)
\end{aligned}
$$

and then, after integration, it can be estimated by the energy on the whole space, while the divergence term $\mathcal{B}$ disappears using the Gauss-Green theorem. In conclusion

$$
\int_2^\infty \int_{\mathcal{M}\times\{|x_1|<t\}} \frac{|u_\omega|^2}{\lambda} + \frac{\alpha}{\alpha+2} |u|^{\alpha+2} g\, dx_1\, dz\, dt \lesssim E.
$$

The Nakanishi/Morawetz-type estimate follows as in [35]:

$$
\int_{\mathbb{R}} \int_{\mathbb{R}^d\times\mathbb{T}} \frac{\min\{|u|^2, |u|^{\alpha+2}\}}{\langle t\rangle \log(|t|+2) \log(\max\{|x_1| - t, 2\})}\, dx\, dy\, dt \lesssim E. \tag{5.7}
$$

We now have all the elements yielding to the exclusion of the soliton-like solution.

5.2. **Extinction of the minimal element.** With the aforementioned tool, we are in position to obtain a contradiction with respect to the hypothesis on the finiteness of the critical energy E_c. Consider the upper bound $C = C(E(u))$ appearing in (5.7), then for any $T > 2$ we can write

$$
\begin{aligned}
C &\geq \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}\times\mathbb{T}} \int_{\mathbb{R}} \frac{\min\{|u|^2, |u|^{\alpha+2}\}}{\langle t\rangle \log(|t|+2)\log(\max\{|x_1|-t, 2\})}\, dx_1\, dz\, dt, \\
&\geq \int_2^T \int_{\mathbb{R}^{d-1}\times\mathbb{T}} \int_{\mathbb{R}} \frac{\min\{|u|^2, |u|^{\alpha+2}\}}{\langle t\rangle \log(|t|+2)\log(\max\{|x_1|-t, 2\})}\, dx_1\, dz\, dt, \\
&\geq \int_2^T \int_{\mathbb{T}} \int_{|x-x(t)|\leq R} \frac{\min\{|u|^2, |u|^{\alpha+2}\}}{\langle t\rangle \log(|t|+2)\log(\max\{|x_1|-t, 2\})}\, dx_1\, dz\, dt.
\end{aligned}
\tag{5.8}
$$

The finite propagation speed implies that $|x(t) - x(0)| \leq t + c_0$ for $t > 0$, then

$$
|x| \leq |x - x(t)| + |x(t) - x(0)| + |x(0)| \leq R + t + c_0 + c_1,
$$

so that $|x_1| - t \leq c + R$. With $[T]$ being the usual floor function of T, we are able to continue the chain above with

$$
\begin{aligned}
(5.8) &\gtrsim \int_2^T \frac{1}{\langle t\rangle \log(|t|+2)} \int_{\mathbb{T}} \int_{|x-x(t)|\leq R} \min\{|u|^2, |u|^{\alpha+2}\}\, dx\, dt \\
&\gtrsim \int_2^{[T]} \frac{1}{\langle t\rangle \log(|t|+2)} \int_{\mathbb{T}} \int_{|x-x(t)|\leq R} \min\{|u|^2, |u|^{\alpha+2}\}\, dx\, dt \\
&= \sum_{j=3}^{[T]} \int_{j-1}^{j} \frac{1}{\langle t\rangle \log(|t|+2)} \int_{\mathbb{T}} \int_{|x-x(t)|\leq R} \min\{|u|^2, |u|^{\alpha+2}\}\, dx\, dt \\
&\gtrsim \sum_{j=3}^{[T]} \frac{1}{\langle j\rangle \log(j+2)} \int_{j-1}^{j} \int_{\mathbb{T}} \int_{|x-x(t)|\leq R} \min\{|u|^2, |u|^{\alpha+2}\}\, dx\, dt \\
&\gtrsim C(1) \sum_{j=3}^{[T]} \frac{1}{\langle j\rangle \log(j+2)} \sim \int_2^T \frac{1}{\langle t\rangle \log(t+2)}\, dt.
\end{aligned}
$$

In the last step we used the property stated in Lemma 5.4 and Corollary 5.5 above (more precisely (5.1) and (5.3)) for a suitable choice of the radius R. This is sufficient to establish a contradiction by taking T large enough, since for $T \to +\infty$

$$
\int_2^T \frac{1}{\langle t\rangle \log t}\, dt \sim \int_2^\infty \frac{1}{t \log t}\, dt,
$$

and the latter diverges, while the chain of inequalities above should imply a uniform bound.

