

MASS-SUBCRITICAL HALF-WAVE EQUATION WITH MIXED NONLINEARITIES: EXISTENCE AND NON-EXISTENCE OF GROUND STATES

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ABSTRACT. We consider the problem of existence of constrained minimizers for the focusing mass-subcritical Half-Wave equation with a defocusing mass-subcritical perturbation. We show the existence of a critical mass such that minimizers do exist for any mass larger than or equal to the critical one, and do not exist below it. At the dynamical level, in the one dimensional case, we show that the ground states are orbitally stable.

1. INTRODUCTION

In this paper, we are motivated by the following Half-Wave type equation

$$i\psi_t = \sqrt{-\Delta}\psi + |\psi|^{q-1}\psi - |\psi|^{p-1}\psi, \quad (1.1)$$

where $\psi(t, x) : [0, T) \times \mathbb{R}^d \rightarrow \mathbb{C}$, with $d \in \mathbb{N}$ and $T > 0$, is a complex wave function and the exponents q, p are of mass subcritical type, meaning that $1 < q < p < 1 + \frac{2}{d}$. The operator $\sqrt{-\Delta}$ in (1.1) is defined by the Fourier multiplier $m(\xi) = |\xi|$, $\xi \in \mathbb{R}^d$, and it acts as $\sqrt{-\Delta}f = \mathcal{F}^{-1}(|\xi|\mathcal{F}(f))$, where $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transform and its inverse, respectively.

In the 1D case, given an initial datum $\psi_0 \in H^{1/2}(\mathbb{R})$ associated to (1.1), it can be proved in the same manner as in [16] (see also Section 3 for details) that the Cauchy problem is (globally) well-posed and satisfies the conservation of mass, energy, and momentum, defined by

$$\begin{aligned} \mathcal{M}(\psi(t)) &= \int_{\mathbb{R}} |\psi(t, x)|^2 dx \\ \mathcal{E}(\psi(t)) &= \frac{1}{2} \int_{\mathbb{R}} \bar{\psi}(t, x) \sqrt{-\partial_x^2} \psi(t, x) dx \\ &\quad + \frac{1}{q+1} \int_{\mathbb{R}} |\psi(t, x)|^{q+1} dx - \frac{1}{p+1} \int_{\mathbb{R}} |\psi(t, x)|^{p+1} dx, \end{aligned}$$

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and

$$\mathcal{P}(\psi(t)) = - \int_{\mathbb{R}} i\bar{\psi}(t, x) \partial_x \psi(t, x) dx,$$

respectively. Particular solutions to (1.1) are the so-called *traveling wave solutions*, which are solutions to (1.1) of the form

$$\psi(t, x) = e^{i\omega t} u(x - vt) \quad (1.2)$$

with $\omega \in \mathbb{R}$, $v \in \mathbb{R}$ such that $|v| < 1$, and $u(x) \in \mathbb{C}$ is a time-independent function belonging to $H^{1/2}(\mathbb{R})$ which satisfies

$$\sqrt{-\partial_x^2} u + iv \partial_x u + \omega u + |u|^{q-1} u - |u|^{p-1} u = 0. \quad (1.3)$$

From a time-independent point of view, we can actually consider (1.3) in arbitrary dimension, namely we consider solutions $u(x) \in \mathbb{C}$, $x \in \mathbb{R}^d$, to

$$\sqrt{-\Delta} u + iv \cdot \nabla u + \omega u + |u|^{q-1} u - |u|^{p-1} u = 0, \quad (1.4)$$

where $\omega \in \mathbb{R}$ and $v \in \mathbb{R}^d$ with $|v| < 1$. Motivated by the 1D case, since in that setting the mass is a physical quantity which is preserved along the flow of (1.1), a natural way to find solutions u to (1.4) is to look for critical points of the functional

$$E_v(u) = \frac{1}{2} T_v(u) + \frac{1}{q+1} \int_{\mathbb{R}^d} |u|^{q+1} dx - \frac{1}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} dx$$

where

$$T_v(u) = \int_{\mathbb{R}^d} \bar{u}(x) (\sqrt{-\Delta} + iv \cdot \nabla) u(x) dx,$$

constrained on the L^2 -spheres of $H^{1/2}(\mathbb{R}^d)$ described by

$$S_\rho = \{u \in H^{1/2}(\mathbb{R}^d) \text{ s.t. } \|u\|_{L^2} = \rho\}.$$

So, by a solution of (1.4) we mean a couple $(\omega_\rho, u_\rho) \in \mathbb{R} \times H^{1/2}(\mathbb{R}^d)$ where ω_ρ appears as the Lagrange multiplier associated to the critical point u_ρ on S_ρ . Note that the functional $E_v(\psi)$ of the time-dependent function ψ defined in (1.2) is preserved along the flow on (1.1) by the fact that

$$E_v(\psi) = \mathcal{E}(\psi) - \frac{v}{2} \mathcal{P}(\psi).$$

For a fixed a mass $\rho > 0$, we introduce the *ground state energy* as the quantity I_{ρ^2} , defined as

$$I_{\rho^2} = \inf_{S_\rho} E_v(u). \quad (1.5)$$

If a minimizer u_ρ to (1.5) exists, we call it *ground state solution*. Another functional which will play a relevant role for our approach is the following Pohozaev functional

$$G_v(u) = T_v(u) + \frac{d(q-1)}{q+1} \int_{\mathbb{R}^d} |u|^{q+1} dx - \frac{d(p-1)}{p+1} \int_{\mathbb{R}^d} |u|^{p+1} dx.$$

It is well-known that if u_ρ is a solution to (1.4) then $u_\rho \in V_\rho$ where

$$V_\rho = \{u \in H^{1/2}(\mathbb{R}^d) \text{ s.t. } \|u\|_{L^2(\mathbb{R}^d)} = \rho, G_v(u) = 0\}.$$

The goal of our paper is to establish a result about the existence and the non-existence of solutions to the minimization problem (1.5). Specifically, for any fixed $v \in \mathbb{R}^d$ with $|v| < 1$, the Theorem below shows the existence of a critical mass ρ_0^v such that ground states exist for any mass $\rho \geq \rho_0^v$, and do not exist if $\rho < \rho_0^v$.

Our main result is as follows.

Theorem 1.1. *Let $d \in \mathbb{N}$, $v \in \mathbb{R}^d$ with $|v| < 1$, and $1 < q < p < 1 + \frac{2}{d}$, then there exists a strictly positive mass ρ_0^v such that:*

- (i) $I_{\rho^2} = 0$ for all $\rho \in (0, \rho_0^v]$;
- (ii) $I_{\rho^2} < 0$ for all $\rho \in (\rho_0^v, \infty)$.

Moreover, there are no constrained minimizers for $0 < \rho < \rho_0^v$, and for all $\rho \in [\rho_0^v, \infty)$ there exists $u_\rho \in S_\rho$ such that $I_{\rho^2} = E(u_\rho)$.

We give some comments about Theorem 1.1.

Remark 1.2. (i) The strategy to prove Theorem 1.1 follows a general scaling argument inspired by [5], and, for $v = 0$, our approach remains valid for any fractional NLS equation with mass-subcritical nonlinearities of power-type as appearing in (1.1).

