Dynamical systems

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Preface

This manuscript contains the notes of the courses in dynamical systems which I have given at the University of Pisa in the last years. Maybe, in the future, it will also contain an organized exposition of (part of) my research on the subject.

I have included the proofs of many results, classical and not. My main fonts of inspiration have been [Aa97], [Ba00], [Gl94], [KH95]. When there is no reference for a proof, either it comes from one of these books or I have rearranged it. But I claim no authorship on any result, except for those which are explicitly referenced to a paper of mine.

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Chapter 1

Introduction and basic concepts

1.1 Dynamical systems

Definition 1.1. A *dynamical system* is a triple $(X, G, \mathcal{S})$ defined as the action $\mathcal{S}$ of a semigroup G with identity e on a set X, that is a function

$$\mathcal{S} : G \times X \to X$$

such that $\mathcal{S}(e, x) = x$ for all $x \in X$, and $\mathcal{S}(g_1, \mathcal{S}(g_2, x)) = \mathcal{S}(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$.

In the following, the set X is assumed to be a locally compact connected metric space.

Two main examples of dynamical system are given in the following definitions.

Definition 1.2. A *discrete-time dynamical system* is defined by the action of $\mathbb{N}_0$ on a set X defined through the iterations of a map $T : X \to X$ by

$$\mathcal{S}(n, x) = T^n(x),$$

where $T^n = T \circ \cdots \circ T$ is the composition of T with itself n times. A discrete-time dynamical system is denoted by the triple $(X, \mathbb{N}_0, T)$.

If the map T is invertible, the system can be extended to the action of the group $\mathbb{Z}$ on X. Examples of a discrete-time dynamical system are sequences defined by a recurrence relation. Let $\{x_n\}$ be a sequence of real numbers defined by

$$x_0 = a \in \mathbb{R}, \qquad x_n = f(x_{n-1}) \quad \forall\, n \geq 1,$$

for a real-valued function f. This corresponds to the dynamical system defined on $X = \mathbb{R}$ through the iterations of the map $f : \mathbb{R} \to \mathbb{R}$, that is $x_n = f^n(a)$.

Definition 1.3. A *continuous-time dynamical system* is defined by the action of $\mathbb{R}$ on a set $X \subset \mathbb{R}^n$ defined through the flow $\phi_t(\underline{x})$ of an autonomous ordinary differential equation $\dot{\underline{x}}(t) = F(\underline{x})$, that is

$$\mathcal{S}(t, \underline{x}) = \phi_t(\underline{x}),$$

where $\phi_t(\underline{x})$ is the solution of an ordinary differential equation[1] with initial condition $\underline{x}$, and $\phi_t : X \to X$ is a continuous function. A continuous-time dynamical system is denoted by the triple $(X, \mathbb{R}, \phi)$.

Definiton 1.3 includes the case of non-autonomous differential equations by using the standard procedure of "enlarging" the space of variables. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ define a time-dependent vector field $F(t, \underline{x})$ on $\mathbb{R}^n$ and consider the Cauchy problem

$$\begin{cases} \dot{\underline{x}}(t) = F(t, \underline{x}(t)) \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

If we let $\underline{y} = (\underline{x}, t) \in \mathbb{R}^{n+1}$ and $\tilde{F}(\underline{y}) = (F(t, \underline{x}), 1)$ be a vector field on $\mathbb{R}^{n+1}$, the previous non-autonomous Cauchy problem is equivalent to the autonomous problem

$$\begin{cases} \dot{\underline{y}}(t) = \tilde{F}(\underline{y}(t)) \\ \underline{y}(0) = (\underline{x}_0, 0) \end{cases}$$

A similar procedure can be applied to the case of sequences defined by a recurrence relation depending on n.

Analogously, it is known that ordinary differential equations of order greater than one can be reduced to systems of ordinary differential equations of order one, hence again included in Definition 1.3. The same is true for the discrete-time case. The following example shows how the procedure works.

[1] All ordinary differential equations we consider are assumed to have the property of local uniqueness of solutions and time-interval of existence of solutions given by $\mathbb{R}$ up to reparametrization.

Example 1.1. Let us consider the sequence $\{x_n\}$ defined as follows

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 1, \quad x_n = x_{n-1} + 2^{n-3}\, x_{n-2} + x_{n-3} \quad \forall\, n \geq 4.$$

We define the vector $\underline{y}_n = (x_n, x_{n-1}, x_{n-2}, n) \in \mathbb{R}^4$. Then using the previous recurrence we have

$$\underline{y}_{n+1} = \Big(x_n + 2^{n-2}\, x_{n-1} + x_{n-2}\,,\, x_n\,,\, x_{n-1}\,,\, n+1\Big) = T(\underline{y}_n) \quad \forall\, n \geq 3$$

with initial condition set to be $y_3 = (1, 1, 0, 3)$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(a, b, c, d) = \Big(a + 2^{d-2}\, b + c\,,\, a\,,\, b\,,\, d+1\Big).$$

The idea of an action of a semigroup on a set X can be used in more abstract contexts. Here we show only one example of algebraic nature that will be studied in more details in part IV of this book.

Example 1.2. Let X be a group, G be $\mathbb{R}$, and consider the action S on X given by multiplication for a one-parameter subgroup of X. For example, if $X = SL(2, \mathbb{R})$ the action of $\mathbb{R}$ defined by

$$S(t, x) = x \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in SL(2, \mathbb{R})$$

represents the geodesic flow on the hyperbolic Poincaré half-plane.

1.2 Basic notions

Definition 1.4. Given a dynamical system $(X, G, \mathcal{S})$, the *orbit* of a point $x \in X$ is the set $\mathcal{O}(x) := \{\mathcal{S}(g, x) \,:\, g \in G\}$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, the orbit of a point $x \in X$ is the set

$$\mathcal{O}(x) = \{T^n(x) \,:\, n \in \mathbb{N}_0\}\,. \tag{1.1}$$

If the map T is invertible, then we can consider the action of the group $\mathbb{Z}$ on X and define the *forward orbit* and *backward orbit* of a point $x \in X$ by

$$\mathcal{O}^+(x) := \{T^n(x) \,:\, n \geq 0\}\,, \quad \mathcal{O}^-(x) := \{T^n(x) \,:\, n \leq 0\}\,.$$

The orbit $\mathcal{O}(x)$ is then given by $\mathcal{O}^+(x) \cup \mathcal{O}^-(x)$.

For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the *forward orbit* and *backward orbit* of a point $\underline{x} \in X$ are defined by

$$\mathcal{O}^+(\underline{x}) := \bigcup_{t \geq 0} \phi_t(\underline{x}), \qquad \mathcal{O}^-(\underline{x}) := \bigcup_{t \leq 0} \phi_t(\underline{x}), \tag{1.2}$$

and the orbit is $\mathcal{O}(\underline{x}) = \mathcal{O}^+(\underline{x}) \cup \mathcal{O}^-(\underline{x})$.

Definition 1.5. Given a dynamical system $(X, G, \mathcal{S})$, the *centralizer* of a point $x \in X$ is the sub-semigroup

$$\mathcal{C}(x) := \{g \in G \,:\, S(g, x) = x\}.$$

A point x is called *fixed* if $\mathcal{C}(x) = G$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, a point $x \in X$ is fixed if and only if $T(x) = x$. If x is not a fixed point but its centralizer is not trivial, x is called *periodic* and the minimum positive element in $\mathcal{C}(x)$ is the *minimal period* of x. For a fixed point $\mathcal{O}(x) = \{x\}$, and for a periodic point of minimal period p

$$\mathcal{O}(x) = \left\{x,\, T(x),\, T^2(x),\, \ldots,\, T^{p-1}(x)\right\}.$$

For a non-invertible map there might be points which are not periodic but are pre-images of a periodic point. For such points x, the centralizer contains only the identity of G, but there exists $k \geq 1$ such that $\mathcal{C}(T^k(x))$ has a minimum positive element p. These points are called *pre-periodic with minimal period p*.

For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$ given by the solutions to $\dot{\underline{x}}(t) = F(\underline{x})$, a point $\underline{x} \in X$ is fixed if and only if $F(\underline{x}) = \underline{0}$. If $\underline{x}$ is not a fixed point but its centralizer is not trivial, $\underline{x}$ is called *periodic* and the minimum positive element in $\mathcal{C}(\underline{x})$ is the *minimal period* of T. A periodic point $\underline{x}$ of minimal period $T > 0$ satisfies

$$\phi_{t+T}(\underline{x}) = \phi_t(\underline{x}), \quad \forall\, t \in \mathbb{R},$$

and

$$\phi_{t+s}(\underline{x}) \neq \phi_t(\underline{x}), \quad \forall\, s \in (0, T),\, t \in \mathbb{R}.$$

For a fixed point $\mathcal{O}(\underline{x}) = \{\underline{x}\}$. For a periodic point of minimal period T

$$\mathcal{O}(\underline{x}) = \bigcup_{0 \leq t \leq T} \phi_t(\underline{x}),$$

and its orbits is called *a periodic orbit of period T*.

Definition 1.6. Given a dynamical system $(X, G, \mathcal{S})$, a set $A \subset X$ is called *invariant* if for each $x \in A$ it holds $\mathcal{S}(g, x) \in A$ for all $g \in G$.

For a continuous-time dynamical system one can introduce a weaker notion. We say that a subset A of X is *forward invariant* if for each $\underline{x} \in A$ it holds $\phi_t(\underline{x}) \in A$ for all $t \geq 0$. Analogously A is called *backward invariant* if the same relation holds for all $t \leq 0$. By definition, A is *invariant* if the previous relation holds for all $t \in \mathbb{R}$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, we consider more situations. We say that a subset A of X is *forward invariant* if $T(A) \subseteq A$, A is called *fully invariant* if $T(A) = A$, A is called *completely invariant* if $T^{-1}(A) = A$. The different notions are useful in different approaches.

Finally, if the action of the group G on X can be interpreted in terms of time evolution, we can introduce notions about the forward and backward evolution of an orbit. In more general situations, one studies the set of all the possible limit points of an orbit as the sequence of the elements of the group acting varies.

Definition 1.7. For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, the *ω-limit set* of a point $x \in X$ is the set

$$\omega(x) := \{y \in X \, : \, \exists \, n_k \to +\infty \text{ such that } T^{n_k}(x) \to y \text{ as } k \to \infty\}.$$

Definition 1.8. For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the *α-limit set* of a point $\underline{x} \in X$ is the set

$$\alpha(\underline{x}) := \left\{\underline{y} \in X \, : \, \exists \, t_k \to -\infty \text{ such that } \phi_{t_k}(\underline{x}) \to \underline{y} \text{ as } k \to \infty\right\}.$$

Analogously the *ω-limit set* of a point $\underline{x} \in X$ is the set

$$\omega(\underline{x}) := \left\{\underline{y} \in X \, : \, \exists \, t_k \to +\infty \text{ such that } \phi_{t_k}(\underline{x}) \to \underline{y} \text{ as } k \to \infty\right\}.$$

Proposition 1.1. *Given a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, let $\underline{x} \in X$ such that $\mathcal{O}^+(\underline{x})$ is bounded. Then the set $\omega(\underline{x})$ is non-empty, compact and invariant. If $\mathcal{O}^-(\underline{x})$ is bounded, the same holds for the set $\alpha(\underline{x})$.*

Proof. Given a point $\underline{x}$ with bounded forward orbit, let us consider a strictly increasing sequence $\{\tau_j\}_{j=0}^{\infty}$ of times in $\mathbb{R}^+$ with $\tau_0 = 0$ and $\tau_j \to +\infty$, and let $\underline{x}_j := \phi_{\tau_j}(\underline{x})$. We first show that

$$\omega(\underline{x}) = \bigcap_{j=0}^{\infty} \overline{\mathcal{O}^+(\underline{x}_j)}. \tag{1.3}$$

By the definition of the ω-limit set, it is immediate that $\omega(\underline{x}) \subset \overline{\mathcal{O}^+(\underline{x}_j)}$ for all $j \geq 0$. Hence it remains to show that if $\underline{y} \in \cap_{j=0}^\infty \overline{\mathcal{O}^+(\underline{x}_j)}$ then $\underline{y} \in \omega(\underline{x})$. By definition of closure of a set, for all $j \geq 0$ there exists a sequence $\{\underline{\xi}_n^j\}_n$ of points in $\mathcal{O}^+(\underline{x}_j)$ such that $\underline{\xi}_n^j \to \underline{y}$, hence there exists a sequence $\{t_n^j\}_n$ such that $\phi_{t_n^j}(\underline{x}_j) \to \underline{y}$. In particular we have proved that there exists a strictly increasing diverging sequence $\{\tau_j\}_{j=0}^\infty$ and sequences $\{t_n^j\}_n$ such that

$$\phi_{\tau_j + t_n^j}(\underline{x}) \xrightarrow[n\to\infty]{} \underline{y}, \quad \forall\, j \geq 0.$$

From $\{\tau_j + t_n^j\}_{j,n}$ we can then extract a diverging sequence $\{\tilde{t}_k\}_k$ such that $\phi_{\tilde{t}_k}(\underline{x}) \to \underline{y}$ as $k \to \infty$. Hence $\underline{y} \in \omega(\underline{x})$, and (1.3) is proved.

The first properties of $\omega(\underline{x})$ follow from (1.3). The sets $\{\overline{\mathcal{O}^+(\underline{x}_j)}\}_j$ define a decreasing sequence of non-empty closed sets, which are bounded because $\mathcal{O}^+(\underline{x})$ is bounded. Hence $\omega(\underline{x})$ is a non-empty compact set. It remains to prove that it is invariant.

Let $\underline{y} \in \omega(\underline{x})$, and let $\{t_k\}_k$ be a positively diverging sequence such that $\phi_{t_k}(\underline{x}) \to \underline{y}$ as $k \to \infty$. By the properties of a continuous-time dynamical system

$$\phi_{t+t_k}(\underline{x}) = \phi_t(\phi_{t_k}(\underline{x})) \xrightarrow[k\to\infty]{} \phi_t(\underline{y}), \quad \forall\, t \in \mathbb{R}.$$

Hence we have shown that $\phi_t(\underline{y}) \in \omega(\underline{x})$ for all $t \in \mathbb{R}$. This concludes the proof for the ω-limit set.