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Appendix A. Smallness of the remainder in the Strichartz norm

Lemma A.1. *Let $\alpha \in \left(\frac{4}{d}, \frac{4}{d-1}\right)$ for $2 \le d \le 4$ or $\alpha > 4$ if $d = 1$. Consider $\{u_n\}_{n\in\mathbb{N}}$ a sequence of solutions to*

$$\begin{cases} \partial_{tt} u_n - \Delta_{x,y} u_n + u_n = 0, & (t,x,y) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{T} \\ u_n(0,x,y) = f_n(x,y) \in H^1(\mathbb{R}^d \times \mathbb{T}) \\ \partial_t u_n(0,x,y) = g_n(x,y) \in L^2(\mathbb{R}^d \times \mathbb{T}) \end{cases},$$

with $\sup_{n\in\mathbb{N}} \| (f_n, g_n) \|_{\mathcal{H}} \le C < \infty$. Suppose that for any $q \in (2, 2^*)$, with 2^* defined in (3.1)

$$\lim_{n\to\infty} \|u_n\|_{L^\infty L^q} = 0.$$

Then

$$\lim_{n\to\infty} \|u_n\|_{L^{\alpha+1} L^{2(\alpha+1)}} = 0.$$

Proof. We drop the subscript n to lighten the notations. We first make a formal computation (from Hölder inequality) without adjusting the parameters:

$$\begin{aligned} \|u\|_{L^{\alpha+1}L^{2(\alpha+1)}} &= \left(\int \left(\int |u|^a |u|^b \,dx\,dy \right)^{1/2} dt \right)^{1/(\alpha+1)} \\ &\le \left(\int \left(\int |u|^{ar} \,dx\,dy \right)^{1/(2r)} \left(\int |u|^{bs} \,dx\,dy \right)^{1/(2s)} dt \right)^{1/(\alpha+1)} \\ &\le \left(\int \|u\|_{L^{ar}}^{a/2} \|u\|_{L^{bs}}^{b/2} \,dt \right)^{1/(\alpha+1)} \\ &\le \|u\|_{L^\infty L^{ar}}^{a/(2\alpha+2)} \|u\|_{L^{b/2} L^{bs}}^{b/(2\alpha+2)}. \end{aligned} \tag{A.1}$$

The claim of Lemma A.1 is satisfied if the following conditions are fulfilled in (A.1):

$$\begin{aligned} &a + b = 2(\alpha+1), \qquad a, b > 0, \\ &r = q/a > 1, \\ &s = r/(r-1) = q/(q-a), \\ &(b/2, bs) \quad \text{is a Strichartz pair as in } \textit{Proposition } 2.6. \end{aligned} \tag{A.2}$$

Under these conditions, we may have by hypothesis along with the energy conservation

$$\|u_n\|_{L^{\alpha+1}L^{2(\alpha+1)}} \le \|u_n\|_{L^\infty L^q}^{\gamma} E^{1-\gamma} \to 0,$$

where $\gamma \in (0,1)$. Note that it is enough to have the convergence to zero in only one $L^\infty L^q$.

Let us now check that all conditions are non-empty.

Case $d = 1$. Let $b = 2\alpha + \epsilon$ and $a = 2 - \epsilon$. We impose that $\epsilon \in (0,2)$ in order to satisfy (A.2). Strichartz admissibility conditions read $b \ge 8$ and $s \ge 2/(b-8)$. We strengthen the first requirement to $b > 8$. By definition of s we have

$$\frac{q}{q-2+\epsilon} \ge \frac{2}{2\alpha+\epsilon-8} \iff q(2\alpha - 10 + \epsilon) \ge 2\epsilon - 4.$$

For $\alpha \ge 5$ and for any $q \in (2, 2^*)$ the l.h.s. of the last inequality is positive for any choice of $\epsilon \in (0,2)$, since the r.h.s. is always negative for such values of ϵ. Then any $q \in (2, 2^*)$ yields to Strichartz admissibility condition.

