Blow-up or global existence for the fractional Ginzburg-Landau equation in multi-dimensional case

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Abstract The aim of this work is to give a complete picture concerning the asymptotic behaviour of the solutions to fractional Ginzburg-Landau equation. In previous works, we have shown global well-posedness for the past interval in the case where spatial dimension is less than or equal to 3. Moreover, we have also shown blow-up of solutions for the future interval in one dimensional case. In this work, we summarise the asymptotic behaviour in the case where spatial dimension is less than or equal to 3 by proving blow-up of solutions for a future time interval in multidimensional case. The result is obtained via ODE argument by exploiting a new weighted commutator estimate.

1 Introduction

In this paper, we consider the following complex Ginzburg – Landau (CGL) equation in a future time interval

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$$\begin{cases} i\partial_t u + Du = i|u|^{p-1}u, & t \in [0,T), \quad T > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where u is a complex valued unknown function, $p > 1$, and $D = (-\Delta)^{1/2}$. The choice of D is closely connected with the recent attempts to develop fractional quantum mechanical approach (see [23]).

We shall observe some new interesting phenomena. On one hand, if we take a future time interval as in (1), then we shall obtain a blow-up result. If, instead, we take past time interval $(-T,0]$, $T > 0$ in the place of the future time interval, then global small data existence for (1) can be proved and therefore we have a similarity to a diffusion type process.

Before giving the main results on the local and global well-posedness for (1), we introduce some notations. For a Banach space X and $1 \le p \le \infty$ let $L^p(\mathbb{R}^n;X)$ be a X-valued Lebesgue space of p-th power. We abbreviate $L^p(\mathbb{R}^n;\mathbb{C})$ as $L^p(\mathbb{R}^n)$. For $f,g \in L^2(\mathbb{R}^n)$, we define the inner product as

$$\langle f,g\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\overline{g}(x)dx.$$

For $s \in \mathbb{R}$, let $H^s(\mathbb{R}^n)$ be the usual inhomogeneous Sobolev space defined as $H^s(\mathbb{R}^n) = (1-\Delta)^{-s/2}L^2(\mathbb{R}^n)$. Let $\dot{H}^s(\mathbb{R}^n)$ be the usual homogeneous Sobolev space defined as $\dot{H}^s(\mathbb{R}^n) = (-\Delta)^{-s/2}L^2(\mathbb{R}^n)$. $H^s_{rad}(\mathbb{R}^n)$ is the restriction to radial functions of $H^s(\mathbb{R}^n)$. Lip refers to space of Lipschitz functions on euclidean space. For $f,g : A \subseteq \mathbb{R}^n \to [0,\infty)$, $f \lesssim g$ means that there exists $C > 0$ such that for any $a \in A$ $f(a) \le Cg(a)$. Given two Banach spaces X,Y, $Y \hookrightarrow X$ means that $Y \subset X$ with continuous embedding. Moreover, we say that a Cauchy problem is locally well-posed forward in time in X, if for any X-valued initial data, there exists $T > 0$ and a Banach space $Y \hookrightarrow C([0,T];X)$ such that there exists a unique solution to the Cauchy problem in Y and $\|u_n - u\|_Y \to 0$ as $\|u_{0,n} - u_0\|_X \to 0$, where u_n and u are solutions for the Cauchy problem for initial data u_0 and $u_{0,n}$, respectively (the last property goes under the name of *continuous dependence on the initial data*). We also say that a Cauchy problem is globally well-posed forward in time in X if the Cauchy problem is locally well-posed for any $T > 0$. Moreover, we also say that a Cauchy problem is globally well-posed in X with sufficiently small data, if we have the property above for sufficiently small data with respect to the X-norm.

Let us notice that equation (1) is invariant under the scale transformation

$$u_\lambda(t,x) = \lambda^{1/(p-1)}u(\lambda t, \lambda x)$$

with $\lambda > 0$. Then

$$\|u_{0,\lambda}\|_{\dot{H}^s(\mathbb{R}^n)} = \lambda^{1/(p-1)+s-n/2}\|u_0\|_{\dot{H}^s(\mathbb{R}^n)}$$

and with

$$s = s_{n,p} := n/2 - 1/(p-1) < n/2,$$

$\dot{H}^s$ norm of initial data is also invariant, for this $s_{n,p}$ is called scale critical exponent. We also call $p_{n,s} = 1+2/(n-2s)$ the $H^s(\mathbb{R}^n)$ scaling critical power. For any s, in the scaling subcritical case where $p < p_{n,s}$ or $s > s_{n,p}$, (1) is expected to have local solution for any $H^s(\mathbb{R}^n)$ initial data on the analogy of scaling invariant Schrödinger equation. For instance, we refer the reader to [4, 6, 5, 16, 17]. However, with power type nonlinearity without gauge invariance, semirelativistic equations could be not locally well-posed even in scaling subcritical case, see [10].

Here we recall local well-posedness results. It is worth mentioning that Borgna and Rial [2] showed that in one dimensional case, CGL equation with cubic nonlinearity is locally well-posed in $H^s(\mathbb{R})$ with $s > 1/2$. They constructed local solutions by a contraction argument based on the unitarity of the propagator and the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Similarly, local solutions may be constructed in the case where uniform control of solutions holds, namely, in $H^s(\mathbb{R}^n)$ with $s > n/2$. On the other hand, for fixed p, $s_{n,p} < n/2$; therefore, the local well-posedness of (1) is expected in wider Sobolev spaces. Indeed, we have the following results that can be established using the approach in [12]:

Proposition 1 ([12]). *Let $n = 2$. For $p > 1$ and $3/4 < s < p < p_{2,s}$, the Cauchy problem* (1) *is locally well-posed in $H^s(\mathbb{R}^2)$.*

Proposition 2 ([12]). *Let $n \geq 3$ and u_0 be radial. For $1 < p < p_{n,1} = 1 + \frac{2}{n-2}$, the Cauchy problem* (1) *is locally well-posed in $H^1_{\mathrm{rad}}(\mathbb{R}^n)$.*

Proposition 3 ([12]). *Let $n = 3$ and u_0 be radial. For $p = p_{3,1} = 3$, the Cauchy problem* (1) *is locally well-posed in $H^1_{\mathrm{rad}}(\mathbb{R}^3)$ with sufficiently small $H^1_{\mathrm{rad}}(\mathbb{R}^3)$ data.*

Remark 1. In Proposition 3, since the local existence result is based on a priori estimate of type

$$\|u\|_{X^1_{\mathrm{rad}}(0,T)} \leq C_0 + C_1 \|u\|^4_{X^1_{\mathrm{rad}}(0,T)}$$

with C_1 which is independent of T, we restrict well-posedness to the small initial data.

We recall that in three dimensional case, $p = p_{3,1} = 3$ is a critical value in view of the result in [18]. However, the result in [18] treats non-gauge invariant nonlinearities having constant sign, for which the test function method works. The question of the existence of local and global solutions for $n \geq 3$ and $p \geq 1 + 2/(n-2)$ seems, at the best of our knowledge, still open.

Proposition 1 may be justified by a Strichartz estimate introduced by Nakamura and Ozawa in [26] or Ginibre and Velo [14]. We remark that they introduced the estimate to study Klein-Gordon equation and it was sufficient to consider Klein-Gordon equation in scaling subcritical case (see Lemma 1 below). On the other hand, for (1), local solutions cannot be constructed based on their Strichartz estimates in general subcritical case. Therefore, in order to consider the well-posedness in $H^1(\mathbb{R}^n)$ for $n \geq 3$, we put radial assumption and apply another Strichartz estimate introduced in [1] by the third author, Bellazzini and Visciglia. For details, see Section 2.