The proof for the α-limit set follows along the same lines. □

Proposition 1.2. *Given a discrete-time dynamical system* $(X, \mathbb{N}_0, T)$*, let* $x \in X$ *such that* $\mathcal{O}(x)$ *is bounded. Then the set* $\omega(x)$ *is non-empty and compact. If* T *is continuous then* $\omega(x)$ *is fully invariant.*

Proof. We can repeat the proof of Proposition 1.1 to show that the ω-limit set is non-empty and compact. In particular the proof follows from the analogue of (1.3).

Let $T : X \to X$ be a continuous map with respect to a topological structure on X. Then given $y \in \omega(x)$, and being $\{n_k\}_k$ the diverging sequence of naturals for which $T^{n_k} \to y$ as $k \to \infty$, we have

$$T^{n_k+1}(x) = T(T^{n_k}(x)) \xrightarrow[k\to\infty]{} T(y).$$

Hence $T(y) \in \omega(x)$, and $\omega(x)$ is a positively invariant set. On the other hand, since $\mathcal{O}(x)$ is bounded, the sequence $\{T^{n_k-1}(x)\}_k$ admits a convergent sub-sequence $\{T^{n_{k_j}-1}(x)\}_j$ with limit point z. Hence $z \in \omega(x)$. Again by continuity of T we find

$$T(z) = T\Big(\lim_{j\to\infty} T^{n_{k_j}-1}(x)\Big) = \lim_{j\to\infty} T^{n_{k_j}}(x) = y$$

since n_{k_j} is a subsequence of n_k. Hence $y \in T(\omega(x))$, and $\omega(x)$ is then fully invariant. □

We remark that the ω-limit set is not completely invariant in general. It is sufficient to think of the case in which the ω-limit set is a fixed point with more than one pre-image.

Definition 1.9. For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the orbit of a point $\underline{y}$ is called *homoclinic* if there exists a fixed point $\underline{x}$ such that

$$\alpha(\underline{y}) = \omega(\underline{y}) = \{\underline{x}\}.$$

If there exist two distinct fixed points $\underline{x}_1$, $\underline{x}_2$ such that

$$\alpha(\underline{y}) = \{\underline{x}_1\} \quad \text{and} \quad \omega(\underline{y}) = \{\underline{x}_2\},$$

then the orbit of the point $\underline{y}$ is called *heteroclinic*.

Definition 1.9 can be adapted verbatim to the case of a discrete-time dynamical system $(X, \mathbb{N}_0, T)$ with invertible T.

1.3 Examples

Here we collect the main examples of discrete dynamical systems that will be used in the following.

Example 1.3 (The roots). Sequences defined by a recurrence are the first very basic example of a discrete-time dynamical system. Let $c > 0$, $k \in [1, 3]$, and consider the sequence $\{a_n\}$ defined by

$$\begin{cases} a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n^k}\right), & \forall\, n \ge 0 \\ a_0 \in (0, +\infty) \end{cases}$$

It is an exercise to prove that for all $a_0 \in \mathbb{R}^+$ it holds $\lim_n a_n = c^{\frac{1}{k+1}}$. This can be read as a result about the asymptotic behaviour of the orbits of points in $\mathbb{R}^+$ for the dynamical system defined by the map

$$T_{c,k} : \mathbb{R}^+ \to \mathbb{R}^+, \quad T_{c,k}(x) = \frac{1}{2}\left(x + \frac{c}{x^k}\right).$$

In fact one can prove that $\omega(x) = c^{\frac{1}{k+1}}$ for all $x \in \mathbb{R}^+$.

Example 1.4 (Rotations of the circle). Let us consider the action of $\mathbb{Z}$ on S^1 given by the rotation of an angle $2\pi\alpha$, for $\alpha \in \mathbb{R}$, that is

$$\mathcal{S}(n, z) = z\, e^{2\pi i n\alpha} \in S^1, \quad \forall\, z \in S^1,\, n \in \mathbb{Z}.$$

By writing $S^1 = \{z \in \mathbb{C} \,:\, z = e^{2\pi i x}, x \in \mathbb{R}\}$, we make the identification of S^1 with $[0,1]/(0 \sim 1)$, the unit interval with end points identified. The rotation of angle $2\pi\alpha$ can then be written as a map on S^1 as

$$R_\alpha : S^1 \to S^1, \quad R_\alpha(x) = \{x + \alpha\}. \tag{1.4}$$

Proposition 1.3. *If α is rational, all orbits of R_α are periodic of the same minimal period. If α is irrational, all orbits of R_α are dense.*

Proof. If $\alpha = p/q \in \mathbb{Q}$ with $(p, q) = 1$, then $R^q_\alpha(x) = \{x + q\alpha\} = x$ for all $x \in [0, 1)$. In addition, if $n \in \mathbb{N}$ and $n < q$, we can write $n\alpha = np/q = m+r/q$ with $m \in \mathbb{Z}$ and $r/q \in \mathbb{Q} \cap (0, 1)$. Hence $R^n_\alpha(x) = \{x + r/q\} \neq x$. It follows that all orbits are periodic of the minimal period q.

Let's now assume that α is irrational. Since R_α is an isometry, it is enough to show that one orbit is dense. In fact, we prove that forward orbits are dense by considering $\{R^n_\alpha(0)\}_{n\geq 0}$.

Let $x \in S^1$, then we show that for any $\varepsilon > 0$ there exists $\bar{n}$ such that $R^{\bar{n}}_\alpha(0) \in (x - \varepsilon\,,\, x + \varepsilon)$. First, by Proposition A.3, we find $p, q \in \mathbb{N}$ such that $0 < q\alpha - p < \varepsilon$. This means that $R^q_\alpha(0) \in (-\varepsilon, \varepsilon)$. If we now consider the points $\{k(q\alpha - p)\}_{k\geq 0}$, it follows that there exists $K > 0$ such that the points $\{k(q\alpha - p)\}_{0\leq k\leq K}$ create a partition of $[0, 1]$ into intervals of length less than ε. Therefore for all $x \in S^1$

$$\min_{0\leq k\leq K} d(x, k(q\alpha - p)) < \varepsilon$$

and the minimum is achieved for some value $\bar{k}$. Hence choosing $\bar{n} = \bar{k}q$ the proof is finished. □

A consequence of the proposition is that if α is rational, then all points have their own periodic orbit as α-limit and ω-limit sets. Instead, if α is irrational, then $\alpha(x) = \omega(x) = S^1$ for all x.

Example 1.5 (The Tent maps). It is a family of maps

$$T_s : [0, 1] \to [0, 1] \quad \text{with } s \in (0, 2]$$

defined as

$$T_s(x) = \begin{cases} s\,x, & \text{if } x \in [0, \frac{1}{2}]; \\ s\,(1 - x), & \text{if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{1.5}$$

Example 1.6 (The Logistic maps). It is a family of maps

$$T_\lambda : [0,1] \to [0,1] \quad \text{with } \lambda \in (0,4]$$

defined as

$$T_\lambda(x) = \lambda\, x\, (1-x). \tag{1.6}$$

Example 1.7 (Linear endomorphisms of the circle). It is a family of maps

$$T_m : S^1 \to S^1 \quad \text{with } m \in \mathbb{N},\ m \geq 2$$

where again we think of S^1 as $[0,1]/(0 \sim 1)$, defined as

$$T_m(x) = \{mx\}. \tag{1.7}$$

Special cases are $m = 2$ which is also called the *Bernoulli map* and is related to the binary expansion of real numbers, and $m = 10$ which is related to the decimal expansion of real numbers.

Example 1.8 (Symbolic dynamics). We now introduce an abstract system. Let $\mathcal{A}$ be a finite or countable alphabet and denote by $N \in \mathbb{N} \cup \{\infty\}$ the number of symbols. Let $\Omega_{\mathcal{A}}$ be the set of all infinite strings with symbols from $\mathcal{A}$, that is

$$\Omega_{\mathcal{A}} = \mathcal{A}^{\mathbb{N}_0} = \{\omega = (\omega_i)_{i \in \mathbb{N}_0} \, : \, \omega_i \in \mathcal{A} \ \forall\, i \in \mathbb{N}_0\}\,.$$

If $N < \infty$, the space X is compact when endowed with the product topology or with the metric

$$d_\theta(\omega, \tilde{\omega}) := \theta^{\min\{i \in \mathbb{N}_0 \,:\, \omega_i \neq \tilde{\omega}_i\}}\,, \quad \text{for a fixed } \theta \in (0,1). \tag{1.8}$$

The space $\Omega_{\mathcal{A}}$ is totally disconnected and a basis of the product topology is given by the *cylinders*: for $k \in \mathbb{N}$, $i_1, i_2, \dots, i_k \in \mathbb{N}_0$, and $a_1, a_2, \dots, a_k \in \mathcal{A}$, we define

$$C_{i_1, i_2, \dots, i_k}(a_1, a_2, \dots, a_k) := \left\{\omega \in \Omega_{\mathcal{A}} \, : \, \omega_{i_j} = a_j \ \forall\, j = 1, \dots, k\right\}.$$

In particular, we use the notations $C(a) = C_1(a)$ and

$$C_{i_1, i_2, \dots, i_k}(\omega) = C_{i_1, i_2, \dots, i_k}(\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_k})$$

for a fixed $\omega \in \Omega_{\mathcal{A}}$.

On $\Omega_{\mathcal{A}}$ we consider the discrete dynamical system given by the action of the continuous map

$$\sigma : \Omega_{\mathcal{A}} \to \Omega_{\mathcal{A}}, \quad (\sigma(\omega))_i = \omega_{i+1} \quad \forall\, i \in \mathbb{N}_0.$$

The system $(\Omega_{\mathcal{A}}, \mathbb{N}_0, \sigma)$ is called the *(one-sided) full shift on* $\mathcal{A}$.

In some situations it is useful to consider a sub-system of the full shift. A first easy example is given by considering infinite strings which cannot contain a given set of words of finite length. For example, let $M = (m_{ij}) \in M(N \times N, \{0,1\})$, a $N \times N$ matrix with coefficients in the set $\{0,1\}$ and rows and columns indexed by $\mathcal{A}$. We set

$$\Omega_{\mathcal{A},M} := \left\{ \omega \in \mathcal{A}^{\mathbb{N}_0} \,:\, m_{\omega_i\, \omega_{i+1}} = 1 \; \forall\, i \in \mathbb{N}_0 \right\},$$

that is, the set $\Omega_{\mathcal{A},M}$ contains the infinite strings in $\mathcal{A}^{\mathbb{N}_0}$ in which the symbol $a \in \mathcal{A}$ may be followed by the symbol $b \in \mathcal{A}$ iff $m_{ab} = 1$. It is immediate to verify that $\Omega_{\mathcal{A},M}$ is forward invariant for the action of σ and it is fully invariant if for each $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ with $m_{ab} = 1$. Hence we can restrict the action of σ to $\Omega_{\mathcal{A},M}$, and the dynamical system $(\Omega_{\mathcal{A},M}, \mathbb{N}_0, \sigma)$ is called *subshift of finite type on* $\mathcal{A}$.

Finally, by considering bi-infinite strings $\mathcal{A}^{\mathbb{Z}}$, one can consider the action of σ on $\mathcal{A}^{\mathbb{Z}}$ and on $\mathcal{A}^{\mathbb{Z}}_M$. In this case the map σ is invertible and the dynamical systems $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z}, \sigma)$ and $(\mathcal{A}^{\mathbb{Z}}_M, \mathbb{Z}, \sigma)$ are called two-sided full shift and subshift of finite type, respectively.

Example 1.9 (Toral translations). Let $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$ be the d-dimensional torus. Given a vector $\underline{v} \in \mathbb{R}^d$, the affine map $\mathbb{R}^d \ni \underline{x} \mapsto \underline{x} + \underline{v}$ may be projected onto the continuous automorphisms of $\mathbb{T}^d$ given by

$$\mathbb{T}^d \ni \underline{x} \mapsto T_v(\underline{x}) := \underline{x} + \underline{v} \pmod{\mathbb{Z}^d}.$$

Proposition 1.4 (see [KH95]). *Given $\underline{v} = (v_1, \dots, v_d)$, all orbits of T_v are dense if and only if the $d+1$ numbers $v_1, \dots, v_d, 1$ are independent on $\mathbb{Z}$.*

Example 1.10 (Toral automorphisms). Let $\mathbb{T}^2 := \mathbb{R}^2 / \mathbb{Z}^2$ be the two dimensional torus. Given a matrix $A \in M(2 \times 2, \mathbb{Z})$ with $\det(A) = 1$, the linear map $\mathbb{R}^2 \ni \underline{x} \mapsto A\, \underline{x}$ may be projected onto a continuous automorphisms of $\mathbb{T}^2$ given by

$$\mathbb{T}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto T_A(x,y) := A \begin{pmatrix} x \\ y \end{pmatrix} \pmod{\mathbb{Z}^2}.$$

The most famous example is the so-called *Arnold's Cat map*, which is the toral automorphism given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Example 1.11 (The Standard map). Let us consider an electron with charge e moving horizontally in a cyclotron thanks to the action of a vertical magnetic field of constant modulus B, and subject to a time-dependent voltage drop $V\sin(\omega t)$ across a narrow azimuthal gap. Let E denote the energy of the electron, then the period of rotation is given by $T = 2\pi\,\frac{E}{eBc}$. We measure energy and time (E,t) just before every voltage drop, hence after one circuit we obtain

$$E' = E - e\,V\sin(\omega t)\,, \qquad t' = t + \frac{2\pi}{eBc}\,E'\,.$$

Using the variables $x := \frac{\omega}{2\pi}t$ and $y := \frac{\omega}{eBc}E$, and setting $k := 2\pi\frac{\omega V}{Bc}$, we have defined the map

$$\tilde{T} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}\,, \quad \tilde{T}(x,y) = \Big(x + y - \frac{k}{2\pi}\,\sin(2\pi x)\,,\, y - \frac{k}{2\pi}\,\sin(2\pi x)\Big)\,.$$

Note that $\tilde{T}(x+1,y) = \tilde{T}(x,y) + (1,0)$, hence given the projection $\pi : \mathbb{R}\times\mathbb{R} \to S^1\times\mathbb{R}$ defined as $\pi(x,y) = (x - \lfloor x\rfloor, y)$, the map

$$T : S^1\times\mathbb{R} \to S^1\times\mathbb{R}\,, \quad T(x,y) = \Big(\Big\{x+y-\frac{k}{2\pi}\,\sin(2\pi x)\Big\}\,,\, y-\frac{k}{2\pi}\,\sin(2\pi x)\Big) \tag{1.9}$$

satisfies $\pi\circ\tilde{T} = T\circ\pi$. Hence $\tilde{T}$ is a lift of T. The map T is known as the *(Chirikov) Standard map.*

Example 1.12 (Bouncing balls). The Standard map of Example 1.11 is an example of a "kicked rotor". Another example of this kind of system is obtained by considering a ball moving under the action of a conservative vertical force field and bouncing on a moving racket.