(ii) We also mention the paper by Jeanjean and Luo in [14] for the Schrödinger-Poisson equation, which was of inspiration for our paper concerning the non-existence results. We recall that in the Schrödinger-Poisson equation

$$-\Delta u + \omega u + (|x|^{-1} * |u|^2)u - |u|^{p-1}u = 0,$$

the nonlocality is in the nonlinearity. On the other hand, the nonlocal nature of equation (1.1) is due to the linear operator defining the kinetic energy. In [14], the analysis is specifically tailored to treat the nonlocal Coulomb-type term, while in our paper we are concerned with local-type nonlinearities in the whole mass-subcritical regime, and our achievements extend to any fractional Laplace operator.

Remark 1.3. We emphasize that when $\rho = \rho_0^v$ the weak subadditivity inequality $I_{(\rho_0^v)^2} \leq I_{\mu^2} + I_{(\rho_0^v)^2 - \mu^2}$ that always holds for this class of translation invariant minimization problem, becomes an equality by the fact that for $0 < \mu < \rho_0^v$ we have $I_{\mu^2} = I_{(\rho_0^v)^2} = 0$. This fact shows that in general the strong subadditivity inequality is only a sufficient condition for the existence of constrained minimizers.

Remark 1.4. (i) After the seminal works by Tao, Viřan, and Zhang [21] on the classical NLS equation with combined nonlinearities, the interest on this type of equation rapidly increased. In particular, after the papers by Soave [18, 19], several scholars treated the problem of the existence of ground states with fixed mass for

NLS-type equations with mixed nonlinearities. We specifically mention the paper by the authors [2], and by Jeanjean and Lu [12], that completed the study on the different scenarios initiated by Soave. We also cite the paper by Jeanjean and Lu [13] that treated general nonlinearities under suitable growth conditions.

We remark that following verbatim the approach in our paper for the classical NLS equation with two competing nonlinearities, we can recover the result of [13]. It is worth mentioning, however, that the authors of [13] consider general mass-subcritical nonlinearities and not only power-like terms.

(ii) Concerning the Half-Wave equation with two competing nonlinearities, we should mention the paper [20], where the authors study the problem of existence and non-existence of traveling waves under the mass constraint. The defocusing-focusing case in the mass-subcritical case (i.e., under the same assumption of Theorem 1.1) is considered in [20, Theorem 1.2]. The authors of that paper claim that ground states cannot exist with sufficiently small mass, as well as they claim the non-existence of a ground state with zero energy. However, the proof of [20, Theorem 1.2] contains a crucial mistake in the sign of some estimates when treating the defocusing term. Furthermore, we actually prove in Theorem 1.1 that a ground state with zero energy do exists, and its corresponding mass is exactly the threshold for the existence of ground states.

(iii) As for the existence of ground states for (1.1) in the mass supercritical regime, we refer to [22].

Remark 1.5. Concerning the existence of ground states, the case with only one focusing term is quite easy, due to the scaling invariance of the equation. On the other hand, the problem becomes very interesting and more difficult from the perspective of the uniqueness and the non-degeneracy of the ground state when $v = 0$, see [1, 9, 10], and also regarding the lack of small data scattering on the dynamical side when $v \neq 0$, see [4].

Our second result concerns the dynamics of the ground states, and specifically it is the following orbital stability property of standing and traveling waves in 1D.

Corollary 1.6. *Fix $d = 1$, $v \in \mathbb{R}$ with $|v| < 1$, and let ρ_0^v be as given in Theorem 1.1. For any $\rho > \rho_0^v$, the set*

$$\mathcal{G} = \{e^{i\gamma}u(\cdot + y) \text{ s.t. } \gamma \in \mathbb{R}, y \in \mathbb{R}, \text{ and } u \in S_\rho \text{ with } I_{\rho^2} = E_v(u)\}$$

is orbitally stable.

Remark 1.7. It is worth noticing that at a dynamical level we can only consider the 1D framework, as is the only case where we can state a local well-posedness theory of the Cauchy problem associated to (1.1).

1.1. Notations. We will work in the whole domain $\mathbb{R}^d$, hence we will omit the dependence on it, and systematically write norms, integrals, etc. without mention of the space we are working on. We will use the short notation $\|f\|_p$ for the L^p -norm of a function f . H^s , for $s \in (0, 1)$, is the standard Sobolev space endowed with norm $\|f\|_{H^s} = \|(1 - \Delta)^{s/2} f\|_2$. $\Re z$, $\Im z$, and $\bar{z}$ are the real part, the imaginary part, and the complex conjugate of z , respectively. If there is no confusion, we suppress the subscript from the functionals E_v, G_v, T_v and the superscript from ρ_0^v from now on.

2. PROOF OF THE MAIN RESULT

As described in the Introduction, the aim is to study the existence of minimizers for

$$E(u) = \frac{1}{2}T(u) + \frac{1}{q+1} \int |u|^{q+1} dx - \frac{1}{p+1} \int |u|^{p+1} dx$$

where

$$T(u) = \int \bar{u}(x)(\sqrt{-\Delta} + iv \cdot \nabla)u(x)dx,$$

for $|v| < 1$, under mass constraint

$$S_\rho = \{u \in H^{1/2} \text{ s.t. } \|u\|_2 = \rho\},$$

for $1 < q < p < 1 + \frac{2}{d}$. Then, the problem is to compute

$$I_{\rho^2} = \inf_{S_\rho} E(u),$$

which is equivalent to

$$I_{\rho^2} = \inf_{V_\rho} E(u),$$

where $V_\rho = \{u \in H^1 \text{ s.t. } \|u\|_2 = \rho, G(u) = 0\}$.

Let us recall that the equation with a single nonlinearity

$$i\psi_t = \sqrt{-\Delta} \pm |\psi|^{p-1}\psi, \tag{2.1}$$

is invariant under the scaling $\psi \mapsto \psi_\lambda = \lambda^{\frac{1}{p-1}}\psi(\lambda t, \lambda x)$, and the L^2 -norm of ψ_λ is left invariant provided that $p = 1 + \frac{2}{d}$. Hence, we say that (2.1) (and also (1.1)) is mass-subcritical.

2.1. Preliminary results. In this subsection we collect some essential tools which will lead to the proof of the main Theorem 1.1.

Let us introduce a general scaling $u \mapsto u_\lambda = \lambda^\alpha u(\lambda x)$ often used along the paper. The quadratic part of the energy rescales as

$$T(u_\lambda) = \lambda^{2\alpha+1-d}T(u),$$

and a general L^{r+1} -norm rescales as

$$\|u_\lambda\|_{r+1} = \lambda^{\alpha-d/(r+1)} \|u\|_{r+1}.$$

Note that by the Plancherel identity, $T(u) = \int (|\xi| - v \cdot \xi) |\mathcal{F}(u)|^2 d\xi$, then the condition $|v| < 1$ ensures that $T(u) > 0$ and that $T(u) \sim \|u\|_{\dot{H}^{1/2}}^2$. We moreover recall the following Gagliardo-Nirenberg-type inequality: for $1 < r < 1 + \frac{2}{(d-1)_+}$,

$$\|u\|_{r+1}^{r+1} \leq C(d, v, r) T(u)^{d(r-1)/2} M(u)^{(r+1)/2-d(r-1)/2},$$

for any $u \in H^{1/2}$.