Next, we review the known blow-up result. In [11], the authors studied the blow-up of solutions to (1) in one dimensional case, by an ordinary differential equation (ODE) argument. In order to review their argument, we define a function space $hL^2(\mathbb{R}^n)$ by

$$hL^2(\mathbb{R}^n) = \{f : \text{mesurable and } \|\frac{1}{h}f\|_{L^2(\mathbb{R}^n)} < \infty\},$$

where h is a mesurable function. In their argument, an ordinary differential inequality (ODI) for the $hL^2(\mathbb{R})$ norm of solutions with some h are shown. In particular, we have the following:

Proposition 4. *Let h be a Lipschitz function satisfying* $1/h \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$ *and*

$$\left\| \frac{1}{h(\cdot)} \int_{\mathbb{R}} \langle \cdot - y \rangle^{-2} h(y) f(y) dy \right\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}. \tag{2}$$

Let $u_0 \in L^2(\mathbb{R})$ *satisfy*

$$\|\frac{1}{h}u_0\|_{L^2(\mathbb{R})} \geq C_1^{\frac{1}{p-1}} \|\frac{1}{h}\|_{L^2(\mathbb{R})}, \tag{3}$$

where $C_1 = \|1/h \cdot [D,h]\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}$*. If there is a solution* $u \in C([0,T); hL^2(\mathbb{R}))$ *to* (1)*, then*

$$\|\frac{1}{h}u(t)\|_{L^2(\mathbb{R})} \geq e^{-C_1 t/2} \left(\|\frac{1}{h}u_0\|_{L^2(\mathbb{R})}^{-p+1} + C_1^{-1} \|\frac{1}{h}\|_{L^2(\mathbb{R})}^{-p+1} \left\{ e^{-C_1(p-1)t/2} - 1 \right\} \right)^{-\frac{1}{p-1}}. \tag{4}$$

Therefore, the lifespan is estimated by

$$T \leq -\frac{2}{p-1} C_1^{-1} \log \left(1 - C_1 \|\frac{1}{h}\|_{L^2(\mathbb{R})}^{p-1} \|\frac{1}{h}u_0\|_{L^2(\mathbb{R})}^{-p+1} \right).$$

Moreover, by scaling argument, the following statement is shown.

Corollary 1 ([11, Corollary 1]). *If* $p < 3$*, then any solutions to* (1) *with non trivial* $L^2(\mathbb{R})$ *initial data cannot stay in* $L^2(\mathbb{R})$ *globally.*

Remark 2. In the Corollary above, $p < 3$ stands for the condition in one dimensional case of the Fujita exponent generally defined in $\mathbb{R}^n$ by $p_F := 1 + 2/n$ (see also Corollary 2). Then the assumption of Corollary 1 is rewritten by $p < p_F$. Under this assumption, by scaling h, (3) holds for any non trivial $L^2(\mathbb{R})$ initial data u_0.

Remark 3. Condition (2) was required to guarantee the commutator estimate:

$$\|[D,h]f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}, \qquad \forall f \in L^2(\mathbb{R}). \tag{5}$$

We remark that Lenzmann and Schikorra [24, Theorem 6.1] showed that (5) holds for any Lipschitz function h, therefore, the assumption (2) can be omitted.

The commutator estimate (5) implies blow-up for solutions to (1) in the following manner. Let $v(t,x) = u(t,x)/h(x)$, where u is a solution to (1). Then, a straight computation shows that v satisfies

$$
\begin{aligned}
i\partial_t v + Dv + \frac{1}{h}[D,h]v &= i\frac{1}{h}\partial_t u + \frac{1}{h}Du \\
&= i\frac{1}{h}|u|^{p-1}u \\
&= i|h|^{p-1}|v|^{p-1}v.
\end{aligned} \tag{6}
$$

Therefore,

$$
\begin{aligned}
\frac{d}{dt}\|v(t)\|^2_{L^2(\mathbb{R})} &= 2\mathrm{Re}\langle v(t), \partial_t v(t)\rangle_{L^2(\mathbb{R})} \\
&= -2\mathrm{Im}\langle v(t), i\partial_t v(t)\rangle_{L^2(\mathbb{R})} \\
&= -2\mathrm{Im}\langle v(t), -Dv(t) - \frac{1}{h}[D,h]v(t) + i|h|^{p-1}|v(t)|^{p-1}v(t)\rangle_{L^2(\mathbb{R})} \\
&= 2\||h|^{(p-1)/(p+1)}v(t)\|^{p+1}_{L^{p+1}(\mathbb{R})} + 2\mathrm{Im}\langle v(t), \frac{1}{h}[D,h]v(t)\rangle_{L^2(\mathbb{R})}.
\end{aligned} \tag{7}
$$

By the Hölder inequality,

$$
\|v(t)\|_{L^2(\mathbb{R})} \le \|\frac{1}{h}\|^{(p-1)/(p+1)}_{L^2(\mathbb{R})}\||h|^{(p-1)/(p+1)}v(t)\|_{L^{p+1}(\mathbb{R})},
$$

which together with (7) implies

$$
\frac{d}{dt}\|v(t)\|^2_{L^2(\mathbb{R})} \ge \|\frac{1}{h}\|^{-p+1}_{L^2(\mathbb{R})}\|v(t)\|^{p+1}_{L^2(\mathbb{R})} - \|\frac{1}{h}[D,h]\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\|v(t)\|^2_{L^2(\mathbb{R})}. \tag{8}
$$

Estimate (8) and Lemma 7 in Section 3 imply that if (3) holds and

$$
\|\frac{1}{h}\cdot[D,h]\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < \infty, \tag{9}
$$

then $\|v(t)\|_{L^2(\mathbb{R})} = \|u(t)/h\|_{L^2(\mathbb{R})}$ blows-up at a finite time. Therefore, if there exists $1/h \in L^2(\mathbb{R})$ satisfying (9), then the argument above works and blow-up of solutions to (1) is shown. In [11], (9) was shown by the boundedness assumption of $1/h$ and (5). We remark that (5) holds in more general situation; for example, in multidimensional case. We also remark that in [9] a generalization of (5) taking the form

$$
\|[(-\mathscr{A})^{1/2}, h]\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \le C\|h\|_{\dot{B}^1_{\infty,1}}
$$

is shown, where $\dot{B}^1_{\infty,1}$ is the standard homogeneous Besov space and

$$
\mathscr{A} := -\nabla\cdot A\nabla + V.
$$

Here A is a smooth positive-definite $n \times n$ matrix and the real-valued potential V satisfies some weak integrability conditions. On the other hand, $h \in \mathrm{Lip}$ is a natural condition for (5). However, there exists some Lipschitz function h satisfying $1/h \in L^2(\mathbb{R}^n)$ only when $n = 1$. This means, we cannot consider blow-up phenomena in multi dimensional case based on (5).

In this paper, we show (9) with polynomial weights which are not Lipschitz in general. In particular, we show the following estimate:

Proposition 5. *Let $n \geq 1$ and $n/2 < q < n/2+1$. Then $\langle \cdot \rangle^{-q}[D, \langle \cdot \rangle^q]$ is bounded operator on $L^2(\mathbb{R}^n)$, where $\langle \cdot \rangle = (1+|x|^2)^{1/2}$.*

Remark 4. Obviously, if $n \geq 1$ and $n/2 < q < n/2+1$, then $\langle \cdot \rangle^{-q} \in L^2(\mathbb{R}^n)$. Moreover, only when $n = 1$, q can be 1.

Then, we have the following blow-up statement:

Proposition 6. *Let $n \geq 1$ and $n/2 < q < n/2+1$. Let $u_0 \in \langle \cdot \rangle^q L^2(\mathbb{R}^n)$ satisfy*

$$\|\langle x \rangle^{-q} u_0\|_{L^2(\mathbb{R}^n)} \geq C_2^{\frac{1}{p-1}} \|\langle x \rangle^{-q}\|_{L^2(\mathbb{R}^n)}, \tag{10}$$

where

$$C_2 = \|\langle \cdot \rangle^{-q}[D, \langle \cdot \rangle^q]\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)}.$$

If there is a solution $u \in C([0,T); \langle \cdot \rangle^q L^2(\mathbb{R}^n))$ to (1), then

$$\begin{aligned} &\|\langle \cdot \rangle^{-q} u(t)\|_{L^2(\mathbb{R}^n)} \\ &\geq e^{-C_2 t/2} \left(\|\langle \cdot \rangle^{-q} u_0\|_{L^2(\mathbb{R}^n)}^{-p+1} + C_2^{-1} \|\langle \cdot \rangle^{-q}\|_{L^2(\mathbb{R}^n)}^{-p+1} \left\{ e^{-(p-1)C_2 t/2} - 1 \right\} \right)^{-\frac{1}{p-1}}. \end{aligned} \tag{11}$$

Therefore, the lifespan is estimated by

$$T \leq -\frac{2}{p-1} C_2^{-1} \log \left(1 - C_2 \|\langle \cdot \rangle^{-q}\|_{L^2(\mathbb{R}^n)}^{p-1} \|\langle \cdot \rangle^{-q} u_0\|_{L^2(\mathbb{R}^n)}^{-p+1} \right). \tag{12}$$

Corollary 2. *Let $n \geq 1$. If $p < p_F := 1 + 2/n$, then any solutions to (1) with non trivial $L^2(\mathbb{R}^n)$ initial data cannot exist globally.*

Remark 5. As Remark 2, under the condition, $p < p_F$, by scaling h, (10) holds for any non trivial $L^2(\mathbb{R}^n)$ data.