For simplicity, we assume that the ball has unit mass and consider the case of vertical force field of constant magnitude g pointing downwards. We let $(0,y(t))$ denote the position of the ball in $\mathbb{R}^2$ at time t, and the force field be given by $\vec{F}(x,y) = (0,-g)$. Let $(0,f(t))$ be the position of the racket at time t, with $f(t)$ be a 1-periodic differentiable function. In addition, we assume that the racket has infinite mass and that the bounces of the ball on it are elastic. In particular, if $z(t) := y(t) - f(t)$ then the momentum conservation law implies $\dot z(t^-) = -\dot z(t^+)$ at a bouncing time t, with $\dot z(t^\pm)$ denoting the derivative immediately after and before the bounce, respectively. Hence,

$$\dot y(t^-) - \dot f(t) = -\dot y(t^+) + \dot f(t) \quad \Leftrightarrow \quad \dot y(t^+) = 2\,\dot f(t) - \dot y(t^-)\,.$$

At this point, using the fact that between two bounces the motion is Hamiltonian, we can reconstruct the motion of the ball by the sequence (t_k, v_k) of bouncing times t_k and velocities v_k of the ball immediately after the bounce at time t_k.

We now make a further simplification. We assume that the racket hits the ball always at the same height, so that $\dot y(t_{k+1}^-) = -\dot y(t_k^+) = -v_k$ because the motion of the ball is free between two consecutive bounces. In conclusion, we obtain the following relations

$$\begin{cases} t_{k+1} = t_k + \frac{2v_k}{g} \\ v_{k+1} = v_k + 2\dot f(t_{k+1}) \end{cases}$$

which define a discrete dynamical system $(\tilde X, \tilde T)$ with $\tilde X = \mathbb{R} \times \mathbb{R}^+$ and

$$\tilde T : \tilde X \to \tilde X\,, \quad \tilde T(t,v) = \Big(t + \frac{2v}{g}\,,\, v + 2\dot f\Big(t + \frac{2v}{g}\Big)\Big)\,.$$

Finally, since $\tilde T(t+1, v) = \tilde T(t,v) + (1,0)$ by the periodicity of f, we can project the system $(\tilde X, \tilde T)$ to the system (X, T), with $X = S^1 \times \mathbb{R}^+$ and

$$T(t,v) = \Big(\Big\{t + \frac{2v}{g}\Big\}\,,\, v + 2\dot f\Big(t + \frac{2v}{g}\Big)\Big)\,.$$

<u>*Example*</u> 1.13 (Birkhoff billiards). Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^3 boundary[2]. Let us normalize the set to $|\partial\Omega| = 1$ and fix the positive orientation of the boundary.

The *mathematical billiard* is the continuous dynamical system given by the frictionless motion of a pointwise ball inside Ω, with elastic specular reflections at $\partial\Omega$. The phase space is then given by $\Omega \times S^1$, since the velocity of the ball is preserved in modulus.

A convenient simpler description of the system is given by the Poincaré map of the flow on the set $\partial\Omega \times [0,\pi]$, described by the evolution of the couples (position,angle) of the subsequent collisions of the ball with the

[2] Thanks to [Ha77] this assumption avoid accumulation of collision times.

boundary of the set. For each collision, its position can be described by the arc-length coordinate $s \in S^1$ and its angle by the angle $\vartheta \in [0, \pi]$ between the trajectory of the ball after the collision and the oriented tangent vector to $\partial\Omega$ at the collision point. We have thus described a map

$$T : S^1 \times (0, \pi) \to S^1 \times (0, \pi)$$

which can be continuously extended to $S^1 \times [0, \pi]$ by $T(s, 0) = (s, 0)$ and $T(s, \pi) = (s, \pi)$.

This map may be defined with some cautions for more general domains $\Omega \subset \mathbb{R}^2$ (see [CM06]).

<u>*Example*</u> 1.14 (Mechanics and Billiards). Let m_1 and m_2 be two distinct point masses moving frictionless on the interval $[0, 1]$, subject to perfectly elastic collisions among them and with two infinite ideal walls at the extremes of the interval. Let $x_1, x_2 \in [0, 1]$ with $x_1 \leq x_2$, and $v_1, v_2 \in \mathbb{R}$, denote the positions and velocities of the masses, and introduce the variables $q_1 := \sqrt{m_1}\, x_1$ and $q_2 := \sqrt{m_2}\, x_2$, and $u_1 = \sqrt{m_1}\, v_1$ and $u_2 := \sqrt{m_2}\, v_2$. The invariances of the kinetic energy K and of the linear momentum P of the system read in the new variables as

$$u_1^2 + u_2^2 = 2K\,, \quad \sqrt{m_1}\, u_1 + \sqrt{m_2}\, u_2 = P\,.$$

In the new variables, the configuration space is given by the triangle

$$A = \left\{ (q_1, q_2) \in \mathbb{R}^2 \,:\, q_1 \geq 0\,,\, q_2 \leq \sqrt{m_2}\,,\, \sqrt{m_2}\, q_1 \leq \sqrt{m_1}\, q_2 \right\}\,.$$

A trajectory $(q_1(t), q_2(t))$ satisfies the following constraints:

$$\dot{q}_1^2(t) + \dot{q}_2^2(t) = 2K \qquad \forall\, t$$

(hence the motion occurs with constant speed);

$$\sqrt{m_1}\, \dot{q}_1 + \sqrt{m_2}\, \dot{q}_2 = P \qquad \forall\, t$$

(hence the velocity vector of the motion has fixed scalar product with the vector $(\sqrt{m_1}, \sqrt{m_2})$).

These properties imply that the motion $(q_1(t), q_2(t))$ in A can be described by the orbit of a mathematical billiard ball inside A.

1.4 Exercises

1.1. Let $T : [0,1] \to [0,1]$ be defined by

$$T(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \in (0,1]; \\ 1, & \text{if } x = 0. \end{cases}$$

Show that for all $x \in [0,1]$ the ω-limit set $\omega(x)$ is non-empty but not forward invariant.

1.2. In Example 1.14 let the masses move in $[0, +\infty)$, and consider the motion with initial positions $q_1(0) < q_2(0)$ and velocities $u_1(0) = 0$ and $u_2(0) = -1$. If $m_2 \geq m_1$, how many collisions among the two balls and among mass m_1 and the wall at $x = 0$ will occur? What happens if $m_2 = 100^n m_1$?

Part I

Topological dynamics

Chapter 2

Continuous-time dynamical systems

In this chapter, we consider the continuous-time dynamical systems as defined in Definition 1.3 with phase space $X = \mathbb{R}^n$.

2.1 Linear systems

The simplest case to study is that of an ordinary differential equation with linear vector field. Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix and consider the ordinary differential equation $\dot{\underline{x}} = A\underline{x}$. It is well known that the flow is given by $\phi_t(\underline{x}) = e^{At}\underline{x}$, and the behaviour of the orbits is determined by the eigenvalues of A. We state a result in the case that all the eigenvalues of A are simple, an analogous result holds counting the multiplicities of the eigenvalues and using the Jordan normal form of A.

Theorem 2.1. *Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix with k distinct real eigenvalues $\lambda_1, \dots, \lambda_k$, and $m = \frac{1}{2}(n-k)$ distinct couples of conjugate complex eigenvalues $a_j \pm i\, b_j$. Then there exists an invertible matrix $P \in M(n \times n, \mathbb{R})$ such that*

$$P^{-1}\, A\, P = \Lambda := diag\Big(\lambda_1, \dots, \lambda_k, B_1, \dots, B_m\Big)$$

where

$$B_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}, \quad \forall\, j = 1, \dots, m,$$

and the flow of the differential equation $\dot{\underline{x}} = A\underline{x}$ is given by

$$\phi_t(\underline{x}) = P\, e^{\Lambda\, t}\, P^{-1}\, \underline{x}$$

where

$$e^{\Lambda\, t} = diag\Big(e^{\lambda_1 t}, \dots, e^{\lambda_k t}, e^{tB_1}, \dots, e^{tB_m}\Big)$$

and

$$e^{tB_j} = e^{a_j t}\begin{pmatrix} \cos(b_j t) & -\sin(b_j t) \\ \sin(b_j t) & \cos(b_j t) \end{pmatrix}, \quad \forall\, j = 1, \dots, m.$$

Remark 2.2. Let us consider the case $n = 2, 3$, so that the matrix A can only have multiple real roots. If $n = 2$ the possible Jordan normal form of a matrix A with a double real eigenvalue λ are

$$\Lambda = \mathrm{diag}\Big(\lambda, \lambda\Big) \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

In the non-diagonal case, one writes $\Lambda = \lambda I + N$, where N is the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for which $N^2 = 0$. So that[1] $e^{\Lambda t} = e^{\lambda t}\, e^{N t}$. It follows that

$$e^{\Lambda t} = \mathrm{diag}\Big(e^{\lambda t}, e^{\lambda t}\Big) \quad \text{or} \quad e^{\lambda t}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Analogously, in the $n = 3$ case, if A has eigenvalues with geometric multiplicities greater than or equal to 2, we are reduced to the previous case. If A has an eigenvalue λ with geometric multiplicity 1 its Jordan normal form is

$$\Lambda = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

and as before we write $\Lambda = \lambda I + N$, where N is a nilpotent matrix such that $N^3 = 0$. Then

$$e^{\Lambda t} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

[1]Here we use the fact that the matrices I and N commute.

In the case of linear ordinary differential equations it is also particularly simple to find fixed points, periodic orbits, and invariant sets. First, using Definition 1.5 we find

Proposition 2.3. *The fixed points of the ordinary differential equation* $\underline{\dot{x}} = A\underline{x}$ *are the points in the kernel of* A.

In particular, the origin $\underline{x}_0 = \underline{0}$ is a fixed point for all A, and the other fixed points come in linear subspaces of $\mathbb{R}^n$. We'll see that the origin plays a special role in characterizing the dynamics of all the non-trivial orbits.

Concerning periodic orbits, it is straightforward from Theorem 2.1 that they can exist only if there is a couple of conjugate complex eigenvalues with null real part. If this holds, all orbits within the relative eigenspace are periodic, as they are of the form $e^{tB}\underline{x}$ with $a = 0$.

In general, the space $\mathbb{R}^n$ can be written as the direct sum of generalised eigenspaces of A, and according to the asymptotic behaviour of the orbits, it makes sense to consider the following decomposition.

Definition 2.1. Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix and let E_λ denote the generalised eigenspace of an eigenvalue λ. We call:
Stable eigenspace of $\underline{0}$ the linear space $E^s(\underline{0})$ defined as

$$E^s(\underline{0}) := \mathrm{Span}\left\{v \in E_\lambda \,:\, \Re(\lambda) < 0\right\} ;$$

Central eigenspace of $\underline{0}$ the linear space $E^c(\underline{0})$ defined as

$$E^c(\underline{0}) := \mathrm{Span}\left\{v \in E_\lambda \,:\, \Re(\lambda) = 0\right\} ;$$

Unstable eigenspace of $\underline{0}$ the linear space $E^u(\underline{0})$ defined as

$$E^u(\underline{0}) := \mathrm{Span}\left\{v \in E_\lambda \,:\, \Re(\lambda) > 0\right\} .$$

Theorem 2.4. *Let* $A \in M(n \times n, \mathbb{R})$ *be a real* $n \times n$ *matrix and consider the ordinary differential equation* $\underline{\dot{x}} = A\underline{x}$. *Then:*

(i) $n = \dim E^s(\underline{0}) + \dim E^c(\underline{0}) + \dim E^u(\underline{0})$*;*

(ii) the eigenspaces $E^s(\underline{0}), E^c(\underline{0}), E^u(\underline{0})$ *are invariant;*

(iii) the following dynamical characterisation holds:

$$E^s(\underline{0}) = \left\{\underline{x} \in \mathbb{R}^n \,:\, \phi_t(\underline{x}) \to \underline{0} \text{ as } t \to +\infty\right\};$$

$$E^u(\underline{0}) = \left\{\underline{x} \in \mathbb{R}^n \,:\, \phi_t(\underline{x}) \to \underline{0} \text{ as } t \to -\infty\right\}.$$

Proof. It is a simple application of Theorem 2.1. □

Remark 2.5. It is interesting to notice that we haven't given a dynamical interpretation for the central eigenspace of $\underline{0}$. The reason is that if $\dim E^c(\underline{0}) \neq 0$ we can find different behaviours for the orbits. Let us consider the simple case $n = \dim E^c(\underline{0}) = 2$ with $\lambda = 0$ being a double eigenvalue. Then there are two possibilities for the matrix A (up to use of the Jordan normal form):

$$A = \operatorname{diag}\Big(0, 0\Big) \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the first case the flow is the identity, that is $\phi_t(x, y) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$, whereas in the second case the flow is given by $\phi_t(x, y) = (x + ty, y)$ for all $(x, y) \in \mathbb{R}^2$. Using Definition 2.2, in the first case $(0, 0)$ is Lyapunov stable and in the second case it is unstable.