The first result is the following key Lemma which will play a key role for the rest of the paper.

Lemma 2.1. *Fix $d \in \mathbb{N}$ and $v \in \mathbb{R}^d$ with $|v| < 1$. Let*

$$A_\delta = \{u \in H^{1/2} \text{ s.t. } E(u) \leq 0, G(u) = 0, T(u) \leq \delta\}.$$

If δ is sufficiently small then $A_\delta = \emptyset$.

Proof. The first observation is that in A_δ , the quadratic term $T(u)$ has a size comparable with the focusing term $\|u\|_{p+1}^{p+1}$. Indeed we have

$$0 \geq E(u) - \frac{1}{d(q-1)} G(u) = \frac{d(q-1)-2}{2d(q-1)} T(u) + \frac{p-q}{(p+1)(q-1)} \|u\|_{p+1}^{p+1}$$

which shows that

$$T(u) \geq \frac{2d(p-q)}{(2-d(q-1))(p+1)} \|u\|_{p+1}^{p+1}.$$

On the other hand, the non positivity of the energy implies that

$$\|u\|_{p+1}^{p+1} \geq T(u), \tag{2.2}$$

therefore

$$\|u\|_{p+1}^{p+1} \sim T(u). \tag{2.3}$$

Now, by computing $E(u) - \frac{1}{2}G(u)$ we get

$$0 \geq E(u) - \frac{1}{2}G(u) = \frac{2-d(q-1)}{2(q+1)} \|u\|_{q+1}^{q+1} - \frac{2-d(p-1)}{2(p+1)} \|u\|_{p+1}^{p+1}.$$

The last inequality guarantees that

$$\|u\|_{q+1}^{q+1} \lesssim \|u\|_{p+1}^{p+1}. \tag{2.4}$$

At this point, by the Gagliardo-Nirenberg interpolation inequality jointly with the norm equivalence $T(u) \sim \|u\|_{\dot{H}^{1/2}}^2$, and (2.4), we have

$$\begin{aligned} \|u\|_{p+1}^{p+1} &\leq \|u\|_{q+1}^{(1-\theta)(p+1)} \|u\|_{\dot{H}^{1/2}}^{\theta(p+1)} \\ &\lesssim \|u\|_{q+1}^{(1-\theta)(p+1)} T(u)^{\theta(p+1)/2} \\ &\lesssim \|u\|_{p+1}^{(1-\theta)(p+1)^2/(q+1)} T(u)^{\theta(p+1)/2} \end{aligned} \quad (2.5)$$

with $\theta = \frac{2d(p-q)}{(p+1)(2d-(d-1)(q+1))}$. Combining (2.2), (2.5), and (2.3), we get

$$T(u) \lesssim T(u)^{(1-\theta)(p+1)/(q+1)} T(u)^{\theta(p+1)/2}.$$

Then, noticing that $1 < \frac{(1-\theta)(p+1)}{q+1} + \frac{\theta(p+1)}{2}$ is always verified, as it is equivalent to $2(q-p) < \theta(p+1)(q-1)$, we have that $T(u)$ cannot be too small, namely $A_\delta = \emptyset$ provided that $\delta \ll 1$. $\square$

Consider now the energy functional without the defocusing term, i.e., let us introduce

$$\tilde{E}(u) := \frac{1}{2}T(u) - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$

and define

$$J_{\rho^2} = \inf_{u \in S_\rho} \tilde{E}(u).$$

Note that when $v = 0$, the functional $\tilde{E}$ corresponds to the energy functional related to the equation

$$i\psi_t = \sqrt{-\Delta}\psi - |\psi|^{p-1}\psi,$$

and the latter is invariant under the scaling $u \mapsto u_\lambda = \lambda^\alpha u(\lambda x)$ with $\alpha = \frac{1}{p-1}$. By this simple observation, we get the following.

Proposition 2.2. $J_{\rho^2} = \rho^{\frac{4-2(d-1)(p-1)}{2-d(p-1)}} J_1$ and $J_1 < 0$.

Proof. The proof follows by a scaling argument. Let us take $u \in S_1$ and define $u_\lambda = \lambda^{\frac{1}{p-1}} u(\lambda x)$. Straightforward computations give

$$\|u_\lambda\|_2^2 = \lambda^{\frac{2}{p-1}-d} \|u\|_2^2 = \lambda^{\frac{2}{p-1}-d},$$

thus, by imposing $\lambda = \lambda(\rho) := \rho^{\frac{2(p-1)}{2-d(p-1)}}$, we obtain that $u_\lambda \in S_\rho$. On the other hand, by the previous observation, $\tilde{E}(u_{\lambda(\rho)}) = \lambda(\rho)^{\frac{2}{p-1}-d+1} \tilde{E}(u) = \rho^{\frac{4-2(d-1)(p-1)}{2-d(p-1)}} \tilde{E}(u)$. The scaling map between S_1 and S_ρ is a bijection, hence $J_{\rho^2} = \rho^{\frac{4-2(d-1)(p-1)}{2-d(p-1)}} J_1$.

Now we prove that $J_1 < 0$. Let us consider the mass-preserving scaling $u_\lambda = \lambda^{\frac{d}{2}} u(\lambda x)$ so that $u_\lambda \in S_1$ and

$$\tilde{E}(u_\lambda) = \frac{\lambda}{2} T(u) - \frac{\lambda^{\frac{d(p-1)}{2}}}{p+1} \|u\|_{p+1}^{p+1}.$$

Note that $1 > \frac{d(p-1)}{2} > 0$ if and only if $1 < p < 1 + \frac{2}{d}$ and that $J_1 \leq \tilde{E}(u_\lambda)$. The claim follows if we choose λ sufficiently small, as $\tilde{E} < 0$ for $\lambda \ll 1$. $\square$

We recall the following well-known facts, that always work for translation invariant minimization problems.

Lemma 2.3. *The function $\rho \rightarrow I_{\rho^2}$ is continuous. Moreover:*

(i) *the ground state energy is weakly subadditive, namely*

$$I_{\rho^2} \leq I_{\mu^2} + I_{\rho^2 - \mu^2} \quad \text{for all } 0 < \mu < \rho;$$

(ii) *if the ground state energy is strongly subadditive, namely*

$$I_{\rho^2} < I_{\mu^2} + I_{\rho^2 - \mu^2} \quad \text{for all } 0 < \mu < \rho$$

then the infimum is achieved.

Proof. For the proof we refer to the classical reference [17]. $\square$

Proposition 2.4. $J_{\rho^2} \leq I_{\rho^2} \leq 0$ and $\lim_{\rho \rightarrow 0} \frac{I_{\rho^2}}{\rho^\alpha} = 0$ for all $\alpha \in [2, \frac{4-2(d-1)(p-1)}{2-d(p-1)})$.