If $\alpha \in (4,5)$ we further impose on ϵ the condition $\epsilon > 10-2\alpha$ beside the upper bound $\epsilon < 2$ so that the l.h.s. is still positive and hence any $0 < q \in (2,2^*)$ is good for our purpose.

Case $d=2$. Recall that in this dimension $2^*=4$. We chose $q=q(\alpha)=2\alpha-2$. We observe that $q(\alpha)\in(2,2^*)$ for any $\alpha\in(2,4)$ which is the range where α is allowed in dimension $d=2$.
Strichartz admissibility reads $b/2>2 \iff b>4$ and $\frac{2}{b-4}\leq s\leq\frac{6}{b-4}$ which is equivalent to

$$\begin{aligned}\frac{2}{b-4}\leq\frac{q}{q-a}\leq\frac{6}{b-4} &\iff \frac{2}{b-4}\leq\frac{2\alpha-2}{2\alpha-2-(2\alpha+2-b)}\leq\frac{6}{b-4}\\ &\iff \frac{2}{b-4}\leq\frac{2\alpha-2}{b-4}\leq\frac{6}{b-4}\\ &\iff 2\leq\alpha\leq 4\end{aligned}$$

which is satisfied for any intra-critical $\alpha\in(2,4)$.

Case $d=3$. In this case $2^*=4$ and $\alpha\in(4/3,2)$. To satisfy the admissibility condition, at first we impose $b\geq 4$. The second Strichartz condition reads

$$\frac{6}{3b-8}\leq\frac{q}{q-a}\leq\frac{4}{b-2}.$$

Let us focus on the l.h.s. condition.

$$\begin{aligned}\frac{6}{3b-8}\leq\frac{q}{q-a} &\iff 6(q-2\alpha-2+b)\leq q(3b-8)\\ &\iff b(6-3q)\leq 12\alpha+12-14q\\ &\iff b\geq\frac{12\alpha+12-14q}{6-3q}:=c_1(\alpha,q)\end{aligned}$$

If we impose $c_1<4$ we are done. But $c_1<4 \iff q<6(\alpha-1)$. So we restrict the upper bound for the choice of q as

$$q<\min\{4,6(\alpha-1)\}.$$

Let us now focus on the r.h.s. condition.

$$\begin{aligned}\frac{q}{q-a}\leq\frac{4}{b-2} &\iff \frac{q}{(q-2\alpha-2+b)}\leq\frac{4}{b-2}\\ &\iff q(b-2)\leq 4(q-2\alpha-2+b)\\ &\iff b(4-q)\geq 8\alpha+8-6q\\ &\iff b\geq\frac{8\alpha+8-6q}{4-q}:=c_2(\alpha,q)\end{aligned}$$

If we impose $c_2<4$ we are done. But this last condition is equivalent to $q>4(\alpha-1)$ and then by considering

$$q>\max\{2,4(\alpha-1)\}$$

we are able to conclude summarizing with

$$\max\{2,4(\alpha-1)\}<q<\min\{4,6(\alpha-1)\}.$$

Case $d = 4$. In this case $2^* = 10/3$ and $\alpha \in (1, 4/3)$. To satisfy the admissibility condition, at first we impose $b \geq 4$. The second Strichartz condition reads

$$\frac{2}{b-2} \leq \frac{q}{q-a} \leq \frac{10}{3b-4}.$$

Let us focus on the l.h.s. condition.

$$\begin{aligned}\frac{2}{b-2} \leq \frac{q}{q-a} &\iff \frac{2}{b-2} \leq \frac{q}{q-2\alpha-2+b}\\ &\iff b(q-2) \geq 4q - 4\alpha - 4\\ &\iff b \geq \frac{4q-4\alpha-4}{q-2} := c_3(\alpha, q)\end{aligned}$$

If we impose $c_3 < 4$ we are done. But $c_3 < 4 \iff \alpha > 1$ which is always satisfied under the intra-criticality condition.