Theorem 2.4 gives the characterisation of the dynamics with respect to the fixed point $\underline{0}$. In particular if $\ker(A) = \{\underline{0}\}$ and $\dim E^c(\underline{0}) = 0$, all orbits converge to $\underline{0}$, either for $t \to +\infty$ or for $t \to -\infty$. If instead the kernel of A consists of a non-trivial linear subspace W with $\dim W = \dim E^c(\underline{0})$, it is easy to see that the dynamics of non-fixed points is determined by that of the points in the space $W^{\perp}$.

Linear systems in the plane

In the case of linear systems in $\mathbb{R}^2$ it is possible to characterise the dynamical properties of the system without explicitly computing the eigenvalues of the matrix A. We also introduce a terminology for fixed points with different local dynamics.

The nature of the origin $\underline{0} = (0, 0)$ as a fixed point of a system $\dot{\underline{x}} = A\underline{x}$, with $\underline{x} = (x, y) \in \mathbb{R}^2$ is determined by the relation between the determinant and the trace of A. Indeed the characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 - \operatorname{tr}(A)\,\lambda + \det(A)\,,$$

so that the eigenvalues are

$$\lambda_{\pm} = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}^2(A) - 4\det(A)}}{2}\,,$$

and we distinguish different cases according to the sign of the determinant of A and of the discriminant $\Delta := \mathrm{tr}^2(A) - 4\det(A)$.

Case 1. $\det(A) > 0$ *and* $\Delta > 0$. The matrix A has two real distinct eigenvalues satisfying $\lambda_+ > \lambda_- > 0$ if $\mathrm{tr}(A) > 0$, and $\lambda_- < \lambda_+ < 0$ if $\mathrm{tr}(A) < 0$.

In both cases the orbits are generalised parabola through $\underline{0}$, at which they are tangent to the line generated by the eigenvector relative to eigenvalue of smallest modulus. If $\mathrm{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable node*. We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\mathrm{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable node*. We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

Note that $\underline{0}$ being a node is an open property since sufficiently small perturbations of A don't change the nature of the origin.

Case 2. $\det(A) > 0$ *and* $\Delta < 0$. The matrix A has a couple of complex conjugate eigenvalues $\lambda_\pm$ with $\Re(\lambda_\pm) = \frac{1}{2}\,\mathrm{tr}(A)$.

If $\mathrm{tr}(A) > 0$ all orbits are spirals out of $\underline{0}$ and they are either clockwise or anti-clockwise according for example to the sign of $\dot{x}$ when $y = 0$. In this case the origin is called an *unstable focus* and $E^u(\underline{0}) = \mathbb{R}^2$. If $\mathrm{tr}(A) < 0$ all orbits are spirals into $\underline{0}$ and as before they are either clockwise or anti-clockwise. In this case the origin is called a *stable focus* and $E^s(\underline{0}) = \mathbb{R}^2$. If $\mathrm{tr}(A) = 0$ all orbits are concentric circles about $\underline{0}$ and again they are either clockwise or anti-clockwise. In this case the origin is called a *center* and $E^c(\underline{0}) = \mathbb{R}^2$.

Notice that $\underline{0}$ being a focus is an open property. Instead $\underline{0}$ being a center is a closed property and arbitrarily small perturbations of A may turn the origin into an unstable or stable focus.

Case 3. $\det(A) > 0$ *and* $\Delta = 0$. The matrix A has one double real eigenvalue $\lambda = \frac{1}{2}\,\mathrm{tr}(A) \neq 0$.

If A is diagonalisable then the orbits lie on straight lines through $\underline{0}$. If $\mathrm{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable star*. We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\mathrm{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable star*. We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

If A is not diagonalisable then we use its Jordan normal form to understand the behaviour of the orbits. The differential equation in normal form reads

$$\begin{cases} \dot{x} = \lambda\, x + y \\ \dot{y} = \lambda y \end{cases}$$

so that there exists an invariant line, which is generated by the eigenvector of A, and the behaviour of the orbits can be found by looking at the sign of the two components of the vector field. If $\operatorname{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable improper node*. We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\operatorname{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable improper node*. We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

Both $\underline{0}$ being a star and being an improper node are closed properties. An arbitrarily small perturbation can turn the origin into a focus or a node, not changing the stability but the nature of the fixed point.

Case 4. $\det(A) < 0$. The matrix A has a couple of distinct real eigenvalues $\lambda_- < 0 < \lambda_+$.

In this case the orbits are generalised hyperbolae, and the origin is called a *saddle*. It holds $\dim E^u(\underline{0}) = \dim E^s(\underline{0}) = 1$, and none of the orbits outside the eigenspaces approaches the origin as $t \to \pm\infty$. Being a saddle is an open property.

Case 5. $\det(A) = 0$. The matrix A has two real eigenvalues, $\lambda_- = 0$ and $\lambda_+ = \operatorname{tr}(A)$.

If $\operatorname{tr}(A) \neq 0$, then A is diagonalisable and there is a line of fixed points. All the other orbits lie in straight lines which are parallel to the eigenspace of λ_+. If $\operatorname{tr}(A) = 0$ we are reduced to the case of Remark 2.5 up to a change of coordinates, hence either all points are fixed or there is a line of fixed points and all other orbits lie in straight lines which are parallel to the eigenspace of λ_-.

Clearly, the properties of the origin considered in this case are closed and can be changed by arbitrarily small perturbations.

2.2 Stability

Let $\dot{\underline{x}} = F(\underline{x})$ be an ordinary differential equation in $\mathbb{R}^n$ with flow $\phi_t(\cdot)$.

Definition 2.2. A point $\underline{x}$ is *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\underline{x}, \underline{y}) < \delta$ implies $d(\phi_t(\underline{x}), \phi_t(\underline{y})) < \varepsilon$ for all $t \geq 0$.

Remark 2.6. Show that it is necessary to introduce also the notion of orbital stability.

Definition 2.3. A point $\underline{x}$ is *Lyapunov asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that $d(\underline{x}, \underline{y}) < \delta$ implies

$$d(\phi_t(\underline{x}), \phi_t(\underline{y})) \xrightarrow[t \to +\infty]{} 0\,.$$

We call *domain of asymptotic stability of* $\underline{x}$ the set $D(\underline{x})$ of points $\underline{y}$ for which $d(\phi_t(\underline{x}), \phi_t(\underline{y})) \to 0$ as $t \to +\infty$. If $D(\underline{x}) = \mathbb{R}^n$ we say that $\underline{x}$ is *globally Lyapunov asymptotically stable.*

Remark 2.7. If in Definition 2.3 we drop the request that the point $\underline{x}$ is Lyapunov stable, then $\underline{x}$ is called *quasi-asymptotically stable.* In this case there exists a neighbourhood $B_\delta(\underline{x})$ so that $d(\phi_t(\underline{x}), \phi_t(\underline{y})) \to 0$ as $t \to +\infty$ for all $\underline{y} \in B_\delta(\underline{x})$, but the orbits of these points may go arbitrarily far from that of $\underline{x}$ before convergence.

It is particularly important to study the stability of a fixed point $\underline{x}_0$ for which $\phi_t(\underline{x}_0) = \underline{x}_0$ for all t in Definitions 2.2 and 2.3.

Example 2.1. Let us consider the following differential equation in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2+y^2}} \\ \dot{y} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2+y^2}} \end{cases}$$

Using polar coordinates (ρ, θ) as shown in Section 2.4 (see (2.10)) with $x = \rho\cos\theta$, $y = \rho\sin\theta$, we are reduced to the equation

$$\begin{cases} \dot{\rho} = \rho(1 - \rho^2) \\ \dot{\theta} = 1 - \cos\theta \end{cases}$$

It is now easy to determine the phase portrait of the equation and deduce that $(x_0, y_0) = (1, 0)$ is a quasi-asymptotically fixed point, but it is not Lyapunov stable.

One first tool to study the stability of a fixed point is to look at the linearisation of the vector field in the point.

Definition 2.4. A fixed point $\underline{x}_0$ of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic* if all the eigenvalues of the Jacobian matrix $JF(\underline{x}_0)$ have real part different from zero.

Theorem 2.8 (Hartman-Grobman)**.** *Let $\underline{x}_0$ be a hyperbolic fixed point of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. Then there exists a neighbourhood $U(\underline{x}_0)$ and a homeomorphism $h : U(\underline{x}_0) \to \mathbb{R}^n$ which sends orbits of the differential equation $\dot{\underline{x}} = F(\underline{x})$ into orbits of the linear differential equation $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$ without changing their direction of time parametrisation*[2]*. In particular the homeomorphism h leaves invariant the stability properties of the fixed point $\underline{y}_0 = \underline{0}$.*

Theorem 2.8 implies that we can characterise a hyperbolic fixed point $\underline{x}_0$ by looking at the linear system $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$. In particular the qualitative behaviour of the orbits in a neighbourhood of $\underline{x}_0$ coincides with that of the orbits in a neighbourhood of $\underline{y}_0 = \underline{0}$. However, in general, the regularity of h in Theorem 2.8 does not increase by increasing the regularity of a general F. Hence, the "shape" of the orbits may change under the action of h.

The situation is easier in dimension two. If $\underline{x}_0 \in \mathbb{R}^2$ is a hyperbolic fixed point, then $JF(\underline{x}_0)$ is in one of the cases 1-4 excluding case 2 with vanishing trace. If we are not in case 3, the fixed point $\underline{x}_0$ can be characterised like $\underline{y}_0 = \underline{0}$ for $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$. Hence we can talk about stable and unstable nodes, stable and unstable foci, and saddles. See [Gl94, Section 5.2].

We now briefly discuss the problem of the regularity of h for $F \in C^\omega$.

Poincaré's Linearisation Theorem

Let us consider a differential equation $\dot{\underline{x}} = F(\underline{x})$ in $\mathbb{R}^n$ for a real analytic vector field F with a fixed point in $\underline{0}$. We then write

$$F(\underline{x}) = JF(\underline{0})\,\underline{x} + \sum_{\ell \geq r} v_\ell(\underline{x}) \tag{2.1}$$

for some $r \geq 2$, where $v_\ell(\underline{x})$ denotes a homogeneous polynomial of degree ℓ.

Definition 2.5. An n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of complex numbers is *resonant* if there exists $m = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ such that

$$(m, \lambda) := \sum_{k=1}^{n} m_k \lambda_k = \lambda_s$$

for some s. The number $|m| = \sum_{k=1}^n m_k$ is called the *order of the resonance*.

[2] A formal statement is that, if ϕ_t is the flow of the original system $\dot{\underline{x}} = F(\underline{x})$ and ψ_t is the flow of the linear system $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$, then for all $\underline{x} \in U(\underline{x}_0)$ we have $h(\phi_t(\underline{x})) = \psi_t(h(\underline{x}))$ for all $t \in \mathbb{R}$ such that $\phi_t(\underline{x}) \in U(\underline{x}_0)$.

Definition 2.6. The Jacobian matrix $JF(\underline{0})$ is called *resonant* if its eigenvalues form a resonant n-tuple.

Theorem 2.9 (Poincaré linearisation)**.** *If a real analytic vector field F in $\mathbb{R}^n$ satisfies $F(\underline{0}) = \underline{0}$ and $JF(\underline{0})$ is diagonal and not resonant of any order $r \geq 2$, then there exists a formal change of coordinates $\underline{y} = h(\underline{x})$ such that $\dot{\underline{y}} = JF(\underline{0})\underline{y}$.*

Proof. Let the vector field F be written as in (2.1), and with standard multi-vector notations let for $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}_0^n$

$$v_r(\underline{x}) = \sum_{|\mu|=r} a_{\mu,r}\, \underline{x}^\mu$$

be the smallest nonlinear term in F. Morevoer, by assumption we have $JF(\underline{0}) = diag(\lambda_1, \dots, \lambda_n)$.

Let us consider a change of coordinates $\underline{y} = h(\underline{x})$ of the form

$$h(\underline{x}) = \underline{x} + \sum_{|\mu|=r} \beta_\mu\, \underline{x}^\mu\,, \tag{2.2}$$

we show that it is possible to choose $\{\beta_\mu\}$ such that $\dot{\underline{y}} = JF(\underline{0})\underline{y} + O(|y|^{r+1})$. By repeating the argument for all nonlinear terms we prove the theorem.

Let $\underline{y} = h(\underline{x})$ with $h(\underline{x})$ as in (2.2), and let $\tilde{h}(\underline{x}) := h(\underline{x}) - \underline{x}$. Then for all $i = 1, \dots, n$ we have

$$\begin{aligned}\dot{y}_i &= \dot{x}_i + \frac{d}{dt}\tilde{h}_i(\underline{x}) = \dot{x}_i + \sum_{|\mu|=r} \beta_{\mu,i} \left(\sum_{k=1}^n \mu_k \frac{\underline{x}^\mu}{x_k} \dot{x}_k \right) = \\ &= \lambda_i x_i + (v_r(\underline{x}))_i + \sum_{|\mu|=r} \beta_{\mu,i} \left[\sum_{k=1}^n \mu_k \frac{\underline{x}^\mu}{x_k} (\lambda_k x_k + (v_r(\underline{x}))_k) \right] + O(|\underline{x}|^{r+1})\end{aligned}$$

Now, using that we can write

$$x_i = y_i - \tilde{h}_i(\underline{x}) = y_i - \sum_{|\mu|=r} \beta_{\mu,i}\, \underline{y}^\mu + O(|\underline{y}|^{r+1})$$

for all $i = 1, \dots, n$, and recalling the expression for the term $v_r(\underline{x})$, we have

$$\begin{aligned}\dot{y}_i &= \lambda_i y_i - \sum_{|\mu|=r} \lambda_i \beta_{\mu,i} \underline{y}^\mu + \sum_{|\mu|=r} a_{\mu,r,i} \underline{y}^\mu + \sum_{|\mu|=r} \beta_{\mu,i} (\mu, \lambda) \underline{y}^\mu + O(|\underline{y}|^{r+1}) = \\ &= \lambda_i y_i + \sum_{|\mu|=r} \underline{y}^\mu \left[-\lambda_i \beta_{\mu,i} + a_{\mu,r,i} + \beta_{\mu,i}(\mu, \lambda) \right] + O(|\underline{y}|^{r+1})\end{aligned}$$

It is then clear that by choosing

$$\beta_{\mu,i} = \frac{a_{\mu,r,i}}{\lambda_i - (\mu, \lambda)}, \quad \forall i = 1, \ldots, n, \forall \mu \in \mathbb{N}_0^n, |\mu| = r$$

the function defined in (2.2) is the change of variable we were looking for. Note that we can choose the $\{\beta_\mu\}$ as above thanks to the fact that $JF(\underline{0})$ is not resonant of order r. $\square$

If in Theorem 2.9 the Jacobian matrix is not diagonalisable the result holds by more complicated computations.