Proof. The fact that $J_{\rho^2} \leq I_{\rho^2}$ follows from the positivity of the defocusing term. The non-positivity of I_{ρ^2} again follows by a scaling argument. Indeed, by taking $u \in S_\rho$ and the mass-preserving scaling $u_\lambda = \lambda^{\frac{d}{2}} u(\lambda x)$ such that $u_\lambda \in S_\rho$ for all $\lambda > 0$, we have

$$E(u_\lambda) = \frac{\lambda}{2} T(u) + \frac{\lambda^{\frac{d(q-1)}{2}}}{q+1} \|u\|_{q+1}^{q+1} - \frac{\lambda^{\frac{d(p-1)}{2}}}{p+1} \|u\|_{p+1}^{p+1}.$$

For $\lambda \rightarrow 0$ we have that $E_v(u_\lambda) \rightarrow 0$, and hence $I_{\rho^2} \leq 0$. The fact that $\lim_{\rho \rightarrow 0} \frac{I_{\rho^2}}{\rho^\alpha} = 0$ for all $\alpha \in [2, \frac{4-2(d-1)(p-1)}{2-d(p-1)})$ follows from Proposition 2.2 and the fact that $I_{\rho^2} \leq 0$. $\square$

We now give a sufficient condition ensuring the strong subadditivity of the function I_ρ .

Lemma 2.5. *Let $\alpha \in [2, \infty)$ and $I_{\rho^2} < 0$. If the function $\rho \rightarrow \frac{I_{\rho^2}}{\rho^\alpha}$ is strictly decreasing, then $I_{\rho^2} < I_{\mu^2} + I_{\rho^2 - \mu^2}$ for all $\mu \in (0, \rho)$.*

Proof. From the inequalities $\frac{\mu^\alpha}{\rho^\alpha} I_{\rho^2} < I_{\mu^2}$ and $\frac{(\rho^2 - \mu^2)^{\frac{\alpha}{2}}}{\rho^\alpha} I_{\rho^2} < I_{\rho^2 - \mu^2}$ we get, by adding term by term,

$$\frac{\mu^\alpha}{\rho^\alpha} I_{\rho^2} + \frac{(\rho^2 - \mu^2)^{\frac{\alpha}{2}}}{\rho^\alpha} I_{\rho^2} < I_{\mu^2} + I_{\rho^2 - \mu^2}.$$

As I_{ρ^2} is non-positive by the Proposition 2.4, to conclude it suffices to have $\mu^\alpha + (\rho^2 - \mu^2)^{\frac{\alpha}{2}} < \rho^\alpha$. Note that

$$\mu^\alpha + (\rho^2 - \mu^2)^{\frac{\alpha}{2}} = (\mu^2)^{\frac{\alpha}{2}} + (\rho^2 - \mu^2)^{\frac{\alpha}{2}} < (\rho^2)^{\frac{\alpha}{2}} = \rho^\alpha$$

by the convexity inequality $x^a + y^a < (x + y)^a$ if $a \geq 1$. $\square$

The next Lemma shows that a strict negativity of the ground state energy is enough to show the existence of constrained minimizers.

Lemma 2.6. *If $I_{\rho^2} < 0$ then exists $u \in S_\rho$ such that $E_v(u) = I_{\rho^2}$.*

Proof. It suffices from Lemma 2.5 that there exists $\alpha \in [2, \infty)$ such that for any $s \in (0, \rho)$, $\frac{I_{s^2}}{s^\alpha} > \frac{I_{\rho^2}}{\rho^\alpha}$. Indeed, the monotonicity of $\frac{I_{s^2}}{s^\alpha}$ implies strong subadditivity by Lemma 2.5, and the latter implies compactness, see Lemma 2.3. See also [5].

Now let us restrict α to the interval $[2, \frac{4-2(d-1)(q-1)}{2-d(q-1)})$, and let us define the quantity

$$Q_\alpha = \inf_{s \in (0, \rho]} \frac{I_{s^2}}{s^\alpha}.$$

From the fact that $I_{\rho^2} < 0$, we have that $Q_\alpha < 0$. Moreover, Proposition 2.4 yields

$$\rho_\alpha^\star = \{\inf s \in (0, \rho] \text{ s.t. } \frac{I_{s^2}}{s^\alpha} = Q_\alpha\} > 0.$$

Clearly, if $\rho_\alpha^\star = \rho$, then the strong subadditivity holds and a minimizer exists. Therefore, let us assume that $\rho_\alpha^\star < \rho$. In the latter case, we have by definition that for any $\mu \in (0, \rho_\alpha^\star)$

$$\frac{\mu^\alpha}{(\rho_\alpha^\star)^\alpha} I_{(\rho_\alpha^\star)^2} < I_{\mu^2}$$

and

$$\frac{((\rho_\alpha^\star)^2 - \mu^2)^{\frac{\alpha}{2}}}{\rho^\alpha} I_{(\rho_\alpha^\star)^2} < I_{(\rho_\alpha^\star)^2 - \mu^2}.$$

Hence, by subadditivity, there exists $u_\alpha \in S(\rho_\alpha^\star)$ with $E(u_\alpha) = I_{(\rho_\alpha^\star)^2}$ and such that for $\theta \in (1 - \epsilon, 1 + \epsilon)$, for some small $\epsilon > 0$,

$$\frac{E(u_\alpha)}{(\rho_\alpha^\star)^\alpha} = \frac{I_{(\rho_\alpha^\star)^2}}{(\rho_\alpha^\star)^\alpha} \leq \frac{I_{\theta^2(\rho_\alpha^\star)^2}}{\theta^\alpha(\rho_\alpha^\star)^\alpha} \leq \frac{E(\theta u_\alpha)}{\theta^\alpha(\rho_\alpha^\star)^\alpha}.$$

Therefore we have

$$\frac{d}{d\theta} (\theta^\alpha E(u_\alpha) - E(\theta u_\alpha)) \big|_{\theta=1} = 0. \quad (2.6)$$

For a minimizer u_α of I_{ρ^2} we have that

$$I_{\rho^2} = E(u_\alpha) = \frac{1}{2}T(u_\alpha) + \frac{1}{q+1}\|u_\alpha\|_{q+1}^{q+1} - \frac{1}{p+1}\|u_\alpha\|_{p+1}^{p+1} \quad (2.7)$$

and

$$T(u_\alpha) + \frac{d(q-1)}{q+1}\|u_\alpha\|_{q+1}^{q+1} - \frac{d(p-1)}{p+1}\|u_\alpha\|_{p+1}^{p+1} = 0. \quad (2.8)$$

Observe from (2.8) that

$$\|u_\alpha\|_{q+1}^{q+1} = \frac{q+1}{d(q-1)} \left(\frac{d(p-1)}{p+1}\|u_\alpha\|_{p+1}^{p+1} - T(u_\alpha) \right), \quad (2.9)$$

and hence from (2.7) we get the identities

$$I_{\rho^2} = E(u_\alpha) = \frac{d(q-1)-2}{2d(q-1)}T(u_\alpha) + \frac{p-q}{(p+1)(q-1)}\|u_\alpha\|_{p+1}^{p+1} \quad (2.10)$$

and

$$T(u_\alpha) + \|u_\alpha\|_{q+1}^{q+1} - \|u_\alpha\|_{p+1}^{p+1} = \frac{d(q-1)-q-1}{d(q-1)}T(u_\alpha) + \frac{2(p-q)}{(p+1)(q-1)}\|u_\alpha\|_{p+1}^{p+1}. \quad (2.11)$$