In [11], so as to show (5), higher frequency part of D is handled by the Coifman-Meyer estimate and lower frequency part is estimated by (2). We remark that (5) is regarded as a Kato-Ponce inequality. For related subjects, we refer the reader to [15, 19, 20, 25] and we remark that Fourier multiplier argument plays a critical role in these references. On the other hand, it seems not easy to obtain (9) based on a Fourier multiplier argument because of the weight function. Therefore, we show Proposition 5 by using the following representation of the commutator:

$$([D,\langle\cdot\rangle^q]f)(x) = C\cdot\text{P.V.}\int_{\mathbb{R}^n}\frac{(\langle x\rangle^q-\langle x+y\rangle^q)f(x+y)}{|y|^{n+1}}dy, \tag{13}$$

where *P.V.* stands for Principal Value (for detail, we refer the reader to [8]). Combining (13) and the Calderón-Zygmund theory, we show (9) with non-Lipschitz weight functions.

Our next step is to study the global existence result for negative times of the following Cauchy problem:

$$\begin{cases} i\partial_t u + Du = i|u|^{p-1}u, & t\in(-T,0], \quad T>0, \quad x\in\mathbb{R}^n,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{14}$$

Making the change of variables $t\to -t$, we reduce this problem to the future time interval for the Cauchy problem

$$\begin{cases} i\partial_t u - Du = -i|u|^{p-1}u, & t\in[0,T), \quad T>0, \quad x\in\mathbb{R}^n,\\ u(0,x)=u_0(x), & x\in\mathbb{R}^n. \end{cases} \tag{15}$$

At least formally, (15) may be rewritten in the following integral equation:

$$u(t) = U(-t)u_0 - \int_0^t U(-t+t')|u(t')|^{p-1}u(t')dt', \tag{16}$$

where $U(t)=e^{itD}$.

Then, Propositions 1, 2, and 3 are valid for (15). Moreover, for (16), we can obtain the following a priori estimates that we include for completeness but detailed proofs can be found in [12].

Proposition 7 ([12]). *Let $n\in\mathbb{N}$ and $p>1$. Let $u_0\in L^2(\mathbb{R}^n)$ and $T>0$. Let $u\in L^\infty(0,T;L^2(\mathbb{R}^n))\cap L^p(0,T;L^{2p}(\mathbb{R}^n))$ be a solution to the integral equation* (16) *for the initial data u_0. Then, for any t_1,t_2 with $0<t_1<t_2<T$,*

$$\|u(t_2)\|^2_{L^2(\mathbb{R}^n)} + 2\|u\|^{p+1}_{L^{p+1}(t_1,t_2;L^{p+1}(\mathbb{R}^n))} = \|u(t_1)\|^2_{L^2(\mathbb{R}^n)}.$$

Proposition 8 ([12]). *Let $n\in\mathbb{N}$ and $p>1$. Let $u_0\in H^1(\mathbb{R}^n)$ and $T>0$. Let $u\in L^\infty(0,T;H^1(\mathbb{R}^n))\cap L^{p-1}(0,T;L^\infty(\mathbb{R}^n))$ be a solution to the integral equation* (16) *for the initial data u_0. Then, for any t_1,t_2 with $0\le t_1<t_2\le T$,*

$$\begin{aligned}&\|\nabla u(t_2)\|^2_{L^2(\mathbb{R}^n)} + 2\||u|^{\frac{p-1}{2}}\nabla u\|^2_{L^2(t_1,t_2;L^2(\mathbb{R}^n))} + \frac{p-1}{2}\||u|^{\frac{p-3}{2}}\nabla|u|^2\|^2_{L^2(t_1,t_2;L^2(\mathbb{R}^n))}\\ &= \|\nabla u(t_1)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{17}$$

Proposition 9 ([12]). *Let $n=1,2$, $p>1$, $n/2<s<\min\{2,p\}$, and $T>0$. Let $u_0\in H^s(\mathbb{R}^n)$ and $u\in L^\infty(0,T;H^s(\mathbb{R}^n))\cap L^2(0,T;L^\infty(\mathbb{R}^n))$ be a solution to* (16) *for the initial data u_0. Then for any t_1,t_2 with $0<t_1<t_2<T$,*

$$\|u(t_2)\|^2_{\dot{H}^s(\mathbb{R}^n)} \leq \|u(t_1)\|^2_{\dot{H}^s(\mathbb{R}^n)} + C\int_{t_1}^{t_2} \|u(t)\|^{p-1}_{L^\infty(\mathbb{R}^n)} \|u(t)\|^2_{\dot{H}^s(\mathbb{R}^n)} dt.$$

Proposition 10 ([12]). *Let* $1 \leq n \leq 3$, $u_0 \in H^2(\mathbb{R}^n)$ *and* $T > 0$. *Let* $u \in C((0,T); H^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ *be a solution to the integral equation* (16) *for the initial data* u_0. *Then, for any* t_1, t_2 *with* $0 < t_1 < t_2 < T$,

$$\begin{aligned} &\|u(t_2)\|^2_{\dot{H}^2(\mathbb{R}^n)} + 2\sum_{j,k=1}^{n} \int_{t_1}^{t_2} \|u(t)\partial_j\partial_k u(t)\|^2_{L^2(\mathbb{R}^n)} dt \\ &\leq \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} + 2n^2(n+1)\int_{t_1}^{t_2} \|u(t)\|^{4-n}_{\dot{H}^1(\mathbb{R}^n)} \|u(t)\|^n_{\dot{H}^2(\mathbb{R}^n)} dt. \end{aligned} \tag{18}$$

Therefore, for (14) we have the following:

Proposition 11. *Under the conditions of Propositions 1, 2, and 3,* (14) *is globally well-posed.*

This paper is composed as follows. In Section 2, we show local well-posedness of (1) by means of Strichartz estimates of [1, 14, 26]. In Section 3, blow-up for (1) is shown with a weighted commutator estimate. In Section 4, a priori estimates for (14) are shown by a direct approach leading to the global well-posedness results.

2 Local well-posedness of (1)

This section is devoted to the proof of the local well-posedness for the Cauchy problem of (1), where $u_0(x) = u(0,x)$ is considered as initial datum. The proof is essentially the same as [12] but for the reader's convenience, we give a proof for Propositions 1, 2, and 3. Here we consider the corresponding integral equation:

$$u(t) = \Phi(u)(t) = U(t)u_0 + \int_0^t U(t-t')|u(t')|^{p-1}u(t')dt'. \tag{19}$$

where $U(t) = e^{itD}$.

2.1 Two dimensional case

In two dimensional case, the local well-posedness may be obtained by the following Strichartz estimates:

Lemma 1 ([26, Lemma 2.1], [14, Remark 3.2]). *Let* (q_1, r_1) *and* (q_2, r_2) *satisfy*

$$\frac{1}{r_j} = \frac{1}{2} - \frac{2}{q_j}, \quad 2 \leq r_j \leq \infty, \quad 4 \leq q_j \leq \infty$$

for $j=1,2$. *Then for* $s\in\mathbb{R}$,

$$\|U(t)\phi\|_{L^{q_1}(0,T;B^{s-\frac{3}{q_1}}_{r_1}(\mathbb{R}^2))}\lesssim\|\phi\|_{H^s(\mathbb{R}^2)},$$
$$\left\|\int_0^t U(t-t')h(t')dt'\right\|_{L^{q_1}(0,T;B^{s-\frac{3}{q_1}}_{r_1}(\mathbb{R}^2))}\lesssim\|h\|_{L^{q_2'}(0,T;B^{s+\frac{3}{q_2}}_{r_2'}(\mathbb{R}^2))},$$

where $B^s_p(\mathbb{R}^2)=B^s_{p,2}(\mathbb{R}^2)$ *is the usual inhomogeneous Besov space.*

Lemma 2 ([12, Lemma 3.2]). *Let* $r>2$*, and* $T>0$*. If*

$$s>\frac{3}{4}+\frac{1}{2r},$$

then $B_r^{s-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})}(\mathbb{R}^2)\hookrightarrow L^\infty(\mathbb{R}^2)$.

We can now proceed with the proof of Proposition 1.