Definition 2.7. An n-tuple $\lambda = (\lambda_1, \ldots, \lambda_n)$ of complex numbers satisfies the *Siegel condition* if there exist $C, \nu > 0$ such that for all $s = 1, \ldots, n$

$$|\lambda_s - (m, \lambda)| > C\,|m|^{-\nu}$$

for all $m \in \mathbb{N}_0^n$ with $|m| \geq 2$.

Theorem 2.10 (Poincaré-Siegel)**.** *Let F be a real analytic vector field in $\mathbb{R}^n$ such that $F(\underline{0}) = \underline{0}$ and $JF(\underline{0})$ is not resonant of any order $r \geq 2$. Let the n-tuple $\lambda \in \mathbb{C}^n$ of eigenvalues of $JF(\underline{0})$ satisfy one of the following conditions:*
(i) $\Re(\lambda_i) < 0$ for all $i = 1, \ldots, n$;
(ii) $\Re(\lambda_i) > 0$ for all $i = 1, \ldots, n$;
(iii) λ satisfies the Siegel condition.
Then the formal change of coordinates $\underline{y} = h(\underline{x})$ obtained in Theorem 2.9 is real analytic on some neighbourhood of $\underline{0}$.

For the proof we suggest to consult the theory of normal forms in [Ar88].

Example 2.2. We give an easy example to show that assumptions in Theorem 2.10 cannot be relaxed. Consider for $(x, y) \in \mathbb{R}^2$ the system

$$\begin{cases} \dot{x} = x \\ \dot{y} = 2y + x^2 \end{cases}$$

The origin $\underline{0} = (0, 0)$ is the unique fixed point with $JF(0, 0) = diag(1, 2)$. Hence the Jacobian matrix is resonant of order 2. The orbits of the system leave on the line $\{x = 0\}$ and on the curves $\{y = x^2(\log(|x|) + const)\}$, hence they are not analytically conjugate to the orbits of the linearised system.

Lyapunov functions

Given a real C^1 function $V(\underline{x})$, we introduce the notation $\dot{V}(\underline{x})$ for its derivative along a vector field F. Namely

$$\dot{V}(\underline{x}) := \langle \nabla V(\underline{x}) \, , \, F(\underline{x}) \rangle \tag{2.3}$$

Notice that $\dot{V}(\underline{x}) = \frac{d}{dt} V(\phi_t(\underline{x}))|_{t=0}$.

Definition 2.8. Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. A C^1 real function $V : U \to \mathbb{R}$ defined in a neighbourhood U of $\underline{x}_0$ is called a *Lyapunov function for* $\underline{x}_0$ if:

(i) $V(\underline{x}) > V(\underline{x}_0)$ for all $\underline{x} \in U \setminus \{\underline{x}_0\}$;

(ii) $\dot{V}(\underline{x}) \leq 0$ for all $\underline{x} \in U$.

If the function $V : U \to \mathbb{R}$ satisfies (i) and

(ii)' $\dot{V}(\underline{x}) < 0$ for all $\underline{x} \in U \setminus \{\underline{x}_0\}$,

it is called a *strict Lyapunov function for* $\underline{x}_0$.

Theorem 2.11 (First Lyapunov stability theorem)**.** *Let $\underline{x}_0 \in \mathbb{R}^n$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a Lyapunov function for $\underline{x}_0$, then $\underline{x}_0$ is Lyapunov stable.*

Proof. Let $V : U \to \mathbb{R}$ be the Lyapunov function for $\underline{x}_0$. Given $\varepsilon > 0$ such that $B_\varepsilon(\underline{x}_0) \subset U$, we let

$$m := \min_{\partial B_\varepsilon(\underline{x}_0)} V \quad \text{and} \quad S_m := \{\underline{x} \in B_\varepsilon(\underline{x}_0) \, : \, V(\underline{x}) < m\}$$

By definition $V(\underline{x}_0) < m$, hence $\underline{x}_0 \in S_m$. Moreover by continuity there exists $\delta > 0$ such that $B_\delta(\underline{x}_0) \subset S_m$. We now show that if $\underline{y} \in B_\delta(\underline{x}_0)$ then $\phi_t(\underline{y}) \in B_\varepsilon(\underline{x}_0)$ for all $t \geq 0$.

Condition (ii) in Definition 2.8 implies that $V(\phi_t(\underline{y})) \leq V(\underline{y}) < m$ for all $t \geq 0$. We conclude that if there exists $t_0 > 0$ such that $\phi_{t_0}(\underline{y}) \notin B_\varepsilon(\underline{x}_0)$, then by continuity of the flow there exists $t_1 \in (0, t_0)$ such that $\phi_{t_1}(\underline{y}) \in \partial B_\varepsilon(\underline{x}_0)$. This is a contradiction to the definition of m. $\square$

Theorem 2.12 (Second Lyapunov stability theorem)**.** *Let $\underline{x}_0 \in \mathbb{R}^n$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a strict Lyapunov function for $\underline{x}_0$, then $\underline{x}_0$ is Lyapunov asymptotically stable.*

Proof. Let $V : U \to \mathbb{R}$ be the strict Lyapunov function for $\underline{x}_0$. By Theorem 2.11 the fixed point $\underline{x}_0$ is Lyapunov stable. We now need to show that the domain of asymptotic stability of $\underline{x}_0$ contains a ball $B_\delta(\underline{x}_0)$.

Let us fix $\varepsilon > 0$, and let $\delta > 0$ be such that $d(\underline{x}_0, \underline{y}) < \delta$ implies $d(\underline{x}_0, \phi_t(\underline{y})) < \varepsilon$ for all $t \geq 0$. Hence $\mathcal{O}^+(\underline{y}) \subset B_\varepsilon(\underline{x}_0)$ for all $\underline{y} \in B_\delta(\underline{x}_0)$, and by Proposition 1.1 we have that $\omega(\underline{y})$ is a non-empty, compact, invariant subset of $B_\varepsilon(\underline{x}_0)$ for all $\underline{y} \in B_\delta(\underline{x}_0)$.

Let us fix $\underline{y} \in B_\delta(\underline{x}_0)$. Condition (ii)' in Definition 2.8 implies that $V(\phi_t(\underline{y}))$ is a decreasing function of t, hence there exists

$$c := \lim_{t\to+\infty} V(\phi_t(\underline{y}))$$

But $V|_{\omega(\underline{y})} \equiv c$ by continuity, in fact for all $\underline{z} \in \omega(\underline{y})$ we have

$$V(\underline{z}) = \lim_{k\to\infty} V(\phi_{t_k}(\underline{y})) = c$$

where $\{t_k\}_k$ is the diverging sequence such that $\phi_{t_k}(\underline{y}) \to z$ as $k \to \infty$. Finally, since $\omega(\underline{y})$ is invariant, we have $V(\phi_t(\underline{z})) = c$ for all t, which by (2.3) implies $\dot{V}(\underline{z}) = 0$ for all $\underline{z} \in \omega(\underline{y})$. Hence $\omega(\underline{y}) \subset \{\dot{V} \equiv 0\}$, and by condition (ii)' $\omega(\underline{y}) = \{\underline{x}_0\}$.

We have thus proved that $B_\delta(\underline{x}_0) \subset D(\underline{x}_0)$. $\square$

Corollary 2.13 (La Salle's Invariance Principle)**.** *Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a Lyapunov function for $\underline{x}_0$ defined on a neighbourhood U of $\underline{x}_0$, then for all $\underline{y} \in U$ such that $\mathcal{O}^+(\underline{y})$ is contained in U and is bounded, we have $\omega(\underline{y}) \subseteq \{\dot{V} \equiv 0\}$.*

Example 2.3. Let us consider the system in $\mathbb{R}^2$ given by

$$\begin{cases} \dot{x} = y \\ \dot{y} = -y^3 - x - x^3 \end{cases}$$

The point $(0,0)$ is the only fixed point and it is not hyperbolic. Looking for a Lyapunov function of the form $V(x,y) = ax^2 + bx^4 + cy^2$ one finds $\dot{V}(x,y) = 2xy(a-c) + 2x^3y(2b-c) - 2cy^4$. Hence

$$V(x,y) = 2x^2 + x^4 + 2y^2$$

is a Lyapunov function for $(0,0)$, with $\{\dot{V} \equiv 0\} = \{y = 0\}$. Hence V is not a strict Lyapunov function. By Theorem 2.11 we have that $(0,0)$ is Lyapunov stable, and applying Corollary 2.13 we also obtain that there exists $\delta > 0$

such that for all $\underline{y} \in B_\delta((0,0))$ it holds $\omega(\underline{y}) \subset \{y = 0\}$. Moreover, since $\omega(\underline{y})$ is an invariant set and the only invariant subset of $\{y = 0\}$ is $\{(0,0)\}$, we have proved that $(0,0)$ is asymptotically stable.

Theorem 2.14 (Bounding functions)**.** *Let F be a vector field in $\mathbb{R}^n$, and assume that there exist a C^1 real function $V : \mathbb{R}^n \to \mathbb{R}$, a compact set $G \subset \mathbb{R}^n$ and $k \in \mathbb{R}$ such that: (a) $G \subset V_k := \{V < k\}$; (b) there exists $\delta > 0$ such that $\dot{V}(\underline{x}) \leq -\delta$ for all $\underline{x} \in \mathbb{R}^n \setminus G$. Then for all $\underline{x} \in \mathbb{R}^n$ there exists $t_0 \geq 0$ such that $\phi_t(\underline{x}) \in V_k$ for all $t > t_0$.*

Proof. If $\underline{x} \in V_k$ we are done, since by assumption (b) $\dot{V}|_{\partial V_k} < 0$, and we can choose $t_0 = 0$. If $\underline{x} \notin V_k$ and $\phi_t(\underline{x}) \notin V_k$ for all $t > 0$

$$V(\phi_t(\underline{x})) - V(\underline{x}) = \int_0^t \frac{d}{ds} V(\phi_s(\underline{x}))\, ds = \int_0^t \dot{V}(\phi_s(\underline{x}))\, ds \leq -\delta t$$

which implies $V(\phi_t(\underline{x})) < k$ for $t > \frac{V(\underline{x})-k}{\delta}$. Hence we find a contradiction, and we have thus proved that there exists $t_0 > 0$ such that $\phi_{t_0}(\underline{x}) \in V_k$, and as before this implies that $\phi_t(\underline{x}) \in V_k$ for all $t \geq t_0$. □

Example 2.4 (Lorenz equations). Let us consider the system in $\mathbb{R}^3$ given by

$$\begin{cases} \dot{x} = \sigma(-x + y) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

with σ, r, b positive constants. We can apply Theorem 2.14 with

$$G = \left\{(x,y,z) \in \mathbb{R}^3 \,:\, rx^2 + y^2 + b(z-r)^2 < 2br^2\right\}$$

$$V(x,y,z) = \frac{1}{2}\Big(rx^2 + \sigma y^2 + \sigma(z - 2r)^2\Big)$$

and $\delta = \sigma b r^2$.

Using the theory of Lyapunov functions we now give a proof of the asymptotic stability of *sinks*, i.e. hyperbolic fixed points of a C^1 vector field with all eigenvalues of the Jacobian matrix of the field with negative real part.

Corollary 2.15. *Let $\underline{x}_0$ be a hyperbolic fixed point of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, and assume that all the eigenvalues of $JF(\underline{x}_0)$ have negative real part. Then $\underline{x}_0$ is asymptotically stable.*

Proof. Let's assume without loss of generality that $\underline{x}_0 = \underline{0}$, then the vector field F satisfies $F(\underline{0}) = \underline{0}$ and can be written as

$$F(\underline{x}) = JF(\underline{0})\underline{x} + G(\underline{x})$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function satisfying $G(\underline{0}) = \underline{0}$ and $JG(\underline{0}) = 0$.

Let $\lambda_1, \dots, \lambda_k$ be the, not necessarily distinct, negative real eigenvalues of $JF(\underline{0})$, and let $a_j \pm ib_j$, with $j = 1, \dots, \frac{1}{2}(n-k)$, be the, not necessarily distinct, couples of conjugate complex eigenvalues with $a_j < 0$. For simplicity we also assume that $JF(\underline{0})$ is written in Jordan normal form, therefore

$$JF(\underline{0}) = diag\Big(\Lambda_1, \dots, \Lambda_h, B_1, \dots B_m\Big)$$

where the Λ_j's are the Jordan blocks relative to the real eigenvalues, and the B_j's are the Jordan blocks relative to the complex eigenvalues.

Let us consider the following change of variables. For $\varepsilon > 0$ let $\underline{y} = (y_1, \dots, y_n)$ be defined as follows:

- if $(x_m, \dots, x_{m+s-1})$ are the components of $\underline{x}$ corresponding to a Jordan block Λ_j, we let $y_{m+\ell} := \varepsilon^{-\ell} x_{m+\ell}$ for $\ell = 0, \dots, s-1$;
- if $(x_p, \dots, x_{p+2s-1})$ are the components of $\underline{x}$ corresponding to a Jordan block B_j, we let $y_{p+2\ell} := \varepsilon^{-\ell} x_{p+2\ell}$ and $y_{p+2\ell+1} := \varepsilon^{-\ell} x_{p+2\ell+1}$, for $\ell = 0, \dots, s-1$.