From (2.6) we obtain

$$\alpha E(u_\alpha) - (T(u_\alpha) + \|u_\alpha\|_{q+1}^{q+1} - \|u_\alpha\|_{p+1}^{p+1}) = 0,$$

which gives, by using (2.9) and (2.11),

$$T(u_\alpha) = \frac{1}{d(\alpha-2)(q-1)-2q-2-2\alpha} \frac{2d(2-\alpha)(p-q)}{p+1} \|u_\alpha\|_{p+1}^{p+1}. \quad (2.12)$$

Note that if $\alpha \in [2, \frac{4-2(d-1)(q-1)}{2-d(q-1)})$, then $d(\alpha-2)(q-1)-2q-2-2\alpha > 0$, while $(2-\alpha)(p-q)$ is always non-positive for $\alpha \geq 2$. Hence, for any $\alpha \in [2, \frac{4-2(d-1)(q-1)}{2-d(q-1)})$ we get an absurd from (2.12) as we would have that $T(u_\alpha) \leq 0$. In the end, $\rho_\alpha^* = \rho$ and hence the strong subadditivity property holds and a minimizer exists. $\square$

In the next Lemma, we show that for a non-empty right neighborhood of $\rho = 0$, the infimum of the energy is zero.

Lemma 2.7. *There exists a strictly positive mass ρ_0 such that:*

- (i) $I_{\rho^2} = 0$ for all $\rho \in (0, \rho_0]$;
- (ii) $I_{\rho^2} < 0$ for all $\rho \in (\rho_0, \infty)$.

Proof. The fact that $I_{\rho^2} \leq 0$, see Proposition 2.4, together with the weak subadditivity implies that if $I_{\rho^2} < 0$ then $I_{s^2} < 0$ all $s > \rho$. The negativity of $I_{\rho^2} < 0$ for sufficiently large ρ follows by a scaling argument. Indeed, let us choose

$$u_\lambda = \lambda^{\frac{1}{q-1}} u(\lambda x);$$

we have

$$E(u_\lambda) = \lambda^{\frac{2}{q-1}-d+1} \left(\frac{1}{2} T(u) + \frac{1}{q+1} \|u\|_{q+1}^{q+1} \right) - \frac{\lambda^{\frac{p+1}{q-1}-d}}{p+1} \|u\|_{p+1}^{p+1}$$

and then $E(u_\lambda) < 0$ by choosing a sufficiently large λ , as $\frac{p+1}{q-1} - d > \frac{2}{q-1} - d + 1$ if and only if $p > q$. On the other hand, $\|u_\lambda\|_2 = \lambda^{\frac{2-d(q-1)}{2(q-1)}} \|u\|_2$ and hence a large λ corresponds to a large mass ρ .

Now we prove (i), i.e., that there exists $\rho_0 > 0$ such that $I_{\rho^2} = 0$ for all $\rho \in (0, \rho_0]$. Note that from the weak subadditivity property, together with $I_{\rho^2} \leq 0$, the function I_{ρ^2} is non-increasing. By defining

$$\rho_0 = \sup\{\rho \text{ s.t. } I_{s^2} = 0 \text{ for all } s \in (0, \rho)\}$$

we have $I_{\rho^2} < 0$ for all $\rho \in (\rho_0, \infty)$. Now we prove that $\rho_0 > 0$.

The idea is to show that I_{ρ^2} cannot be attained in S_ρ if ρ is sufficiently small. As a byproduct we will have that

$$\rho_0 = \sup\{\rho \text{ s.t. } I_{s^2} = 0 \text{ for all } s \in (0, \rho]\} \quad (2.13)$$

is strictly positive, because the negativity of I_{ρ^2} implies existence of minimizers thanks to Lemma 2.6.

Therefore, let us assume that there exists a sequence $\rho_n \rightarrow 0$ such that $I_{\rho_n^2}$ is attained by a ground states u_{ρ_n} . By the fact that $E(u_{\rho_n}) \leq 0$ we have (2.2), so jointly with the Gagliardo-Nirenberg interpolation inequality we get

$$T(u_{\rho_n}) \lesssim \|u_{\rho_n}\|_{p+1}^{p+1} \lesssim T(u_{\rho_n})^{d(p-1)/2} \rho_n^{p+1-d(p-1)}$$

and so, as $1 < p < 1 + \frac{2}{d}$,

$$T(u_{\rho_n}) = o_n(1). \quad (2.14)$$

On the other hand, a ground state fulfills $G(u_{\rho_n}) = 0$, hence (2.14) contradicts Lemma 2.1. □

The next Lemma is a non-existence result that shows that if the ground state energy is zero in a open interval then the ground state energy is never achieved.

Lemma 2.8. *If $I_{\rho^2} = 0$ in an interval $\mathcal{I} = (0, \rho_1)$, then for any $\rho \in \mathcal{I}$, I_{ρ^2} is not attained in S_ρ .*

Proof. Let us assume that exists $\rho \in \mathcal{I}$ such that $I_{\rho^2} = 0 = E(u)$ with $u \in S_{\rho}$. Then

$$E(u) = I_{\rho^2} \leq I_{\theta^2 \rho^2} \leq E(\theta u)$$

for $\theta \in (1 - \epsilon, 1 + \epsilon)$, for some small $\epsilon > 0$, and then

$$\frac{d}{d\theta} E(\theta u) \Big|_{\theta=1} = 0,$$

which implies

$$T(u) + \|u\|_{q+1}^{q+1} - \|u\|_{p+1}^{p+1} = 0.$$

The above condition, which tells us that u is a static solution solving

$$\sqrt{-\Delta}u + iv \cdot \nabla u + |u|^{q-1}u - |u|^{p-1}u = 0,$$

is not compatible with the condition $E(u) = I_{\rho^2} = 0$. Indeed, thanks to (2.10) we get

$$\|u\|_{p+1}^{p+1} = \frac{(2 - d(q-1))(p+1)}{2d(p-q)} T(u)$$

and thanks to (2.11)

$$T(u) + \|u\|_{q+1}^{q+1} - \|u\|_{p+1}^{p+1} = -\frac{1}{d} T(u) \neq 0.$$

This proves that for any $\rho \in \mathcal{I}$ a minimizer for E on S_{ρ} cannot exist. $\square$

The last Lemma guarantees the existence of a ground state at the critical mass ρ_0 .