Proof (Proof of Proposition 1). At first we fix $3/4<s<p<p_{2,s}$. Let (q_1,r_1) satisfy the conditions of Lemma 1, Lemma 2 and $q_1>p-1$. We remark that such a pair exists under the assumption $s<p<p_{2,s}$. Let $X^s(0,T)=L^\infty(0,T;H^s(\mathbb{R}^2))\cap L^{q_1}(0,T;B^{s-3/q_1}_{r_1}(\mathbb{R}^2))$. Then, for a fixed T,

$$\begin{aligned}\|\Phi(u)\|_{X^s(0,T)}&\le\|u_0\|_{H^s(\mathbb{R}^2)}+C\||u|^{p-1}u\|_{L^1(0,T;H^s(\mathbb{R}^2))}\\&\le\|u_0\|_{H^s(\mathbb{R}^2)}+CT^{1-(p-1)/q_1}\|u\|^p_{X^s(0,T)},\end{aligned}\tag{20}$$

and

$$\begin{aligned}&\|\Phi(u)-\Phi(v)\|_{X^s(0,T)}\\&\le C\||u|^{p-1}u-|v|^{p-1}v\|_{L^1(0,T;H^s(\mathbb{R}^2))}\\&\le CT^{1-(p-1)/q_1}(\|u\|_{X^s(0,T)}+\|v\|_{X^s(0,T)})^{p-1}\|u-v\|_{X^s(0,T)}\\&+CT^{1-(p-1)/q_1}(\|u\|_{X^s(0,T)}+\|v\|_{X^s(0,T)})^{\max(1,p-1)}\|u-v\|^{\min\{1,p-1\}}_{X^s(0,T)}.\end{aligned}$$

This means that if T is sufficiently small, then Φ is a map from

$$B_{X^s(0,T)}(2\|u_0\|_{H^s(\mathbb{R}^2)}):=\left\{f\in X^s(0,T)\mid\|f\|_{X^s(0,T)}\le2\|u_0\|_{H^s(\mathbb{R}^2)}\right\}.$$

into itself. Moreover, if $p\ge2$, Φ is a contraction map in $X^s(0,T)$. If $p<2$, Φ may not be a contraction map on $X^s(0,T)$ for any $T>0$. On the other hand, it is not difficult to see that

$$\begin{aligned}&\|\Phi(u)-\Phi(v)\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}\\&\lesssim T^{1-(p-1)/q_1}(\|u\|_{X^s(0,T)}+\|v\|_{X^s(0,T)})^{p-1}\|u-v\|_{L^\infty(0,T;L^2(\mathbb{R}^2))}.\end{aligned}\tag{21}$$

Therefore (20) and (21) imply that if $u_1 \in B_{X^s(0,T)}(2\|u_0\|_{H^s(\mathbb{R}^2)})$ and $u_k = \Phi(u_{k-1})$ for $k \geq 2$, then there exists $u^* = \lim_{k\to\infty} u_k$ in $L^\infty(0,T;L^2(\mathbb{R}^2))$. Since $\Phi(u_k) \to \Phi(u^*)$ in $L^\infty(0,T;L^2(\mathbb{R}^2))$ as $k \to \infty$, u^* is a solution of (19). Moreover, since $X^s(0,T) \hookrightarrow L^\infty(0,T;H^s(\mathbb{R}^2))$, u^* is also in $L^\infty(0,T;H^s(\mathbb{R}^2))$, which and (20) imply

$$u^* \in L^{q_1}(0,T;B_{r_1}^{s-\frac{3}{q_1}}(\mathbb{R}^2)).$$

If $s > 1$, by the Gagliardo-Nirenberg inequality, for some $0 < \theta < 1$,

$$\|u-v\|_{L^\infty(0,T;L^\infty(\mathbb{R}^2))} \lesssim \|u-v\|^{\theta}_{L^\infty(0,T;L^2(\mathbb{R}^2))}\|u-v\|^{1-\theta}_{L^\infty(0,T;H^s(\mathbb{R}^2))}$$

and therefore the solution map depends continuously on the initial data in $H^s(\mathbb{R}^2)$. In the case where $s \leq 1$, by (21), the solution map depends continuously on the initial data in $L^2(\mathbb{R}^2)$. We define $s_3, s_4 > 0$ so that they satisfy the following:

$$\max\left\{\frac{3}{4}+\frac{1}{2r_1}, s_4-\frac{3}{4}(p-1)\right\} < s_3 < s_4 < \min\left\{s, s_3+\frac{3}{4}\right\},$$
$$r_3 = \frac{3}{2}\left(s_3-s_4+\frac{3}{4}\right)^{-1},$$

and $q_3 = \frac{3}{s_4-s_3}$, where (q_3, r_3) satisfy the condition of Lemma 1. Let u and v be solutions of (1) for initial data u_0 and v_0, respectively. Then by Lemma 1,

$$\begin{aligned}&\|u-v\|_{L^{q_1}(0,T;B_{r_1}^{s_3-\frac{3}{q_1}}(\mathbb{R}^2))}\\ &\leq \|u_0-v_0\|_{H^{s_3}(\mathbb{R}^2)} + C\||u|^{p-1}u-|v|^{p-1}v\|_{L^{q_3'}(0,T;B^{s_4}_{r_3'}(\mathbb{R}^2))}.\end{aligned} \tag{22}$$

For $z_j \in \mathbb{C}$ with $j = 1,2,3,4$, with $w_1 = z_2 - z_1$ and $w_2 = z_4 - z_3$,

$$\begin{aligned}&|z_4|^{p-1}z_4 - |z_3|^{p-1}z_3 - |z_2|^{p-1}z_2 + |z_1|^{p-1}z_1\\ &= \frac{p+1}{2}\int_0^1 |z_3+\theta w_2|^{p-1}d\theta w_2 - \frac{p+1}{2}\int_0^1 |z_1+\theta w_1|^{p-1}d\theta w_1\\ &+\frac{p-1}{2}\int_0^1 |z_3+\theta w_2|^{p-3}(z_3+\theta w_2)^2 d\theta \overline{w_2}\\ &-\frac{p-1}{2}\int_0^1 |z_1+\theta w_1|^{p-3}(z_1+\theta w_1)^2 d\theta \overline{w_1}.\end{aligned}$$

Then a direct computation implies that

$$
\begin{aligned}
&\big||z_4|^{p-1}z_4-|z_3|^{p-1}z_3-|z_2|^{p-1}z_2+|z_1|^{p-1}z_1\big|\\
&\lesssim (|z_3|^{p-1}+|z_4|^{p-1})|w_2-w_1|\\
&+\frac{p+1}{p}|w_1||z_3-z_1|^{p-1}+\frac{1}{p}|w_1||z_4-z_2|^{p-1}+(|z_3|^{p-1}+|z_4|^{p-1})|w_2-w_1|\\
&+|w_1||z_3-z_1|^{p-1}+|w_1||z_4-z_2|^{p-1}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\big\||u(t,\cdot+h)|^{p-1}u(t,\cdot+h)-|v(t,\cdot+h)|^{p-1}v(t,\cdot+h)\\
&\quad -|u(t)|^{p-1}u(t)+|v(t)|^{p-1}v(t)\big\|_{L^{r_3'}(\mathbb{R}^2)}\\
&=\big\||u(t,\cdot+h)|^{p-1}u(t,\cdot+h)-|u(t)|^{p-1}u(t)\\
&\quad -|v(t,\cdot+h)|^{p-1}v(t,\cdot+h)+|v(t)|^{p-1}v(t)\big\|_{L^{r_3'}(\mathbb{R}^2)}\\
&\le 4\|u(t)\|^{p-1}_{L^{\frac{2r_3(p-1)}{r_3-2}}(\mathbb{R}^2)}\|u(t,\cdot+h)-v(t,\cdot+h)-u(t)+v(t)\|_{L^2(\mathbb{R}^2)}\\
&+\frac{2(p+2)}{p}\|v(t,\cdot+h)-v(t)\|_{L^2(\mathbb{R}^2)}\|u(t)-v(t)\|^{p-1}_{L^{\frac{2r_3(p-1)}{r_3-2}}(\mathbb{R}^2)},
\end{aligned}
$$