Then it is a standard computation to verify that $\underline{y}$ satisfies the ODE

$$\dot{\underline{y}} = A_\varepsilon \underline{y} + \tilde{G}(\underline{y}),$$

with $\tilde{G}(\underline{0}) = \underline{0}$ and $J\tilde{G}(\underline{0}) = 0$ and

$$A_\varepsilon = diag\Big(\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_h, \tilde{B}_1, \dots \tilde{B}_m\Big),$$

where

$$\tilde{\Lambda}_j = \begin{pmatrix} \lambda_j & \varepsilon & 0 & \dots & 0 & 0 \\ 0 & \lambda_j & \varepsilon & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_j & \varepsilon & 0 \\ 0 & \dots & \dots & 0 & \lambda_j & \varepsilon \\ 0 & \dots & \dots & \dots & 0 & \lambda_j \end{pmatrix}$$

and

$$\tilde{B}_j = \begin{pmatrix} R_j & \varepsilon I_2 & 0 & \dots & 0 & 0 \\ 0 & R_j & \varepsilon I_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & R_j & \varepsilon I_2 & 0 \\ 0 & \dots & \dots & 0 & R_j & \varepsilon I_2 \\ 0 & \dots & \dots & \dots & 0 & R_j \end{pmatrix}, \quad \text{with } R_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}$$

and I_2 the 2×2 identity matrix.

We now show that $V(\underline{y}) = \sum_{i=1}^n y_i^2$ is a strict Lyapunov function for $\underline{0}$. It is enough to study the derivative $\dot{V}(\underline{y}) = 2\sum_{i=1}^n y_i \dot{y}_i$.

If $(y_m, \dots, y_{m+s-1})$ are the components of $\underline{y}$ corresponding to a Jordan block $\tilde{\Lambda}_j$ we have

$$\sum_{\ell=0}^{s-1} y_{m+\ell}\,\dot{y}_{m+\ell} = \sum_{\ell=0}^{s-2} y_{m+\ell}(\lambda_j y_{m+\ell} + \varepsilon y_{m+\ell+1}) + \lambda_j y_{m+s-1}^2 + O(|\underline{y}|^3) =$$

$$\le \lambda_j \sum_{\ell=0}^{s-1} y_{m+\ell}^2 + \frac{\varepsilon}{2}(y_m^2 + y_{m+s-1}^2) + \varepsilon \sum_{\ell=1}^{s-2} y_{m+\ell}^2 + O(|\underline{y}|^3) \le$$

$$\le (\lambda_j + \varepsilon) \sum_{\ell=0}^{s-1} y_{m+\ell}^2 + O(|\underline{y}|^3).$$

With an analogous argument, if $(y_m, \dots, y_{m+2s-1})$ are the components of $\underline{y}$ corresponding to a Jordan block $\tilde{B}_j$ we have

$$\sum_{\ell=0}^{s-1}(y_{m+2\ell}\,\dot{y}_{m+2\ell} + y_{m+2\ell+1}\,\dot{y}_{m+2\ell+1}) =$$

$$= \sum_{\ell=0}^{s-2} y_{m+2\ell}\,(a_j y_{m+2\ell} - b_j y_{m+2\ell+1} + \varepsilon y_{m+2\ell+2}) +$$

$$+ \sum_{\ell=0}^{s-2} y_{m+2\ell+1}\,(b_j y_{m+2\ell} + a_j y_{m+2\ell+1} + \varepsilon y_{m+2\ell+3}) +$$

$$+ y_{m+2s-2}\,(a_j y_{m+2s-2} - b_j y_{m+2s-1}) + y_{m+2s-1}\,(b_j y_{m+2s-2} + a_j y_{m+2s-1}) +$$

$$+ O(|\underline{y}|^3) =$$

$$= a_j \sum_{\ell=0}^{s-1}(y_{m+2\ell}^2 + y_{m+2\ell+1}^2) + \varepsilon \sum_{\ell=0}^{s-2}\Big(y_{m+2\ell}\, y_{m+2\ell+2} + y_{m+2\ell+1}\, y_{m+2\ell+3}\Big) +$$

$$+ O(|\underline{y}|^3) \leq$$

$$\leq (a_j + \varepsilon) \sum_{\ell=0}^{s-1}(y_{m+2\ell}^2 + y_{m+2\ell+1}^2) + O(|\underline{y}|^3).$$

If we fix $\varepsilon > 0$ such that $(\lambda_j + \varepsilon) < 0$ and $(a_j + \varepsilon) < 0$ for all eigenvalues of $JF(\underline{0})$, letting $\mu \in \mathbb{R}^-$ satisfy $(\lambda_j + \varepsilon) \leq \mu < 0$ and $(a_j + \varepsilon) \leq \mu < 0$ for all j, we have proved that

$$\dot{V}(\underline{y}) \leq 2\mu|\underline{y}|^2 + O(|\underline{y}|^3).$$

We need to show that there exists $\delta > 0$ such that $\dot{V}(\underline{y}) < 0$ for all $\underline{y} \in B_\delta(\underline{0})$ and $\underline{y} \neq \underline{0}$. By definition of $O(\cdot)$ functions, there exist $c > 0$ and $\tilde{\delta} > 0$ such that

$$O(|\underline{y}|^3) \leq c|\underline{y}|^3\,, \quad \forall\, \underline{y} \in B_{\tilde{\delta}}(\underline{0}).$$

If we choose $\delta = \min\{-\frac{2\mu}{c}, \tilde{\delta}\}$ it follows

$$\dot{V}(\underline{y}) \leq 2\mu|\underline{y}|^2 + c|\underline{y}|^3 = |\underline{y}|^2(2\mu + c|\underline{y}|) < 0, \quad \forall\, \underline{y} \in B_\delta(\underline{0}) \setminus \{\underline{0}\},$$

and the proof is finished. □

2.3 Integrals of motion and invariant sets

Conservative systems and first integrals

Definition 2.9. A C^1 function $I : \mathbb{R}^n \to \mathbb{R}$ is a *first integral* for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if $\dot{I}(\underline{x}) = 0$ for all $\underline{x} \in \mathbb{R}^n$, with $\dot{I}(\underline{x})$ defined as in (2.3).

If $I : \mathbb{R}^n \to \mathbb{R}$ is a first integral for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, then its level sets are invariant for the differential equation $\dot{\underline{x}} = F(\underline{x})$, so that in particular orbits of $\dot{\underline{x}} = F(\underline{x})$ lie in the level sets of I.

An important example of differential equations with a first integral are Hamiltonian systems with Hamiltonian function independent of time.

Definition 2.10. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^1 function and use the notation $(\underline{x}, \underline{y})$ for points in $\mathbb{R}^{2n}$, with $\underline{x}, \underline{y} \in \mathbb{R}^n$. The *Hamiltonian vector field associated to* H is $F_H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given for $i = 1, \ldots, n$, by $(F_H)_i = \partial H/\partial y_i$ and $(F_H)_{(n+i)} = -\partial H/\partial x_i$, and H is called the *Hamiltonian function* of the field. The system of differential equations in $\mathbb{R}^{2n}$ with field F_H is called the *Hamiltonian system of* H.

A particular case are conservative mechanical systems with one degree of freedom, systems which describe for example the motion in $\mathbb{R}$ of a point of mass m under conservative forces. In this case the Hamiltonian function has the form

$$H : \mathbb{R}^2 \to \mathbb{R}\,, \quad H(x,y) = \frac{1}{2m}\, y^2 + W(x) \tag{2.4}$$

where $W(x) \in C^1$ is the potential energy of the system. We recall that in this case the Hamiltonian system associated to H is

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x,y) = \frac{1}{m}\, y \\ \dot{y} = -\frac{\partial H}{\partial x}(x,y) = -W'(x) \end{cases}$$

and corresponds to the second-order differential equation $m\ddot{x} = -W'(x)$.

Proposition 2.16. *A C^1 function H is a first integral for the Hamiltonian vector field F_H.*

Proof. A simple computation gives

$$\dot{H} = \langle \nabla H\,,\, F_H \rangle = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial x_i}\, \frac{\partial H}{\partial y_i} - \frac{\partial H}{\partial y_i}\, \frac{\partial H}{\partial x_i} \Big) \equiv 0.$$

□

Theorem 2.17 (Liouville theorem)**.** *A Hamiltonian system in $\mathbb{R}^{2n}$ with C^2 Hamiltonian function H preserves the $2n$-dimensional Lebesgue measure of the sets.*

Proof. For $A \subset \mathbb{R}^{2n}$ let $\phi_t(A)$ be the evolution of the set at time t, and let m be the $2n$-dimensional Lebesgue measure. Then

$$m(\phi_t(A)) = \int_{\phi_t(A)} 1\, dm = \int_A |\det(J\phi_t)|\, dm.$$

The variation equation of a differential equation shows that $J\phi_t$ satisfies the Cauchy problem

$$\begin{cases} \frac{d}{dt}\, J\phi_t(\underline{x}) = JF_H(\phi_t(\underline{x}))\, J\phi_t(\underline{x}) \\ J\phi_t(\underline{x})|_{t=0} = I \end{cases}$$

where I is the identity matrix. The solution to the previous Cauchy problem is then

$$J\phi_t(\underline{x}) = \exp\Big(\int_0^t JF_H(\phi_s(\underline{x}))\, ds \Big)\, I,$$

and using the identity $\det(\exp(M)) = \exp(\mathrm{tr}(M))$, valid for any finite square matrix M, we obtain

$$\det(J\phi_t(\underline{x})) = \exp\Big(\int_0^t \mathrm{tr}(JF_H(\phi_s(\underline{x})))\,ds\Big).$$

Then

$$m(\phi_t(A)) = \int_A \exp\left(\int_0^t \mathrm{div}(F_H)(\phi_s(\underline{x}))\,ds\right) dm.$$

Since

$$\mathrm{div}(F_H) = \sum_{i=1}^n \Big(\frac{\partial^2 H}{\partial x_i \partial y_i} - \frac{\partial^2 H}{\partial y_i \partial x_i}\Big) \equiv 0,$$

it follows that

$$m(\phi_t(A)) = m(A)\,, \quad \forall\, t \in \mathbb{R}$$

and the proof is finished. □

Corollary 2.18. *A Hamiltonian system in $\mathbb{R}^{2n}$ with C^2 Hamiltonian function H cannot have fixed points which are sinks or sources.*

Let us consider mechanical Hamiltonian systems with one degree of freedom with Hamiltonian function $H(x, y)$ as in (2.4). Applying the general theory of the previous sections and the results in this section, one can easily prove the following characterisation of the fixed points.

Proposition 2.19. *Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function written as in (2.4). Then the fixed points of the associated Hamiltonian system are of the form $(x_0, 0)$ with $W'(x_0) = 0$.*
If $W''(x_0) < 0$ then $(x_0, 0)$ is a hyperbolic fixed point of saddle type, if $W''(x_0) > 0$ it is not hyperbolic and it is a center.
If $W''(x_0) = 0$ the point $(x_0, 0)$ is not hyperbolic and one needs to use the level sets of $H(x, y)$ to study the dynamics in a neighbourhood of the point.

Example 2.5. The Hamiltonian function of a pendulum of mass m and length ℓ in a vertical gravitational field with potential energy $W(h) = mgh$ is

$$H(x, y) = \frac{1}{2m\ell^2}\, y^2 + mg\ell(1 - \cos x)\,,$$

Consider the motion of this pendulum in presence of a constant friction given by $-\mu\, y$, with $\mu \geq 0$.

Example 2.6. Study the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 - \mu\, y \end{cases}$$

with $\mu \in \mathbb{R}$.

Invariant sets

It is in general difficult to find explicit expressions for invariant sets. However, there are particular easy situations. For example, given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \dots, F_n)$, if there exists $c \in \mathbb{R}$ such that $F_i(x_1, \dots, x_{i-1}, c, x_{i+1}, \dots, x_n) = 0$ for all $x_j \in \mathbb{R}$ with $j \neq i$, then the hyperplane $\{x_i = c\}$ is an invariant set. This can be proved by the following method.

Proposition 2.20. *Let $I : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and for $c \in \mathbb{R}$ let $I_c := \{I(\underline{x}) = c\}$ be a non-empty level set of I such that $\nabla I|_{I_c} \not\equiv 0$. The level set I_c is invariant for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if $\dot{I}|_{I_c} \equiv 0$.*

Proof. Let $\underline{x}_0 \in I_c$ such that $\nabla I(\underline{x}_0) \neq 0$. Then there exists a local differentiable change of coordinates $\underline{y} = h(\underline{x})$ such that in a neighbourhood $U(\underline{x}_0)$ we have $I_c \cap U = \{y_n = 0\}$ and let $\underline{x}_0 = (\tilde{\underline{y}}_0, 0)$ with $\tilde{\underline{y}}_0 \in \mathbb{R}^{n-1}$. Hence, in these new coordinates $\nabla I \in \mathrm{Span}\{(0, \dots, 0, 1)\}$ in U.

Then, from $\dot{I}|_{I_c} \equiv 0$, we have that $F_n|_U \equiv 0$. Let $\tilde{F} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be defined as $\tilde{F}(y_1, \dots, y_{n-1}) = (F_1(y_1, \dots, y_{n-1}, 0), \dots, F_{n-1}(y_1, \dots, y_{n-1}, 0))$. Then by the local uniqueness of the solutions to the system $\dot{\underline{x}} = F(\underline{x})$, the solution with initial condition in $\underline{x}_0$ coincides in U with $(\tilde{\phi}_t(\tilde{\underline{y}}_0), 0)$, where $\tilde{\phi}_t$ is the flow of the system $\dot{\tilde{\underline{y}}} = \tilde{F}(\tilde{\underline{y}})$. Hence, the solution is in I_c. This proves the invariance of I_c. $\square$

Example 2.7. Given the system

$$\begin{cases} \dot{x} = x^2 - y - 1 \\ \dot{y} = (x - 2)y \end{cases}$$

the lines $y = 0$, $y = x + 1$ and $y = 3x - 3$ are invariant sets.

Stable and unstable manifolds

An important example of invariant sets is given by the *stable* and *unstable manifolds* of a hyperbolic fixed point.