Lemma 2.9. *Let ρ_0 be defined as in (2.13). Then there exists $u \in S_{\rho_0}$ such that $I_{\rho_0^2} = E(u)$.*

Proof. Let us consider a sequence $\rho_n \rightarrow \rho_0$ with $\rho_n > \rho_0$. We have $I_{\rho_n^2} < 0$ and let us call u_{ρ_n} a ground states that belongs to S_{ρ_n} . Clearly u_{ρ_n} is a bounded sequence in $H^{1/2}$ and $\liminf_{n \rightarrow \infty} \|u_{\rho_n}\|_{p+1}^{p+1} > 0$. Indeed, if along some subsequence ρ_{n_k} , $\lim_{k \rightarrow \infty} \|u_{\rho_{n_k}}\|_{p+1}^{p+1} = 0$, then by the negativity of the energy we obtain $\lim_{k \rightarrow \infty} T(u_{n_k}) = 0$, and the latter is in contrast with Lemma 2.1. By the nonlocal version of the well-known Lieb Translation Lemma, see [3], up to a space translation and up to a subsequence, $u_{\rho_n} \rightharpoonup \bar{u}$ with $\bar{u} \neq 0$, the weak convergence being in $H^{1/2}$. To prove that $\bar{u} \in S_{\rho_0}$ it suffices to observe that if $\|\bar{u}\|_2^2 = \mu^2 < \rho_0^2$ then

$$I_{\rho_0^2 - \mu^2} + I_{\mu^2} + o_n(1) \leq E(u_{\rho_n} - \bar{u}) + E(\bar{u}) = E(u_{\rho_n}) = I_{\rho_n^2} + o_n(1)$$

and hence, by the weak subadditivity inequality, $E(\bar{u}) = I_{\mu^2}$. By Lemma 2.8 this is a contradiction. $\square$

2.2. Proof of Theorem 1.1. By means of the tools developed above, we can conclude the proof of Theorem 1.1. Indeed, it is now an immediate consequence of Lemma 2.6, Lemma 2.7, Lemma 2.8, and Lemma 2.9.

3. LWP AND DYNAMICAL RESULTS IN 1D

In this section we give a proof of the local well-posedness of (1.1) in the space domain $\mathbb{R}$. Although it follows the same lines of the scheme as in [16], we give a proof here for the sake of clarity and to keep the paper self-contained.

3.1. H^s solutions, $s > \frac{1}{2}$. Let us recall the following estimate, see [8], which holds for any function $f \in H^s(\mathbb{R})$, $s > \frac{1}{2}$:

$$\| |f|^{p-1} f \|_{H^s} \leq C \|f\|_{\infty}^{p-1} \|f\|_{H^s}. \quad (3.1)$$

We write a solution to (1.1) in its Duhamel formulation,

$$\psi(t) = e^{-it\sqrt{-\Delta}} \psi_0 + i \int_0^t e^{-i(t-\tau)\sqrt{-\Delta}} (|\psi|^{q-1} \psi - |\psi|^{p-1} \psi)(\tau) d\tau,$$

and we define the ball of radius $R > 0$ in the space of functions $C(I, H^s)$, $I = (-T, T)$ being a time interval containing $t = 0$, as

$$B_R = \{f \in L^\infty(I, H^s) \text{ s.t. } \|f\|_{L^\infty(I, H^s)} \leq R\}.$$

The latter is a Banach space, and we can perform a fixed point argument in B_R for the map

$$\Phi(\psi(t)) = e^{-it\sqrt{-\Delta}} \psi_0 + i \int_0^t e^{-i(t-\tau)\sqrt{-\Delta}} (|\psi|^{q-1} \psi - |\psi|^{p-1} \psi)(\tau) d\tau.$$

By using the unitary property of the linear propagator $e^{-it\sqrt{-\Delta}}$ in any H^s , the Minkowski inequality, the estimate (3.1), and the embedding $H^s \hookrightarrow L^\infty$ for $s > \frac{1}{2}$, we straightforwardly have

$$\|\Phi(\psi)\|_{L^\infty(I, H^s)} \leq \|\psi_0\|_{H^s} + CT(\|\psi\|_{L^\infty(I, H^s)}^q + \|\psi\|_{L^\infty(I, H^s)}^p).$$

Hence, for $R = 2\|\psi_0\|_{H^s}$ and $T < \frac{1}{2C(R^{q-1} + R^{p-1})}$ we have that Φ maps B_R into itself. A similar estimate applies for the difference $\Phi(\psi_1(t)) - \Phi(\psi_2(t))$, thus Φ is a contraction in B_R and a solution $\psi(t)$ to (1.1) with $\psi(0) = \psi_0$ exists in $C(I, H^s)$ at least for small times $T > 0$. The conservation of energy and mass is a consequence of a standard regularization argument. The blow-up alternative in $H^{1/2}$ holds also for initial data in H^s , as we can see by employing the Brezis-Gallouët inequality (see [16, Appendix D]): there exists $C > 0$ such that for any $f \in H^s$, $s > \frac{1}{2}$,

$$\|f\|_{\infty} \leq C \|f\|_{H^{1/2}} \ln^{1/2} \left(2 + \frac{\|f\|_{H^s}}{\|f\|_{H^{1/2}}} \right). \quad (3.2)$$

By the fixed point argument above, the blow-up alternative in H^s holds, and precisely $T_{\max} < \infty$ if and only if $\lim_{t \rightarrow T_{\max}^-} \|\psi(t)\|_{H^s} = \infty$, where $T_{\max}$ is the maximal forward time of existence (similarly for the backward time direction). Let us now introduce the

quantity $K = \sup_{t \in I} \|\psi(t)\|_{H^{1/2}}$. By repeating the estimate of the contraction argument, and by exploiting (3.2), we have that

$$\|\psi(t)\|_{H^s} \leq \|\psi_0\|_{H^s} + C \int_0^t \|\psi(\tau)\|_{H^{1/2}}^{p-1} \ln^{\frac{p-1}{2}} \left(2 + \frac{\|\psi(\tau)\|_{H^s}}{\|\psi(\tau)\|_{H^{1/2}}} \right) \|\psi(\tau)\|_{H^s} d\tau,$$

and by noting that $g(x) = x^{p-1} \ln^{\frac{p-1}{2}}(2 + a/x)$ is monotone increasing, we get

$$\|\psi(t)\|_{H^s} \leq \|\psi_0\|_{H^s} + CK^{p-1} \int_0^t \ln^{\frac{p-1}{2}} \left(2 + \frac{\|\psi(\tau)\|_{H^s}}{K} \right) \|\psi(\tau)\|_{H^s} d\tau.$$

By setting $h(t) = \frac{\|\psi(t)\|_{H^s}}{K}$, we rewrite the estimate above as

$$h(t) \leq \frac{\|\psi_0\|_{H^s}}{K} + CK^{p-1} \int_0^t \ln^{\frac{p-1}{2}}(2 + h(\tau))h(\tau) d\tau := H(t).$$

Observe that

$$\frac{d}{dt}H(t) = K^{p-1}h(t) \ln^{\frac{p-1}{2}}(2 + h(t)) \leq K^{p-1}(2 + H(t)) \ln^{\frac{p-1}{2}}(2 + H(t)),$$

and then

$$\frac{d}{dt} \left(\ln^{\frac{3-p}{2}}(2 + H(t)) \right) \leq \frac{3-p}{2} K^{p-2}$$

which easily gives

$$2 + H(t) \leq \exp \left(\frac{3-p}{2} K^{p-2} t + \ln^{\frac{3-p}{2}} \left(2 + \frac{\|\psi_0\|_{H^s}}{K} \right) \right)^{\frac{2}{3-p}}$$

and then the norm $\|\psi(t)\|_{H^s}$ remains bounded if T is bounded. This means that the blow-up alternative holds at the regularity $H^{1/2}$. At this point, by using the blow-up alternative in $H^{1/2}$, and the mass-subcritical nature of the nonlinearities, we can infer that the solutions can be extended globally in time.