and this means

$$
\begin{aligned}
&\||u|^{p-1}u-|v|^{p-1}v\|_{L^{q_3'}(0,T;B^{s_4}_{r_3'}(\mathbb{R}^2))}\\
&\lesssim \Big\|\|u\|^{p-1}_{L^{\frac{2r_3(p-1)}{r_3-2}}(\mathbb{R}^2)}\|u-v\|_{H^{s_4}(\mathbb{R}^2)}+\|v\|_{H^{s_4}(\mathbb{R}^2)}\|u-v\|^{p-1}_{L^{\frac{2r_3(p-1)}{r_3-2}}(\mathbb{R}^2)}\Big\|_{L^{q_3'}(0,T)}\\
&\le \Big\|\|u\|^{p-1-\frac{r_3-2}{r_3}}_{L^\infty(\mathbb{R}^2)}\|u\|^{\frac{r_3-2}{r_3}}_{L^2(\mathbb{R}^2)}\|u-v\|_{H^{s_4}(\mathbb{R}^2)}\Big\|_{L^{q_3'}(0,T)}\\
&+\Big\|\|v\|_{H^{s_4}(\mathbb{R}^2)}\|u-v\|^{p-1-\frac{r_3-2}{r_3}}_{L^\infty(\mathbb{R}^2)}\|u-v\|^{\frac{r_3-2}{r_3}}_{L^2(\mathbb{R}^2)}\Big\|_{L^{q_3'}(0,T)}\\
&\le \|u\|^{p-1-\frac{r_3-2}{r_3}}_{L^{q_3'(p-1-\frac{r_3-2}{r_3})}(0,T;L^\infty(\mathbb{R}^2))}\|u\|^{\frac{r_3-2}{r_3}}_{L^\infty(0,T;L^2(\mathbb{R}^2))}\|u-v\|_{L^\infty(0,T;H^{s_4}(\mathbb{R}^2))}\\
&+\|v\|_{L^\infty(0,T;H^{s_4}(\mathbb{R}^2))}\|u-v\|^{p-1-\frac{r_3-2}{r_3}}_{L^{q_3'(p-1-\frac{r_3-2}{r_3})}(0,T;L^\infty(\mathbb{R}^2))}\|u-v\|^{\frac{r_3-2}{r_3}}_{L^\infty(0,T;L^2(\mathbb{R}^2))}\\
&\le \|u\|^{p-1-\frac{r_3-2}{r_3}}_{L^{q_1}(0,T;L^\infty(\mathbb{R}^2))}\|u\|^{\frac{r_3-2}{r_3}}_{L^\infty(0,T;L^2(\mathbb{R}^2))}\|u-v\|_{L^\infty(0,T;H^{s_4}(\mathbb{R}^2))}\\
&+\|v\|_{L^\infty(0,T;H^{s_4}(\mathbb{R}^2))}\|u-v\|^{p-1-\frac{r_3-2}{r_3}}_{L^{q_1}(0,T;L^\infty(\mathbb{R}^2))}\|u-v\|^{\frac{r_3-2}{r_3}}_{L^\infty(0,T;L^2(\mathbb{R}^2))},
\end{aligned}
$$

where $q_1,q_3>4>q_3'>q_3'\left(p-1-\frac{r_3-2}{r_3}\right)$. This and (22) imply that $u\to v$ in $L^{q_1}(0,T;B^{s_3-\frac{3}{q_1}}_{r_1}(\mathbb{R}^2))$ as $u_0\to v_0$ in $H^s(\mathbb{R}^2)$ because $u\to v$ in $(L^\infty(0,T;L^2(\mathbb{R}^2)))$

and u, v are uniformly bounded in $(L^\infty(0,T;H^s(\mathbb{R}^2)))$ as $u_0 \to v_0$ in $H^s(\mathbb{R}^2)$. Moreover,

$$\begin{aligned}
&\|u-v\|_{L^\infty(0,T;H^s(\mathbb{R}^2))}\\
&\lesssim \|u_0-v_0\|_{H^s(\mathbb{R}^2)} + (\|u_0\|_{H^s(\mathbb{R}^2)} + \|v_0\|_{H^s(\mathbb{R}^2)})\|u-v\|^{p-1}_{L^{p-1}(0,T;L^\infty(\mathbb{R}^2))}\\
&\lesssim \|u_0-v_0\|_{H^s(\mathbb{R}^2)} + (\|u_0\|_{H^s(\mathbb{R}^2)} + \|v_0\|_{H^s(\mathbb{R}^2)})\|u-v\|^{p-1}_{L^{q_1}(0,T;B^{s_3-\frac{3}{q_1}}_{r_1}(\mathbb{R}^2))}.
\end{aligned}$$

Therefore, the solution map is also continuously dependent in $L^\infty(0,T;H^s(\mathbb{R}^2))$.

2.2 The case $n \geq 3$. Local H^1 existence result

In the case where $n \geq 3$, the Strichartz estimate Lemma 1 doesn't seem sufficient to obtain a uniform control of solutions in the $H^1(\mathbb{R}^3)$ setting. So here, we consider radial data and use the following Strauss lemma.

Lemma 3 ([28, Theorems 1,2], [7, Proposition 1]). *Let $n \geq 2$ and let $1/2 < s < n/2$. Then for a radial function f*

$$\||\cdot|^{\frac{n}{2}-s} f\|_{L^\infty(\mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^s_{\mathrm{rad}}(\mathbb{R}^n)}.$$

Since solutions are not uniformly controlled at the origin by the Strauss lemma above, we apply the following weighted Strichartz estimate:

Lemma 4 ([1, Propositions 2.2 and 2.3]). *Let $n \in \mathbb{N}$. Let $\delta > 0$ and $[x]_\delta = |x|^{1-\delta} + |x|^{1+\delta}$. The for any $q_1 \in [2,\infty]$ and $q_2 \in (2,\infty]$,*

$$\|[x]_\delta^{-1/q_1} U(t) f\|_{L^{q_1}(\mathbb{R};L^2(\mathbb{R}^n))} \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$
$$\left\|[x]_\delta^{-1/q_1}\int_0^t U(t-t')F(t')dt'\right\|_{L^{q_1}(0,T;L^2(\mathbb{R}^n))} \lesssim \|[x]_\delta^{1/q_2} F\|_{L^{q_2'}(0,T;L^2(\mathbb{R}^n))}.$$

We can now prove Proposition 2.

Proof (Proof of Proposition 2). By using the uniform $H^1(\mathbb{R}^n)$ control obtained in (17), we reduce the proof to the local well-posedness in $H^1(\mathbb{R}^n)$. Let $\delta > 0$, $1/2 < s < 1$, and $2 < q_1, q_2 < \infty$ satisfy

$$-(p-1)\left(\frac{n}{2}-s\right) + \frac{1-\delta}{q_2} = -\frac{1-\delta}{q_1}. \tag{23}$$

We remark that there exist δ, q_1, q_2, s if $1 < p < 1 + 2/(n-2)$ since,

$$(p-1)\left(\frac{n}{2}-s\right) < 1 \Longrightarrow p < 1 + \frac{2}{n-2s} < 1 + \frac{2}{n-2}.$$

We define the norm $Y^1(T)$ as

$$\begin{aligned}\|u\|_{Y^1(T)} &= \|u\|_{L^\infty(0,T;H^1_{\mathrm{rad}}(\mathbb{R}^n))} \\ &\quad + \left\| [x]_\delta^{-1/q_1} u \right\|_{L^{q_1}(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^n))} + \left\| [x]_\delta^{-1/q_1} \nabla u \right\|_{L^{q_1}(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^n))}.\end{aligned}$$

Let $\psi \in \mathscr{S}(\mathbb{R}^n;[0,1])$ be radial and satisfy

$$\psi(x) = \begin{cases} 1 & \text{if} \quad |x| \le 1, \\ 0 & \text{if} \quad |x| \ge 2. \end{cases} \tag{24}$$

Then by Lemmas 3 and 4 and (23),

$$\begin{aligned}&\|\Phi(u)\|_{Y^1(T)} \\ &\lesssim \|u_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} + \left\| \int_0^t U(t-t') \left(\psi |u(t')|^{p-1} u(t') \right) dt' \right\|_{Y^1(T)} \\ &+ \left\| \int_0^t U(t-t') \left((1-\psi) |u(t')|^{p-1} u(t') \right) dt' \right\|_{Y^1(T)} \\ &\lesssim \|u_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} \\ &+ \| |x|^{-(p-1)(\frac{n}{2}-s)+\frac{1-\delta}{q_2}} ||x|^{\frac{n}{2}-s} u|^{p-1} u \|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\ &+ \| |x|^{-(p-1)(\frac{n}{2}-s)+\frac{1-\delta}{q_2}} ||x|^{\frac{n}{2}-s} u|^{p-1} \nabla u \|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\ &+ \| |u|^{p-1} u \|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))} + \| \nabla (|u|^{p-1} u) \|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))} \\ &\lesssim \|u_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} + T^{1-\frac{1}{q_1}-\frac{1}{q_2}} \|u\|^p_{Y^1(T)}\end{aligned}$$

and therefore for some T and R, Φ is a map from $B_{Y^1(T)}(R)$ into itself. Moreover,

$$\begin{aligned}&\|\Phi(u)-\Phi(v)\|_{Y^1(T)} \\ &\lesssim \| |x|^{-\frac{1-\delta}{q_1}} (||x|^{\frac{n}{2}-s} u|^{p-1} - ||x|^{\frac{n}{2}-s} v|^{p-1}) (|\nabla u| + |u|) \|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\ &+ \| |x|^{-\frac{1-\delta}{q_1}} ||x|^{\frac{n}{2}-s} v|^{p-1} (|\nabla (u-v)| + |u-v|) \|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\ &+ \| (||x|^{\frac{n}{2}-s} u|^{p-1} - ||x|^{\frac{n}{2}-s} v|^{p-1}) |x|^{-\frac{1+\delta}{q_1}} (|\nabla u| + |u|) \|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))} \\ &+ \| ||x|^{\frac{n}{2}-s} v|^{p-1} |x|^{-\frac{1+\delta}{q_1}} (|\nabla (u-v)| + |u-v|) \|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))}.\end{aligned} \tag{25}$$