Definition 2.11. Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with flow $\phi_t(\cdot)$, and let U be a neighbourhood of $\underline{x}_0$. The *local stable manifold* $W^s_{loc}(\underline{x}_0)$ *of* $\underline{x}_0$ *in* U is the set

$$W^s_{loc}(\underline{x}_0) := \{\underline{x} \in U \; : \; \phi_t(\underline{x}) \in U \text{ for all } t \geq 0, \; \phi_t(\underline{x}) \to \underline{x}_0 \text{ as } t \to +\infty\}$$

Analogously, the *local unstable manifold* $W^u_{loc}(\underline{x}_0)$ *of* $\underline{x}_0$ *in* U is the set

$$W^u_{loc}(\underline{x}_0) := \{\underline{x} \in U \; : \; \phi_t(\underline{x}) \in U \text{ for all } t \leq 0, \; \phi_t(\underline{x}) \to \underline{x}_0 \text{ as } t \to -\infty\}$$

Theorem 2.21 (Stable and unstable manifolds)**.** *Let* $\underline{x}_0$ *be a fixed point of a* C^k*,* $k \geq 1$*, vector field* $F : \mathbb{R}^n \to \mathbb{R}^n$ *with flow* $\phi_t(\cdot)$*. Let's assume that* $\underline{x}_0$ *is hyperbolic and let* $E^s(\underline{0})$ *and* $E^u(\underline{0})$ *be the stable and unstable eigenspaces associated to the linear system* $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$*. Then there exists* $\varepsilon > 0$ *such that there exist local stable and unstable manifolds,* $W^s_{loc}(\underline{x}_0)$ *and* $W^u_{loc}(\underline{x}_0)$*, of* $\underline{x}_0$ *in* $B_\varepsilon(\underline{x}_0)$ *with the following properties:*

(i) $W^s_{loc}(\underline{x}_0)$ *and* $W^u_{loc}(\underline{x}_0)$ *are unique in* $B_\varepsilon(\underline{x}_0)$*;*

(ii) $W^s_{loc}(\underline{x}_0)$ *is forward invariant, and* $W^u_{loc}(\underline{x}_0)$ *is backward invariant;*

(iii) $W^s_{loc}(\underline{x}_0)$ *and* $W^u_{loc}(\underline{x}_0)$ *are* C^k *manifolds,* $\dim W^s_{loc}(\underline{x}_0) = \dim E^s(\underline{0})$ *and* $\dim W^u_{loc}(\underline{x}_0) = \dim E^u(\underline{0})$*;*

(iv) $W^s_{loc}(\underline{x}_0)$ *is tangential to* $\underline{x}_0 + E^s(\underline{0})$ *at* $\underline{x}_0$*, and* $W^u_{loc}(\underline{x}_0)$ *is tangential to* $\underline{x}_0 + E^u(\underline{0})$ *at* $\underline{x}_0$*.*

Proof of Theorem 2.21 in $\mathbb{R}^2$ *(see [HSD]).* Without loss of generality, let's assume that the fixed point is $(x_0, y_0) = (0, 0)$. If $\dim E^s = 2$ or $\dim E^u = 2$, the proof is trivial since in these cases either $W^s_{loc}(0, 0)$ or $W^u_{loc}(0, 0)$, respectively, coincide with a ball around $(0, 0)$ and the properties of the statement follow from the Hartman-Grobman Theorem 2.8.

The interesting case is when $\dim E^s = \dim E^u = 1$ and $(0, 0)$ is a saddle. Up to a change of variables, we can assume that the system is written as

$$\begin{cases} \dot{x} = -\lambda\, x + f(x, y) \\ \dot{y} = \mu\, y + g(x, y) \end{cases} \tag{2.5}$$

with $\lambda, \mu > 0$, $f, g \in C^k$ with $f(0,0) = g(0,0) = 0$, and $f, g = O(x^2 + y^2)$ if $k \geq 2$, and $f, g, \partial_x f, \partial_y f, \partial_x g, \partial_y g = o(\sqrt{x^2 + y^2})$ if $k = 1$. Hence, $E^s = \mathrm{Span}\{(1,0)\}$ and $E^u = \mathrm{Span}\{(0,1)\}$.

We give the proof for the local stable manifold, it follows analogously for the local unstable one. For any $\varepsilon > 0$ and $M > 1$, introduce the following notations:

$$\begin{aligned} &D_\varepsilon := \{|x| \leq \varepsilon,\, |y| \leq \varepsilon\}, \quad C_M := \{|x| \geq M\,|y|\}, \\ &S_\varepsilon^\pm := C_M \cap \{x = \pm\varepsilon\}, \quad C_M^\pm := C_M \cap \{x \gtrless 0\}. \end{aligned} \tag{2.6}$$

The proof is divided into different steps.

Step I. There exists $\varepsilon_0 > 0$ such that for all $M > 1$ we have $\dot{x}|_{D_\varepsilon \cap C_M^\pm} \lessgtr 0$.
By assumption, there exists $\varepsilon > 0$ such that

$$|f(x,y)| \leq \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2 + y^2}\,, \quad \forall\,(x,y) \in D_\varepsilon.$$

Then, on $D_\varepsilon \cap C_M^+$, we have

$$\begin{aligned} &\dot{x} = -\lambda\,x + f(x,y) \leq -\lambda\,x + \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2 + y^2} \leq \\ &\quad \leq -\lambda\,x + \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2\left(1 + \frac{1}{M^2}\right)} \leq x\left(-\lambda + \frac{\lambda}{2}\right) = -\frac{\lambda}{2}\,x < 0. \end{aligned}$$

Similarly, on $D_\varepsilon \cap C_M^-$, we have

$$\begin{aligned} &\dot{x} = -\lambda\,x + f(x,y) \geq -\lambda\,x - \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2 + y^2} \geq \\ &\quad \geq -\lambda\,x - \frac{\lambda}{2\sqrt{2}}\,\sqrt{x^2\left(1 + \frac{1}{M^2}\right)} \geq |x|\left(\lambda - \frac{\lambda}{2}\right) = \frac{\lambda}{2}\,|x| > 0. \end{aligned}$$

Step II. For any $M > 1$, there exists $\varepsilon_1 = \varepsilon_1(M) > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ on the boundary of $D_\varepsilon \cap C_M$ the field F points towards the outside of C_M.
First of all, we consider $\varepsilon \in (0, \varepsilon_0)$ with ε_0 from Step I. Let us study the case $x, y > 0$. We have $\dot{x}|_{\partial(D_\varepsilon \cap C_M^+)} < 0$. It is then enough to prove that $\dot{y}|_{\partial(D_\varepsilon \cap C_M^+)} > 0$. Fixed $M > 1$, by assumption, there exists $\varepsilon_1 < \varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_1)$

$$|g(x,y)| \leq \frac{\mu}{2\sqrt{1 + M^2}}\,\sqrt{x^2 + y^2}\,, \quad \forall\,(x,y) \in D_\varepsilon.$$

Then, on $\partial(D_\varepsilon \cap C_M^+)$, we have

$$\dot{y} = \mu\, y + g(x,y) \geq \mu\, y - \frac{\mu}{2\sqrt{1+M^2}}\,\sqrt{x^2+y^2} =$$
$$= \mu\, y - \frac{\mu}{2\sqrt{1+M^2}}\,\sqrt{y^2(M^2+1)} = y\left(\mu - \frac{\mu}{2}\right) = \frac{\mu}{2}\, y > 0.$$

The other cases follow analogously.

Step III. For $\varepsilon \in (0,\varepsilon_1)$, on S_ε^+ there exist non-empty open intervals I_+ and I_- such that for all $(x_0,y_0) \in I_\pm$ the orbit $\phi_t(x_0,y_0)$ intersects $\partial(D_\varepsilon \cap C_M^+) \cap \{y \gtrless 0\}$.
The existence and the properties of the intervals I_+ and I_- follow from Steps I and II, and from the local uniqueness and the continuity with respect to the initial conditions of the solutions to (2.5).

Step IV. There exists $\varepsilon_2 < \varepsilon_1$ such that for $\varepsilon \in (0,\varepsilon_2)$ The set $S_\varepsilon^+ \setminus (I_+ \cup I_-)$ consists of a single point $(\varepsilon, \bar{y}_+(\varepsilon))$.
By the properties of the solutions to (2.5), there exist y_1, y_2 such that

$$S_\varepsilon^+ \setminus (I_+ \cup I_-) = \{(\varepsilon, y)\ : y \in [y_1, y_2]\}\,.$$

We need to show that $y_1 = y_2 = \bar{y}_+$.

Let's assume that $y_1 < y_2$. It is known that multiplying a vector field $F(x,y)$ by a non-vanishing function $h(x,y)$, the orbits of the system do not change but only their time-parametrisation is affected. Let $h(x,y) = 1/(\lambda - f(x,y)/x)$ in $D_\varepsilon \cap C_M$. Then the system in $D_\varepsilon \cap C_M$ becomes

$$\begin{cases} \dot{x} = -x \\ \dot{y} = \frac{\mu\, y + g(x,y)}{\lambda - \frac{f(x,y)}{x}} = \frac{\mu}{\lambda}\, y + \tilde{g}(x,y) \end{cases} \tag{2.7}$$

with $\tilde{g}, \partial_y \tilde{g} = o(\sqrt{x^2+y^2})$. There exists $\varepsilon_2 < \varepsilon_1$ such that for all $M > 1$ and all $\varepsilon \in (0,\varepsilon_2)$ we have

$$\left|\frac{\partial \tilde{g}}{\partial y}(x,y)\right| \leq \frac{\mu}{2\lambda\sqrt{2}}\,\sqrt{x^2+y^2}\,, \quad \forall\,(x,y) \in D_\varepsilon.$$

Then the solutions to (2.7) with initial condition (ε, y) are of the form $(\varepsilon\, e^{-t}, y(t))$. Hence, we can compute the vertical distance between the orbits $\phi_t(\varepsilon, y_1)$ and $\phi_t(\varepsilon, y_2)$ by computing the distance $y_2(t) - y_1(t)$ of the second components of the solutions to (2.7) with initial conditions (ε, y_1)

and (ε, y_2). We have

$$\begin{aligned}\frac{d}{dt}(y_2(t) - y_1(t)) &= \frac{\mu}{\lambda}\, y_2(t) + \tilde{g}(\varepsilon\, e^{-t}, y_2(t)) - \frac{\mu}{\lambda}\, y_1(t) + \tilde{g}(\varepsilon\, e^{-t}, y_1(t)) = \\ &= \frac{\mu}{\lambda}\,(y_2(t) - y_1(t)) + \frac{\partial \tilde{g}}{\partial y}(\varepsilon\, e^{-t}, \xi(t))\,(y_2(t) - y_1(t)) \geq \\ &\geq (y_2(t) - y_1(t))\left(\frac{\mu}{\lambda} - \frac{\mu}{2\lambda\sqrt{2}}\,\varepsilon\, e^{-t}\sqrt{1 + \frac{1}{M^2}}\right) \geq \\ &\geq \frac{\mu}{2\lambda}\,(y_2(t) - y_1(t)).\end{aligned}$$

Hence, $(y_2(t) - y_1(t)) \to +\infty$, which contradicts that the orbits of the set $S_\varepsilon^+ \setminus (I_+ \cup I_-)$ are forward asymptotic to $(0,0)$. We have thus proved that $y_1 = y_2 = \bar{y}_+$.

Conclusion part I.
By Steps I-IV, for all $M > 1$ there exists $\varepsilon_2 = \varepsilon_2(M) > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$ we obtain the existence of a unique point $(\varepsilon, \bar{y}_+(\varepsilon))$ in S_ε^+ whose orbit is forward asymptotic to $(0,0)$. Therefore, fixing a $\bar{M} > 1$ the local stable manifold $W^s_{loc}(0,0)$ in $D_\varepsilon \cap C_M^+$ for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ is given by the forward orbit of the point $(\varepsilon_2(\bar{M}), \bar{y}_+(\varepsilon_2(\bar{M})))$. An analogous argument shows the existence of the local stable manifold $W^s_{loc}(0,0)$ in $D_\varepsilon \cap C_M^-$ for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$.

This shows the uniqueness of $W^s_{loc}(0,0)$, its forward invariance, its regularity since the orbits of a system inherit the regularity of the vector field, and that its dimension is 1. It remains to prove that $W^s_{loc}(0,0)$ is tangent at $(0,0)$ to $E^s = \mathrm{Span}\{(1,0)\}$, that is to the x-axis.

Step V. Fixing a $\bar{M} > 1$, for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ the orbit $\phi_t(\varepsilon, \bar{y}_+(\varepsilon))$ has vanishing angular coefficient as $t \to +\infty$.
Let $(x_\varepsilon(t), y_\varepsilon(t))$ denote the two components of $\phi_t(\varepsilon, \bar{y}_+(\varepsilon))$. We need to show that $y_\varepsilon(t)/x_\varepsilon(t) \to 0$ as $t \to +\infty$.

By the local uniqueness of the solutions to (2.5), for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ the point $(\varepsilon, \bar{y}_+(\varepsilon))$ is in the orbit of $(\varepsilon', \bar{y}_+(\varepsilon'))$ for all $\varepsilon' \in (\varepsilon, \varepsilon_2(\bar{M}))$. This shows that, for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$, for all $t \geq 0$ there exists $\tilde{\varepsilon}(t) < \varepsilon$ such that $(x_\varepsilon(t), y_\varepsilon(t)) = (\tilde{\varepsilon}(t), \bar{y}_+(\tilde{\varepsilon}(t)))$. Hence, as $t \to +\infty$ we have $\tilde{\varepsilon}(t) \to 0^+$ so that $(x_\varepsilon(t), y_\varepsilon(t))$ is in $\cap_{\bar{M} \leq M \leq \tilde{M}(t)} C_M^+$ for some $\tilde{M}(t) \to +\infty$. This shows that $y_\varepsilon(t)/x_\varepsilon(t) \leq 1/\tilde{M}(t) \to 0^+$ as $t \to +\infty$.

Conclusion part II.
Step V concludes the proof of the theorem. $\square$

Given a hyperbolic fixed point $\underline{x}_0$, one can introduce a notion of *global* stable and unstable manifolds. However, these sets in general have weaker properties than the local counterparts.