3.2. $H^{1/2}$ solutions. The existence of solutions at the $H^{1/2}$ regularity uses a regularization argument, see [16] (see also [7] for a Half-Wave-Schrödinger equation in $\mathbb{R}^2$). Let us consider a sequence of initial data $\{\psi_{0,n}\} \subset H^s$, $s > \frac{1}{2}$, converging to ψ_0 in the $H^{1/2}$ -topology. The H^s solutions $\psi_n(t)$ to (1.1) corresponding to these initial data are global by the previous discussion, and moreover the $H^{1/2}$ -norm of these solutions remains bounded by the conservation of the energy, as well as remains bounded the $H^{-1/2}$ -norm of $\partial_t \psi_n(t)$. The bound is actually uniform in n , and then locally in time we can extract a weakly convergent subsequence $\psi_n(t) \rightharpoonup \psi(t)$, and by compact embedding the convergence is actually strong in the L_{loc}^r -topology and then ψ is a weak solution to (1.1).

By employing an argument due to Judovic [15], see [11] and [16] for the application to the Szegő equation and to the cubic Half-Wave equation, respectively, we prove

uniqueness of weak solutions. Let ψ_1 and ψ_2 be two solutions emanating from the same initial datum ψ_0 , and define the function $h(t) = \|\psi_1(t) - \psi_2(t)\|_2^2$. Using the equation solved by ψ_1 and ψ_2 , by the fact that $\Im \int (\psi_1 - \psi_2) \sqrt{-\Delta} (\psi_1 - \psi_2) = 0$, we have

$$\begin{aligned} \frac{d}{dt} h(t) &= 2\Re \int (\partial_t \psi_1 - \partial_t \psi_2)(\bar{\psi}_1 - \bar{\psi}_2) \\ &= 2\Im \int (\bar{\psi}_1 - \bar{\psi}_2)(|\psi_2|^{p-1} \psi_2 - |\psi_1|^{p-1} \psi_1) \\ &\quad + 2\Im \int (\bar{\psi}_1 - \bar{\psi}_2)(|\psi_2|^{q-1} \psi_2 - |\psi_1|^{q-1} \psi_1). \end{aligned}$$

The analysis of the term involving $(|\psi_2|^{\bullet-1} \psi_2 - |\psi_1|^{\bullet-1} \psi_1)$, $\bullet \in \{p, q\}$ is analogous, so we consider just one of them. By the following easy computations, we have that

$$\begin{aligned} |\psi_1 - \psi_2|^2 (|\psi_1|^{p-1} + |\psi_2|^{p-1}) &= |\psi_1 - \psi_2|^{2-\delta} |\psi_1 - \psi_2|^\delta (|\psi_1|^{p-1} + |\psi_2|^{p-1}) \\ &\lesssim |\psi_1 - \psi_2|^{2-\delta} (|\psi_1|^\delta + |\psi_2|^\delta) (|\psi_1|^{p-1} + |\psi_2|^{p-1}) \\ &\lesssim |\psi_1 - \psi_2|^{2-\delta} (|\psi_1|^{p-1+\delta} + |\psi_2|^{p-1+\delta} + |\psi_2|^\delta |\psi_1|^{p-1} + |\psi_1|^\delta |\psi_2|^{p-1}) \\ &\lesssim |\psi_1 - \psi_2|^{2-\delta} (|\psi_1|^{p-1+\delta} + |\psi_2|^{p-1+\delta}), \end{aligned} \tag{3.3}$$

for any $\delta \in (0, 2)$, where in the last step we used the Young inequality with exponents $p = \frac{p-1+\delta}{\delta}$ and $p' = \frac{p-1+\delta}{p-1}$. By using in order $\| |\psi_1|^{p-1} - |\psi_2|^{p-1} \| \lesssim |\psi_1 - \psi_2| (|\psi_1|^{p-1} + |\psi_2|^{p-1})$, (3.3), and the Hölder inequality, we get

$$\begin{aligned} &2\Im \int (\bar{\psi}_1 - \bar{\psi}_2)(|\psi_2|^{p-1} \psi_2 - |\psi_1|^{p-1} \psi_1) \\ &\lesssim \int |\psi_1 - \psi_2|^2 (|\psi_1|^{p-1} + |\psi_2|^{p-1}) \\ &\lesssim \int |\psi_1 - \psi_2|^{2-\delta} (|\psi_1|^{p-1+\delta} + |\psi_2|^{p-1+\delta}) \\ &\lesssim \|\psi_1 - \psi_2\|_{L^{r(2-\delta)}}^{2-\delta} (\|\psi_1\|_{L^{r'(p-1+\delta)}}^{p-1+\delta} + \|\psi_2\|_{L^{r'(p-1+\delta)}}^{p-1+\delta}). \end{aligned}$$

Choosing $r = \frac{2}{2-\delta}$ we then have

$$\begin{aligned} &2\Im \int (\bar{\psi}_1 - \bar{\psi}_2)(-|\psi_1|^{p-1} \psi_1 + |\psi_2|^{p-1} \psi_2) \\ &\lesssim \|\psi_1 - \psi_2\|_{L^2}^{2/r} (\|\psi_1\|_{L^{r'(p-1+\delta)}}^{p-1+\delta} + \|\psi_2\|_{L^{r'(p-1+\delta)}}^{p-1+\delta}). \end{aligned}$$

We recall now that there exists a constant $C > 0$ such that for any $1 < r < \infty$, $\|f\|_r \leq C\sqrt{r}\|f\|_{H^{1/2}}$, for any $f \in H^{1/2}$, see [16, Appendix D]. It follows that

$$\begin{aligned} \frac{d}{dt}h(t) &\lesssim (r'(p-1+\delta))^{\frac{p-1+\delta}{2}}h(t)^{1/r}(\|\psi_1\|_{H^s}^{p-1+\delta} + \|\psi_2\|_{H^s}^{p-1+\delta}) \\ &\lesssim (r'(p-1+\delta))^{\frac{p-1+\delta}{2}}h(t)^{1/r} \end{aligned}$$

where in the last step we used the uniform bound $\sup_t(\|\psi_1(t)\|_{H^s} + \|\psi_2(t)\|_{H^s}) \leq C$, which implies

$$\frac{d}{dt}\left(r'h(t)^{1/r'}\right) \lesssim (r'(p-1+\delta))^{\frac{p-1+\delta}{2}}$$

and then, for any t ,

$$h(t) \lesssim \left(\frac{t}{r'}(r'(p-1+\delta))^{\frac{1}{2}}\right)^{(p-1+\delta)r'} = \left(\frac{t\sqrt{p-1+\delta}}{\sqrt{r'}}\right)^{(p-1+\delta)r'} \rightarrow 0$$

as $r' \rightarrow \infty$. Note that $r' \rightarrow \infty$ if and only if $\delta \rightarrow 0$. Hence, the uniqueness of weak solutions is proved.