Let us now focus on the r.h.s. condition.

$$\begin{aligned}\frac{q}{q-a} \leq \frac{10}{3b-4} &\iff \frac{q}{(q-2\alpha-2+b)} \leq \frac{10}{3b-4}\\ &\iff q(3b-4) \leq 10(q-2\alpha-2+b)\\ &\iff b(10-3q) \geq 20\alpha + 20 - 14q\\ &\iff b \geq \frac{20\alpha+20-14q}{10-3q} := c_4(\alpha, q)\end{aligned}$$

If we impose $c_4 < 4$ we are done. But this last condition is equivalent to $q > 10(\alpha - 1)$ and then by considering

$$q > \max\{2, 10(\alpha - 1)\}$$

we are able to conclude summarizing with

$$\max\{2, 4(\alpha - 1)\} < q < \frac{10}{3}.$$

□

Appendix B. Decay property of the linear flow

We use a decay property from [9]. The key argument is, again, a scaling argument as in the Section 2 and [19]. We will briefly sketch the proof given in [9, Example 1.2].
By means of the basis $\{\Phi_j(y)\}_{j\in\mathbb{N}}$ given in (2.11) and (2.12) we decompose

$$e^{it\sqrt{1-\Delta_{x,y}}} f(x, y) = \sum_{j\in\mathbb{N}} e^{it\sqrt{1+\lambda_j-\Delta_x}} f_j(x)\Phi_j(y).$$

Thus we get

$$\left\| e^{it\sqrt{1-\Delta_{x,y}}} f \right\|_{L^\infty(\mathbb{R}^d\times\mathbb{T})} \leq \sum_{j\in\mathbb{N}} \left\| e^{it\sqrt{1+\lambda_j-\Delta_x}} f_j(\cdot) \right\|_{L^\infty(\mathbb{R}^d)} \|\Phi_j(\cdot)\|_{L^\infty_y}.$$

From [9], we have

$$\left\| e^{it\sqrt{1-\Delta_x}} f \right\|_{L^\infty_x} \leq C|t|^{-d/2} \|f\|_{B^{\frac{d}{2}+1}_{1,1}}.$$

The function $w_m(t, x) = e^{it\sqrt{m-\Delta_x}} f$ satisfies the equation $\partial_{tt} w_m - \Delta_x w_m + m w_m = 0$ with $w(0, x) = f(\sqrt{m}x) := f_m$, with $w_m := w(\sqrt{m}t, \sqrt{m}x)$ and w satisfying $\partial_{tt} w - \Delta_x w + w = 0$ with $w(0, x) = f(x)$. We use a scaling argument to deduce an estimate for f_m, noticing that for $m \geq 1$, the Besov norm of a rescaled function can be bounded by:

$$\|f_m\|_{B^{\frac{d}{2}+1}_{1,1}} \leq m^{\frac{d+2}{4}} \|f\|_{B^{\frac{d}{2}+1}_{1,1}},$$

giving the following estimate with $m = 1 + \lambda_j > 1$

$$\begin{aligned}\left\|e^{it\sqrt{1-\Delta_{x,y}}} f\right\|_{L^\infty(\mathbb{R}^d\times\mathbb{T})} &\leq C|t|^{-\frac{d}{2}} \sum_{j\in\mathbb{N}} \sqrt{1+\lambda_j} \|f_j\|_{B^{\frac{d}{2}+1}_{1,1}} \|\Phi_j(y)\|_{L^\infty_y} \\ &= C|t|^{-\frac{d}{2}} \sum_{j\in\mathbb{N}} (1+\lambda_j)^{d+1}(1+\lambda_j)^{-d-1/2} \|f_j\|_{B^{\frac{d}{2}+1}_{1,1}} \|\Phi_j(y)\|_{L^\infty_y} \\ &\lesssim |t|^{-\frac{d}{2}} \sum_{j\in\mathbb{N}} (1+\lambda_j)^{d+1} \|f_j\|_{B^{\frac{d}{2}+1}_{1,1}} \|\Phi_j(y)\|_{L^\infty_y}.\end{aligned}$$

Noticing that the r.h.s. can be expressed a term involving derivatives in (x, y), one can find $N \in \mathbb{N}$ large enough to have

$$\left\|e^{it\sqrt{1-\Delta_{x,y}}} f\right\|_{L^\infty(\mathbb{R}^d\times\mathbb{T})} \leq C|t|^{-\frac{d}{2}} \|f\|_{W^{N,1}(\mathbb{R}^d\times\mathbb{T})}. \tag{B.1}$$

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