Then for $p \ge 2$, Φ is a contraction map on $B_{Y^1(T)}(R)$. Similarly, for $1 < p < 2$, we define the auxiliary norm $Y^0(T)$ as

$$\|u\|_{Y^0(T)} := \|u\|_{L^\infty(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^n))} + \| [x]_\delta^{-1/q_1} u \|_{L^{q_1}(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^n))}.$$

Then for $1 < p < 2$,

$$
\begin{aligned}
&\|(\Phi(u) - \Phi(v))\|_{Y^0(T)} \\
&\lesssim \left\| [x]_\delta^{-1/q_1} \left(\left| |x|^{\frac{n}{2}-s} v \right| + \left| |x|^{\frac{n}{2}-s} v \right| \right)^{p-1} |u-v| \right\|_{L^{q_2'}(0,T;L^2_{\mathrm{rad}}(|x|\le 2))} \\
&+ \left\| \left(\left| |x|^{\frac{n}{2}-s} v \right| + \left| |x|^{\frac{n}{2}-s} v \right| \right)^{p-1} |u-v| \right\|_{L^1(0,T;L^2_{\mathrm{rad}}(|x|>1))} \\
&\lesssim T^{1-\frac{1}{q_1}-\frac{1}{q_2}} (\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)})^{p-1} \|u-v\|_{Y^0(T)}.
\end{aligned}
$$

Therefore Φ is a contraction map on $Y^0(T)$ for some T and R, which implies that (1) posses a unique solution in $Y^1(T)$. Moreover, by Lemma 3 and (25), with some $0 < \theta < 1$, for solutions u and v of (4) for initial data u_0 and v_0, respectively,

$$
\begin{aligned}
&\|u-v\|_{Y^1(T)} \\
&\lesssim \|u_0 - v_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} + T^{1-\frac{1}{q_1}-\frac{1}{q_2}} (\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)})^{p-1} \|u-v\|_{Y^1(T)} \\
&+ T^{1-\frac{1}{q_1}-\frac{1}{q_2}} (\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)}) \left\| |x|^{\frac{n}{2}-s}(u-v) \right\|^{p-1}_{L^\infty(0,T;L^\infty_{\mathrm{rad}}(\mathbb{R}^n))} \\
&\lesssim \|u_0 - v_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} + T^{1-\frac{1}{q_1}-\frac{1}{q_2}} (\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)})^{p-1} \|u-v\|_{Y^1(T)} \\
&+ T(\|u\|_{Y^1(T)} + \|v\|_{Y^1(T)}) \|u-v\|^{p-1}_{Y^1(T)}
\end{aligned}
$$

and therefore $\|u-v\|_{Y^1(T)} \to 0$ as $\|u_0 - v_0\|_{H^1_{\mathrm{rad}}(\mathbb{R}^n)} \to 0$.

2.3 *Three dimensional case, small H^1 data solutions for $p = 3$*

In the three dimensional scaling critical case, the weighted Strichartz estimate Lemma 4 doesn't seem sufficient to control solutions uniformly. So here, we transform (1) into the corresponding wave equation.

The Cauchy problem (1) with initial data $u(0) = u_0$ is rewritten as the following:

$$
\begin{aligned}
\Box u &= i(-i\partial_t + D)|u|^{p-1}u \\
&= i\frac{p+1}{2}|u|^{p-1}(Du - i|u|^{p-1}u) \\
&- i\frac{p-1}{2}|u|^{p-3}u^2\overline{(Du - i|u|^{p-1}u)} + iD(|u|^{p-1}u) \\
&= i\left(D(|u|^{p-1}u) + \frac{p+1}{2}|u|^{p-1}Du - \frac{p-1}{2}|u|^{p-3}u^2 D\overline{u} \right) + p|u|^{2p-2}u \\
&=: F_p(u).
\end{aligned}
$$

Then the corresponding integral equation is the following:

$$\begin{aligned} u(t) = \cos(tD)u_0 + \frac{\sin(tD)}{D}(iDu_0 + |u_0|^{p-1}u_0) \\ + \int_0^t \frac{\sin((t-t')D)}{D} F_p(u)(t')dt'. \end{aligned} \tag{26}$$

For any radially symmetric function f, we define $\widetilde{f}$ as $\widetilde{f}(|x|) = f(x)$. Then for any radial data, (26) is rewritten as

$$\widetilde{u}(t) = \partial_t J[u_0](t) + J[iDu_0 + |u_0|^{p-1}u_0](t) + \int_0^t J[F_p(u)(t')](t-t')dt' \tag{27}$$

where

$$J[f](t,r) = \frac{1}{2r}\int_{|r-t|}^{r+t} \lambda \widetilde{f}(\lambda)d\lambda.$$

This transformation is justified as follows:

Lemma 5 ([12, Lemma 3.5]). *Let $1 < p \leq 3$ and $u_0 \in H^1_{\mathrm{rad}}(\mathbb{R}^3)$ and $u \in C(0,T;H^1_{\mathrm{rad}}(\mathbb{R}^3))$ be the solution of* (16)*. Then u is also the solution of* (27)*.*

To obtain the uniform control, we use the estimates below regarding J. For any $f : [0,\infty) \to \mathbb{C}$, we define $A[f] : \mathbb{R} \to \mathbb{C}$ as $A[f](\lambda) = f(|\lambda|)$. See also [21].

Lemma 6 ([12, Lemma 3.6]). *Let $f : [0,\infty) \to \mathbb{C}$. Then*

$$\left\| \frac{1}{2\cdot}\int_{|\cdot-t|}^{\cdot+t} f(\lambda)d\lambda \right\|_{L^\infty(0,\infty)} \leq M[A[f]](t),$$

where M is the Hardy-Littlewood-Maximal operator defined by

$$M[h](x) = \sup_{r>0} \frac{1}{2r}\int_{|x-y|<r} |h(y)|dy$$

for $h : \mathbb{R} \to \mathbb{C}$.

Corollary 3 ([12, Corollary 3.7]). *Let $f : \mathbb{R}^3 \to \mathbb{C}$ be radial. Then*

$$\|J[f]\|_{L^2(0,T;L^\infty(\mathbb{R}^3))} \leq C\|f\|_{L^2_{\mathrm{rad}}(\mathbb{R}^3)}.$$

Corollary 4 ([12, Corollary 3.8]). *Let $h : [0,\infty) \times \mathbb{R}^3 \to \mathbb{C}$ be radial. Then*

$$\left\| \int_0^t J[h(t')](t-t')dt' \right\|_{L^2(0,T;L^\infty(0,\infty))} \leq C\|h\|_{L^1(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^3))}.$$

Corollary 5 (Hardy, [12, Corollary 3.9]). *Let $f \in C^1([0,\infty);\mathbb{C})$. Then*

$$\left\| \frac{d}{dt}\left(\frac{1}{2r}\int_{|r-t|}^{r+t} \lambda f(\lambda)d\lambda \right) \right\|_{L^2(0,\infty;L^\infty(0,\infty))} \leq C\|rf'\|_{L^2(0,\infty)}.$$

Proof. Let g be even extension of f.

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2r}\int_{|r-t|}^{r+t}\lambda f(\lambda)d\lambda\right) &= \frac{(r+t)f(r+t)-(t-r)f(|r-t|)}{2r}\\ &= \frac{(r+t)f(r+t)-(t-r)g(t-r)}{2r}\\ &= \frac{1}{2r}\int_{-r}^{r}\{g(t+\tau)+(t+\tau)g'(t+\tau)\}d\tau.\end{aligned}$$

Then

$$\left\|\frac{d}{dt}\left(\frac{1}{2r}\int_{|r-t|}^{r+t}\lambda f(\lambda)d\lambda\right)\right\|_{L^2(0,\infty;L^\infty(0,\infty))} \le \|M[g]\|_{L^2(\mathbb{R})}+\|M[\cdot g']\|_{L^2(\mathbb{R})}. \tag{28}$$

Therefore, (28) and the following Hardy estimate([22, (0.2)]) imply Corollary 5:

$$\|g\|_{L^2(\mathbb{R})} \lesssim \|\cdot g'\|_{L^2(\mathbb{R})}.$$

We can now give the proof of Proposition 3.