We can pass from weak to strong solutions by the following argument. We know that

$$\|\psi(t)\|_2 \leq \liminf_{n \rightarrow \infty} \|\psi_n(t)\|_2 = \liminf_{n \rightarrow \infty} \|\psi_{0,n}\|_2 = \|\psi_0\|_2,$$

where we used the weak convergence of $\psi_n(t)$ to $\psi(t)$, the conservation of the mass for the $\psi_n(t)$ and the strong convergence of $\psi_{0,n}$ to ψ_0 , in L^2 . Since the non-linear flow (1.1) is time reversible, one gets the converse inequality, and thus $\|\psi(t)\|_2 = \|\psi_0\|_2$. By this fact,

$$\|\psi(t)\|_2 = \lim_{n \rightarrow \infty} \|\psi_{0,n}\|_2 = \lim_{n \rightarrow \infty} \|\psi_{0,n}(t)\|_2,$$

always by conservation of the mass, and then $\lim_{n \rightarrow \infty} \|\psi_n(t)\|_2 = \|\psi(t)\|_2$. This implies that locally in time, we have strong convergence $\psi_n(t) \rightarrow \psi(t)$ in the L^2 -topology. By invoking the Gagliardo-Nirenberg inequality, and the uniform bound of the $H^{1/2}$ -norm of the weak solutions, uniformly in time, we obtain that $\psi_n(t) \rightarrow \psi(t)$ strongly in $L^{q+1} \cap L^{p+1}$ as well, locally in time. By strong convergence in $L^{q+1} \cap L^{p+1}$

and the conservation of mass and energy, we have

$$\begin{aligned}
& \frac{1}{2} \|\psi(t)\|_{H^{1/2}}^2 + \frac{1}{q+1} \|\psi(t)\|_{q+1}^{q+1} - \frac{1}{p+1} \|\psi(t)\|_{p+1}^{p+1} \\
& \leq \lim_{n \rightarrow \infty} \left(\frac{1}{2} \|\psi_n(t)\|_{H^{1/2}}^2 + \frac{1}{q+1} \|\psi_n(t)\|_{q+1}^{q+1} - \frac{1}{p+1} \|\psi_n(t)\|_{p+1}^{p+1} \right) \\
& = \lim_{n \rightarrow \infty} \left(\frac{1}{2} \|\psi_{0,n}\|_{H^{1/2}}^2 + \frac{1}{q+1} \|\psi_{0,n}\|_{q+1}^{q+1} - \frac{1}{p+1} \|\psi_{0,n}\|_{p+1}^{p+1} \right) \\
& = \frac{1}{2} \|\psi_0\|_{H^{1/2}}^2 + \frac{1}{q+1} \|\psi_0\|_{q+1}^{q+1} - \frac{1}{p+1} \|\psi_0\|_{p+1}^{p+1}.
\end{aligned}$$

By the time reversibility of the equation and uniqueness, again, we find the converse inequality for any t , and then, locally uniformly in time, we have $\lim_{n \rightarrow \infty} \|\psi_n(t)\|_{H^{1/2}} = \lim_{n \rightarrow \infty} \|\psi(t)\|_{H^{1/2}}$, then we upgrade the weak converge of $\psi_n(t) \rightharpoonup \psi(t)$ to a strong convergence, i.e., $\psi_n(t) \rightarrow \psi(t)$ in $H^{1/2}$. This completes the existence theory in $H^{1/2}$.

3.3. Stability result. We conclude with the proof of Corollary 1.6. In this section, we prove Corollary 1.6 following the ideas of [6]. Fix $d = 1$, $v \in \mathbb{R}$ with $|v| < 1$, and let ρ_0^v be as given in Theorem 1.1. For any $\rho > \rho_0^v$, the set

$$\mathcal{G} = \{e^{i\gamma} u(\cdot + y) \text{ s.t. } \gamma \in \mathbb{R}, y \in \mathbb{R}, \text{ and } u \in S_\rho \text{ with } I_{\rho^2} = E_v(u)\}$$

is *orbitally stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\psi_0 \in H^{\frac{1}{2}}(\mathbb{R})$ with $\inf_{w \in \mathcal{G}} \|w - \psi_0\|_{H^{\frac{1}{2}}} < \delta$ we have

$$\sup_{t \in \mathbb{R}} \inf_{w \in \mathcal{G}} \|\psi(t, \cdot) - w\|_{H^{1/2}(\mathbb{R})} < \varepsilon$$

where $\psi(t, \cdot)$ is the solution of (1.1) with initial datum ψ_0 . Note that $\mathcal{G}$ is invariant by translation, namely if $w \in \mathcal{G}$ then also $w(\cdot - y) \in \mathcal{G}$ for any $y \in \mathbb{R}$.

We also notice that our functional E_v is invariant along the time evolution. We argue by contradiction assuming that there exists a $\rho > \rho_0^v$ such that $\mathcal{G}$ is not orbitally stable. This means that there exists $\varepsilon > 0$, a sequence of initial data $\{\psi_{n,0}\} \subset H^{1/2}(\mathbb{R})$, and a sequence of times $\{t_n\} \subset \mathbb{R}$, such that the global solution $\psi_n(t)$ with $\psi_n(0, \cdot) = \psi_{n,0}$, satisfies

$$\lim_{n \rightarrow \infty} \inf_{w \in \mathcal{G}} \|\psi_{n,0} - w\|_{H^{1/2}(\mathbb{R})} = 0 \quad \text{and} \quad \inf_{w \in \mathcal{G}} \|\psi_n(t_n, \cdot) - w\|_{H^{1/2}(\mathbb{R})} \geq \varepsilon.$$

Then there exists a minimizer $u_\rho \in H^{1/2}(\mathbb{R})$ of E_v and $\theta \in \mathbb{R}$ such that $w = e^{i\theta} u_\rho$ and

$$\|\psi_{n,0}\|_2 \rightarrow \|v\|_2 = \rho \quad \text{and} \quad E_v(\psi_{n,0}) \rightarrow E_v(u_\rho).$$

Note that we can assume that $\psi_{n,0} \in S_\rho$. Indeed, by setting $\alpha_n = \rho/\|\psi_{n,0}\|_2$, we have that $\alpha_n \rightarrow 1$, $\alpha_n \psi_{n,0} \in S_\rho$ and $E_v(\alpha_n \psi_{n,0}) \rightarrow I_{\rho^2}$, and then we can replace $\psi_{n,0}$ with $\alpha_n \psi_{n,0}$. Hence, $\{\psi_{n,0}\}$ is a minimizing sequence for I_{ρ^2} , and since $E_v(\psi_n(t_n)) =$

$E_v(\psi_{n,0})$, also $\{\psi_n(t_n, \cdot)\}$ is a minimizing sequence for I_{ρ^2} . Since we proved that every minimizing sequence has a converging subsequence (up to translation) in the strong $H^{\frac{1}{2}}$ -topology to a minimum on the sphere S_{ρ} , we have a contradiction and the proof is complete.

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