Proof (Proof of Proposition 3). Let

$$X^1_{\mathrm{rad}}(0,T) = L^\infty(0,T;H^1_{\mathrm{rad}}(\mathbb{R}^3))\cap L^2(0,T;L^\infty_{\mathrm{rad}}(\mathbb{R}^3)).$$

For $0<T<1$ and $p=3$, By Corollaries 3, 4, 5, and the Hölder and Gagliardo-Nirenberg inequalities imply that, if initial data u_0 sufficiently small, then Φ maps $B_{X^1_{\mathrm{rad}}(0,T)}(R)$ into itself with some T and R. Since

$$\begin{aligned}&|F_3(u)-F_3(v)|\\ &= \big|i\left(D(|u|^2u)-2|u|^2Du-u^2D\overline{u}\right)+3|u|^4u\\ &-i\left(D(|v|^2v)-2|v|^2Dv-v^2D\overline{v}\right)-3|v|^4v\big|\\ &\lesssim |D(|u|^2u-|v|^2v)|+|u|^2|D(u-v)|\\ &+\left(\big||u|^2-|v|^2\big|+\big|u^2-v^2\big|\right)|Dv|+\big||u|^4u-|v|^4v\big|,\end{aligned}$$

we have

$$\begin{aligned}&\|F_3(u)-F_3(v)\|_{L^1(0,T;L^2_{\mathrm{rad}}(\mathbb{R}^3))}\\ &\lesssim (\|u\|_{X^1_{\mathrm{rad}}(0,T)}+\|v\|_{X^1_{\mathrm{rad}}(0,T)})^2\|u-v\|_{X^1_{\mathrm{rad}}(0,T)}\\ &+(\|u\|_{X^1_{\mathrm{rad}}(0,T)}+\|v\|_{X^1_{\mathrm{rad}}(0,T)})^4\|u-v\|_{X^1_{\mathrm{rad}}(0,T)}.\end{aligned}$$

This means Φ is a contraction map on $B_{X^1_{\mathrm{rad}}(0,T)}(R)$ for sufficiently small u_0.

3 Blow-up for (1)

At first, we recall the following ODE argument:

Lemma 7 ([11, Lemma 2.1]). *Let $C_1, C_2 > 0$ and $q > 1$. If $f \in C^1([0,T);\mathbb{R})$ satisfies $f(0) > 0$ and*

$$f' + C_1 f = C_2 f^q \quad \text{on } [0,T) \text{ for some } T > 0,$$

then

$$f(t) = e^{-C_1 t}\left(f(0)^{-(q-1)} + C_1^{-1}C_2 e^{-C_1(q-1)t} - C_1^{-1}C_2\right)^{-\frac{1}{q-1}}.$$

Moreover, if $f(0) > C_1^{\frac{1}{q-1}} C_2^{-\frac{1}{q-1}}$, then $T < -\frac{1}{C_1(q-1)}\log(1 - C_1 C_2^{-1} f(0)^{-q+1})$.

Next, we recall Calderón-Zygmund argument. We call K, a mesurable function on $\mathbb{R}^n$, Calderón-Zygmund (CZ) kernel if K satisfies

$$|K(x)| \le |x|^{-n}, \quad |\nabla K(x)| \le |x|^{-n+1}, \quad \int_{\varepsilon<|x|<R} K(x) = 0, \qquad 0 < \forall\varepsilon < \forall R.$$

Then CZ kernel is known to give a $L^p(\mathbb{R}^n)$ bounded operator as follows:

Lemma 8 ([3, Theorem 1]). *Let K be a CZ kernel. Then for $1 < p < \infty$, there exists a positive constant C such that*

$$\left\| \text{P.V.} \int_{\mathbb{R}^n} K(x-y) f(y) dy \right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$$

for any $f \in L^p(\mathbb{R}^n)$.

Now we are in position to show Proposition 5.

Proof. Thanks to Lemma 7, it is enough to show

$$\left\| \langle\cdot\rangle^{-q}[D, \langle\cdot\rangle^q] \right\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} < \infty.$$

At first, We divide the operator into the following two pieces:

$$\langle x\rangle^{-q}[(-\Delta)^{1/2}, \langle x\rangle^q] = CT_1 + CT_2,$$

where ψ is a cut-off function defined by (24).

$$T_1(f)(x) = \langle x\rangle^{-q}\int_{\mathbb{R}^n} \frac{(1-\psi(y))(\langle x\rangle^q - \langle x+y\rangle^q)}{|y|^{n+1}} f(x+y)dy,$$
$$T_2(f)(x) = \langle x\rangle^{-q}\, \text{P.V.}\int_{\mathbb{R}^n} \frac{\psi(y)(\langle x\rangle^q - \langle x+y\rangle^q)}{|y|^{n+1}} f(x+y)dy.$$

In order to estimate T_1 by dividing into two pieces:

$$T_1 = T_3 + T_4,$$

where

$$T_3(f)(x) = \langle x\rangle^{-q} \int_{|x|\le|y|} \frac{(1-\psi(y))(\langle x\rangle^q - \langle x+y\rangle^q)}{|y|^{n+1}} f(x+y)dy,$$
$$T_4(f)(x) = \langle x\rangle^{-q} \int_{|x|\ge|y|} \frac{(1-\psi(y))(\langle x\rangle^q - \langle x+y\rangle^q)}{|y|^{n+1}} f(x+y)dy.$$

By the Hölder and Young inequalities,

$$\begin{aligned}
&\|T_3(f)\|_{L^2(\mathbb{R}^n)}\\
&\le (1+2^q)\left\| \langle x\rangle^{-q} \int_{|x|\le|y|} \frac{\langle y\rangle^q(1-\psi(y))}{|y|^{n+1}} f(x+y)dy \right\|_{L^2(\mathbb{R}^n)}\\
&\le (1+2^q)\|\langle\cdot\rangle^{-q}\|_{L^2(\mathbb{R}^n)} \left\| \int_{\mathbb{R}^n} \frac{\langle y\rangle^q(1-\psi(y))}{|y|^{n+1}} f(x+y)dy \right\|_{L^\infty(\mathbb{R}^n)}\\
&\le (1+2^q)\|\langle\cdot\rangle^{-q}\|_{L^2(\mathbb{R}^n)} \|\langle\cdot\rangle^q|\cdot|^{-n-1}(1-\psi)\|_{L^2(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Similarly by the Young inequality,

$$\begin{aligned}
\|T_4(f)\|_{L^2(\mathbb{R}^n)} &\le (1+2^q)\left\| \int_{\mathbb{R}^n} \frac{1-\psi(y)}{|y|^{n+1}} |f(x+y)|dy \right\|_{L^2(\mathbb{R}^n)}\\
&\le (1+2^q)\||\cdot|^{-n-1}(1-\psi)\|_{L^1(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Next, in order to estimate T_2, we recall that

$$\begin{aligned}
\langle x+y\rangle^q &= \langle x\rangle^q + \frac{q}{2}\langle x\rangle^{q-2}(|x+y|^2 - |x|^2) + R_1(x,y),\\
&= \langle x\rangle^q + q\langle x\rangle^{q-2} x\cdot y + R_2(x,y),
\end{aligned} \tag{29}$$

where $R_2(x,y) = R_1(x,y) + q\langle x\rangle^{q-2}|y|^2/2$ and

$$R_1(x,y) = \frac{q(q-2)}{2^2} \int_{|x|^2}^{|x+y|^2} (1+\rho)^{q/2-2}(|x+y|^2-\rho)d\rho.$$

By combining (13) and (29), we have

$$T_2 = -qT_5 - T_6,$$

where

$$T_5(f)(x) = \frac{x}{\langle x\rangle^2} \cdot \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{y\psi(y)}{|y|^{n+1}} f(x+y)dy,$$
$$T_6(f)(x) = \frac{1}{\langle x\rangle^q} \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{R_2(x,y)\psi(y)}{|y|^{n+1}} f(x+y)dy.$$

It is easy to see that $K(y) = y|y|^{-n-1}\psi(y)$ is a CZ kernel. Therefore

$$\|T_5(f)\|_{L^2(\mathbb{R}^n)} \le \left\| \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{y\psi(y)}{|y|^{n+1}} f(\cdot + y)dy \right\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}.$$

Moreover, since

$$\begin{aligned} |y|^{-n-1}|R_1(x,y)| &\le (\langle x\rangle^{q-2} + \langle x+y\rangle^{q-2})(|x+y|^2 - |x|^2)^2|y|^{-n-1} \\ &\le (\langle x\rangle^{q-2} + \langle x+y\rangle^{q-2})(|x+y| + |x|)^2|y|^{-n+1}, \end{aligned}$$

by the Young inequality,

$$\|T_6(f)\|_{L^2(\mathbb{R}^n)} \le C \left\| \int_{\mathbb{R}^n} \frac{\psi(y)}{|y|^{n-1}} f(x+y)dy \right\|_{L^2(\mathbb{R}^n)} \le \||\cdot|^{-n+1}\psi\|_{L^1(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.$$

4 A priori estimates

This last section is devoted to the proofs of Propositions 7, 8, 9, 10, and 11. The proofs are essentially the same in [12], but we report them here for sake of completeness.

Proof (Proof of Proposition 7). The proposition follows from a standard argument, so we omit the proof.

Proof (Proof of Proposition 8). The proposition follows from a standard argument, so we omit the proof.

Proof (Proof of Proposition 9). Here we give a direct proof based on the integral equation by using the method in [27].

$$\begin{aligned} &\|u(t_2)\|_{\dot{H}^s(\mathbb{R}^n)}^2 \\ &= \|u(t_1)\|_{\dot{H}^s(\mathbb{R}^n)}^2 - 2\mathrm{Re}\int_{t_1}^{t_2} \langle D^s(|u(t)|^{p-1}u(t)), D^s u(t)\rangle_{L^2(\mathbb{R}^n)} dt \\ &\le \|u(t_1)\|_{\dot{H}^s(\mathbb{R}^n)}^2 + 2\int_{t_1}^{t_2} \|D^s(|u(t)|^{p-1}u(t))\|_{L^2(\mathbb{R}^n)} \|u(t)\|_{\dot{H}^s(\mathbb{R}^n)} dt \\ &\le \|u(t_1)\|_{\dot{H}^s(\mathbb{R}^n)}^2 + C\int_{t_1}^{t_2} \|u(t)\|_{L^\infty(\mathbb{R}^n)}^{p-1} \|u(t)\|_{\dot{H}^s(\mathbb{R}^n)}^2 dt, \end{aligned}$$

where we used the nonlinear estimate

$$\||f|^{p-1}f\|_{\dot{H}^s(\mathbb{R}^n)} \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}^{p-1}\|f\|_{\dot{H}^s(\mathbb{R}^n)}$$

(see [13, Lemma 3.4]).

Proof (Proof of Proposition 10). Since $|u|^2u \in C((0,T);H^2(\mathbb{R}^n))$, the following calculation is justified by the Plancherel identity:

$$
\begin{aligned}
&\|u(t_2)\|^2_{\dot{H}^2(\mathbb{R}^n)} \\
&= \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} - 2\mathrm{Re}\int_{t_1}^{t_2} \left\langle \Delta |u(t)|^2 u(t), \Delta u(t) \right\rangle_{L^2(\mathbb{R}^n)} dt \\
&= \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} - 2\mathrm{Re}\sum_{j,k=1}^{n}\int_{t_1}^{t_2} \left\langle |u(t)|^2 \partial_j\partial_k u(t), \partial_j\partial_k u(t) \right\rangle_{L^2(\mathbb{R}^n)} dt \\
&- 4\mathrm{Re}\sum_{j,k=1}^{n}\int_{t_1}^{t_2} \left\langle \partial_k u(t)\partial_j |u(t)|^2, \partial_j\partial_k u(t) \right\rangle_{L^2(\mathbb{R}^n)} dt \\
&- 2\mathrm{Re}\sum_{j,k=1}^{n}\int_{t_1}^{t_2} \left\langle \partial_j\partial_k |u(t)|^2, \overline{u(t)}\partial_j\partial_k u(t) \right\rangle_{L^2(\mathbb{R}^n)} dt
\end{aligned}
$$

$$
\begin{aligned}
&= \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} - 2\sum_{j,k=1}^{n}\int_{t_1}^{t_2} \|u(t)\partial_j\partial_k u(t)\|^2_{L^2(\mathbb{R}^n)} dt \\
&+ 2\sum_{j,k=1}^{n}\int_{t_1}^{t_2} \left\langle \partial_j^2 |u(t)|^2, |\partial_k u(t)|^2 \right\rangle_{L^2(\mathbb{R}^n)} dt \\
&- \sum_{j,k=1}^{n}\int_{t_1}^{t_2} \left\langle \partial_j\partial_k |u(t)|^2, \partial_j\partial_k |u(t)|^2 - 2\mathrm{Re}(\overline{\partial_j u(t)}\partial_k u(t)) \right\rangle_{L^2(\mathbb{R}^n)} dt.
\end{aligned}
$$

By the Hölder,Young, and Sobolev inequalities,

$$
\begin{aligned}
&\|u(t_2)\|^2_{\dot{H}^2(\mathbb{R}^n)} \\
&\le \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} - 2\sum_{j,k=1}^{n}\int_{t_1}^{t_2} \|u(t)\partial_j\partial_k u(t)\|^2_{L^2(\mathbb{R}^n)} dt \\
&+ 2n^2 \sum_{k=1}^{n}\int_{t_1}^{t_2} \|\partial_k u(t)\|^4_{L^4(\mathbb{R}^n)} dt + 2\sum_{j,k=1}^{n}\int_{t_1}^{t_2} \|\partial_j u(t)\|^2_{L^4(\mathbb{R}^n)} \|\partial_k u(t)\|^2_{L^4(\mathbb{R}^n)} dt \\
&\le \|u(t_1)\|^2_{\dot{H}^2(\mathbb{R}^n)} - 2\sum_{j,k=1}^{n}\int_{t_1}^{t_2} \|u(t)\partial_j\partial_k u(t)\|^2_{L^2(\mathbb{R}^n)} dt \\
&+ 2n^2(n+1)\int_{t_1}^{t_2} \|u(t)\|^{4-n}_{\dot{H}^1(\mathbb{R}^n)} \|u(t)\|^{n}_{\dot{H}^2(\mathbb{R}^n)} dt.
\end{aligned}
$$

We can now conclude the paper by showing Proposition 11.

Proof (Proof of Proposition 11). When $s = 1$ and when $s = 2$ and $p = 3$, a priori estimates shows the global well-posedness by the blow-up alternative argument. Here we consider the case where $p = 3$ and $1 < s < 2$. Let $[a]$ be the floor function of a. Let $T_1 = \min\{1, T_0\}$. By using the H^1 a priori estimate, for any $t > 0$,

$$\begin{aligned}\|u\|_{L^4(0,t;L^\infty(\mathbb{R}^2))} &\le \sum_{k=0}^{[t/T_1]+1} \|u\|_{L^4(kT_1,(k+1)T_1;L^\infty(\mathbb{R}^2))} \\ &\le \sum_{k=0}^{[t/T_1]+1} \|u\|_{X^1(kT_1,(k+1)T_1)} \\ &\le 2T_1^{-1}(1+t)\|u_0\|_{H^1(\mathbb{R}^2)}.\end{aligned}$$

Then by using Proposition 10,

$$\begin{aligned}\|u(t)\|^2_{\dot{H}^s(\mathbb{R}^2)} &\lesssim \|u_0\|^2_{H^s(\mathbb{R}^2)} + \int_0^t \|u(t')\|^2_{L^\infty(\mathbb{R}^n)} \|u(t')\|^2_{\dot{H}^s(\mathbb{R}^2)} dt \\ &\lesssim \|u_0\|^2_{H^s(\mathbb{R}^2)} + \|u(t')\|^2_{L^4(0,t;L^\infty(\mathbb{R}^2))} \|u\|^2_{L^4(0,t;\dot{H}^s(\mathbb{R}^2))} \\ &\lesssim \|u_0\|^2_{H^s(\mathbb{R}^2)} + \|u_0\|^2_{H^1(\mathbb{R}^2)}(1+t)^2 \|u\|^2_{L^4(0,t;\dot{H}^s(\mathbb{R}^2))}.\end{aligned}$$

This shows

$$\|u(t)\|^4_{\dot{H}^s(\mathbb{R}^2)} \lesssim \|u_0\|^4_{H^s(\mathbb{R}^2)} + \|u_0\|^4_{H^1(\mathbb{R}^2)}(1+t)^4 \|u\|^4_{L^4(0,t;\dot{H}^s(\mathbb{R}^2))}.$$

Therefore Gronwall inequality imply the global well-posedness in $H^s(\mathbb{R}^2)$.

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