Definition 2.12. Let $\underline{x}_0$ be a hyperbolic fixed point of a C^k, $k \geq 1$, vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with flow $\phi_t(\cdot)$. The *global stable* and *unstable manifolds* of $\underline{x}_0$ are defined as

$$W^s(\underline{x}_0) := \bigcup_{t \leq 0} \phi_t(W^s_{loc}(\underline{x}_0))\,, \qquad W^u(\underline{x}_0) := \bigcup_{t \geq 0} \phi_t(W^u_{loc}(\underline{x}_0)), \tag{2.8}$$

where $W^{s,u}_{loc}(\underline{x}_0)$ are the local manifolds in $B_\varepsilon(\underline{x}_0)$ for some $\varepsilon > 0$.

It is interesting to analyse the possible intersection of the global stable and unstable manifolds. By the local uniqueness of the solutions to an ODE, the two global manifolds cannot intersect transversally. In $\mathbb{R}^2$ they can coincide or end up at another saddle fixed point, giving rise to a homoclinic or two heteroclinic orbits respectively. In $\mathbb{R}^n$ with $n \geq 3$ more interesting phenomena occurs, and some imply the existence of "chaotic" phenomena.

2.4 Motion in the plane and periodic orbits

In this section we consider a system of differential equations in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{2.9}$$

with C^k, $k \geq 1$, functions $f, g : \mathbb{R}^2 \to \mathbb{R}$. We discuss methods to study the phase space of (2.9) which work in two dimensions.

Polar coordinates

In $\mathbb{R}^2$, it is sometimes easier to study the phase space of a system when using polar coordinates. Let

$$\Omega := \left\{(\rho, \theta) \in \mathbb{R}^2 \,:\, \rho > 0\,,\, 0 \leq \theta \leq 2\pi\right\} / (\{\theta = 0\} = \{\theta = 2\pi\}),$$

that is Ω is a strip in the plane with the upper and the lower boundary identified, hence it is an open cylinder. The map $\psi : \Omega \to \mathbb{R}^2$, $(x, y) = \psi(\rho, \theta)$, with

$$\begin{cases} x(\rho, \theta) = \rho \cos\theta \\ y(\rho, \theta) = \rho \sin\theta \end{cases}$$

is a diffeomorphism from Ω to $\mathbb{R}^2$, with Jacobian $\det J\psi(\rho,\theta) = \rho$. We can then use ψ and its inverse to push a vector field $F(x,y) = (f(x,y), g(x,y))$ back to a vector field on Ω. An easy computation shows that the system (2.9) when written in polar coordinates reads

$$\begin{cases} \dot\rho = f(\rho\cos\theta, \rho\sin\theta)\,\cos\theta + g(\rho\cos\theta, \rho\sin\theta)\,\sin\theta \\ \dot\theta = \frac{g(\rho\cos\theta,\rho\sin\theta)\,\cos\theta - f(\rho\cos\theta,\rho\sin\theta)\,\sin\theta}{\rho} \end{cases} \tag{2.10}$$

for all $(\rho,\theta) \in \Omega$. In general one should not expect that the vector field in polar coordinates can be continuously extended to the boundary $\{\rho = 0\}$ of Ω. This is true only under particular conditions on the functions f, g.

An important application of the use of polar coordinates is the identification of circular periodic orbits. Using Proposition 2.20 one can show that

Proposition 2.22. *If there exists $\rho_0 > 0$ such that*

$$f(\rho_0\cos\theta, \rho_0\sin\theta)\,\cos\theta + g(\rho_0\cos\theta, \rho_0\sin\theta)\,\sin\theta = 0\,, \quad \forall\,\theta \in [0, 2\pi]$$

and $\dot\theta \neq 0$ for all $(\rho,\theta) \in \{\rho = \rho_0\}$, then the set

$$\Gamma = \{\rho = \rho_0\} = \left\{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = \rho_0^2\right\}$$

is a periodic orbit.

Isoclines

Here we introduce a method to find an analytic expression for the orbits of (2.9) in special situations.

Proposition 2.23. *Let (x_0, y_0) be a non-fixed point for (2.9). Then there exists a neighbourhood $U(x_0,y_0)$ such that the set $\mathcal{O}(x_0,y_0) \cap U$, that is the orbit of (x_0,y_0) in U, is the graph of a function.*
In particular, if $f(x_0,y_0) \neq 0$ there exist $\varepsilon > 0$ and a C^k function $h : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ such that

$$\mathcal{O}(x_0,y_0) \cap \Big((x_0 - \varepsilon, x_0 + \varepsilon) \times \mathbb{R}\Big) = \{(x, h(x)) \,:\, x \in (x_0 - \varepsilon, x_0 + \varepsilon)\}\,,$$

and $h(x)$ satisfies the Cauchy system

$$\begin{cases} \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \\ y(x_0) = y_0 \\ x \in (x_0 - \varepsilon, x_0 + \varepsilon) \end{cases} \tag{2.11}$$

Instead, if $g(x_0, y_0) \neq 0$ the analogous statement holds by interchanging the roles of x and y and of f and g.

Proof. If $f(x_0, y_0) \neq 0$ there exists $\varepsilon > 0$ such that $f(x, y) \neq 0$ for all $(x, y) \in B_{2\varepsilon}(x_0, y_0)$. Let $h(x)$ be a solution to system (2.11) and define the C^k function $I(x, y) = y - h(x)$ on $\{x \in (x_0 - \varepsilon, x_0 + \varepsilon)\} \cap B_{2\varepsilon}(x_0, y_0)$. Then with respect to system (2.9)

$$\begin{aligned}\dot{I}|_{\{I=0\}} &= \big(\dot{y} - h'(x)\,\dot{x}\big)\,|_{\{I=0\}} = \big(g(x, y) - h'(x)\,f(x, y)\big)\,|_{y=h(x)} = \\ &= g(x, h(x)) - h'(x)\,f(x, h(x)) \equiv 0.\end{aligned}$$

Therefore $I_0 := \{y = h(x)\}$ is an invariant set in a neighbourhood $U(x_0, y_0)$ containing (x_0, y_0). Then $I_0 = \mathcal{O}(x_0, y_0) \cap U$, and the proposition is proved.

The analogous argument works if $g(x_0, y_0) \neq 0$. $\square$

The solutions to system (2.11) are called *isoclines* for (2.9).

<u>*Example*</u> 2.8 (Predator-prey Lotka-Volterra models). We apply the method of finding isoclines to prove the existence of periodic orbits in a predator-prey Lotka-Volterra system. Let $x, y \in \mathbb{R}_0^+$ denote the population of two species in a predator-prey relationship. The population x predates on the population y, hence the system of differential equations for x and y is of the form

$$\begin{cases} \dot{x} = x\,(-A + b_1\,y) \\ \dot{y} = y\,(B - b_2\,x) \end{cases} \tag{2.12}$$

with $A, B, b_1, b_2 > 0$. The system has two fixed points, $P_0 = (0, 0)$ and $P_1 = (B/b_2\,,\, A/b_1)$. The point P_0 is hyperbolic and it is a saddle with stable and unstable manifolds given by the x and y axis respectively, whereas the point P_1 is not hyperbolic being a center.

Let us find the isoclines of (2.12). When $x_0 \neq 0$ and $y_0 \neq A/b_1$ we can write

$$\begin{cases} \frac{dy}{dx} = \frac{y\,(B - b_2\,x)}{x\,(-A + b_1\,y)} \\ y(x_0) = y_0 \end{cases}$$

which has a local solution given implicitly by the equality

$$\int_{y_0}^{y} \frac{-A + b_1\,s}{s}\,ds = \int_{x_0}^{x} \frac{B - b_2\,t}{t}\,dt \quad \Leftrightarrow \quad I(x, y) = I(x_0, y_0)$$

where

$$I(x, y) := A\,\log y + B\,\log x - b_1\,y - b_2\,x\,.$$

We have thus found that $I(x, y)$ is a first integral for (2.12), hence the orbits lie on its level sets. Then, it is immediate to find that $I(x, y)$ has a point of global minimum at P_1, therefore the levels sets $\{I(x, y) = c\}$ are closed curves for c bigger than $I(P_1)$ but sufficiently close to it. Hence, the orbits on these level sets are periodic[3].

The field and the symmetries of the system

Here we introduce two ideas to draw the phase portrait of a system. Both ideas work in all dimensions but are particularly simple to apply in the two dimensional case.

The first idea uses the property of the field to be tangent to the orbits of a system. Therefore, in principle, one can obtain the orbits of a system simply by drawing the field in all the points of the phase space. In practice, it is useful to draw the behaviour of the field on some curves. For example, it is a good idea to draw the lines on which the single components of the field vanish (the intersection of these lines give the fixed points) and to obtain the direction of the field in all the regions of the phase space between these lines.

A more theoretical idea to apply is to look for symmetries of the system. Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, the associated system $\dot{\underline{x}} = F(\underline{x})$, and a diffeomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$, we give the following definition.

Definition 2.13. Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ and a diffeomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$, we say that the system $\dot{\underline{x}} = F(\underline{x})$ is *symmetric* with respect to S if

$$d_{\underline{x}} S(F(\underline{x})) = \pm F(S(\underline{x}))\,, \quad \forall\, \underline{x} \in \mathbb{R}^n\,.$$

A simple case is the case of systems symmetric with respect to linear transformations. That is there exists an invertible matrix $S \in M(n \times n, \mathbb{R})$ such that $S\,F(\underline{x}) = \pm F(S\,\underline{x})$.

Proposition 2.24. *If the system* (2.9) *in* $\mathbb{R}^2$ *is symmetric with respect to a diffeomorphism* S*, given a trajectory* $(x(t), y(t))$ *of the system, the curve* $(\tilde{x}(t), \tilde{y}(t))$ *defined as*

$$(\tilde{x}(t), \tilde{y}(t)) = \begin{cases} S(x(t), y(t))\,, & \textit{if } d_{\underline{x}} S(F(\underline{x})) = F(S(\underline{x})), \\ S(x(-t), y(-t))\,, & \textit{if } d_{\underline{x}} S(F(\underline{x})) = -F(S(\underline{x})), \end{cases}$$

is a solution to (2.9).

[3] It can be proved that all level sets are closed curves, therefore all orbits different from the axes and the fixed points are periodic.

Proof. It is enough to compute $(\dot{\tilde{x}}(t), \dot{\tilde{y}}(t))$. $\square$

Example 2.9. We show how the proposition works in two easy cases. Let us consider the system (2.9) with the assumption that $f(-x,-y) = -f(x,y)$ and $g(-x,-y) = -g(x,y)$. The field $F(x,y) = (f(x,y), g(x,y))$ satisfies $F(-x,-y) = -F(x,y)$, hence it is symmetric with respect to the linear transformation $S(x,y) = (-x,-y)$ and

$$d_{(x,y)}S(F(x,y)) = -F(x,y) = F(S(x,y)).$$

Then, given a trajectory $(x(t), y(t))$ of the system, we show that another trajectory is given by $(\tilde{x}(t), \tilde{y}(t)) = (-x(t), -y(t))$. Indeed, we have

$$\dot{\tilde{x}}(t) = -\dot{x}(t) = -f(x(t), y(t)) = f(-x(t), -y(t)) = f(\tilde{x}(t), \tilde{y}(t)),$$

$$\dot{\tilde{y}}(t) = -\dot{y}(t) = -g(x(t), y(t)) = g(-x(t), -y(t)) = g(\tilde{x}(t), \tilde{y}(t)).$$

The other case considered in the proposition is obtained for Hamiltonian systems in $\mathbb{R}^2$ with Hamiltonian function of the form (2.4). In this case the field is $F(x,y) = (y, -W'(x))$ and the system is symmetric with respect to the linear transformation $S(x,y) = (x,-y)$ since

$$d_{(x,y)}S(F(x,y)) = (y, W'(x)) = -F(x,-y) = -F(S(x,y)).$$

Then, given a trajectory $(x(t), y(t))$ of the system, we show that another trajectory is given by $(\tilde{x}(t), \tilde{y}(t)) = (x(-t), -y(-t))$. Indeed,

$$\dot{\tilde{x}}(t) = -\dot{x}(-t) = -y(-t) = \tilde{y}(t),$$

$$\dot{\tilde{y}}(t) = \dot{y}(-t) = -W'(x(-t)) = -W'(\tilde{x}(t)).$$

Periodic orbits: non-existence

We describe two methods to prove non-existence of periodic orbits in a region of the phase space. The first is of pure topological nature and the second uses the analytical nature of the differential equation (2.9).

Definition 2.14. Let $\Gamma \subset \mathbb{R}^2$ be a simple closed curve. Given a vector field $F(x,y) = (f(x,y), g(x,y))$ without fixed points on Γ, the *Poincaré index of* Γ, denoted by $I_F(\Gamma)$, is the number of turns that F makes counterclockwise as a point goes round Γ. It can be computed as

$$I_F(\Gamma) := \frac{1}{2\pi} \int_\Gamma d\left(\arctan \frac{g}{f}\right) = \frac{1}{2\pi} \int_\Gamma \frac{f\,dg - g\,df}{f^2 + g^2}$$

Proposition 2.25. *Given a vector field F on $\mathbb{R}^2$, the Poincaré index of a curve has the following properties:*

(i) let $t \mapsto \Gamma_t$ be a continuous family of simple closed curves, then $I_F(\Gamma_t)$ is constant as long as no Γ_t contains a fixed point of F;

(ii) let Γ be a simple closed curve not containing fixed points of F which can be written as $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 and Γ_2 are two simple closed curves not containing fixed points of F. Then $I_F(\Gamma) = I_F(\Gamma_1) + I_F(\Gamma_2)$;

(iii) if Γ is a periodic orbit then $I_F(\Gamma) = +1$.

Definition 2.15. Let (x_0, y_0) be an isolated fixed point of a vector field F. The *Poincaré index of* (x_0, y_0), $I_F(x_0, y_0)$, is the Poincaré index of any simple closed curve Γ encircling (x_0, y_0) and no other fixed point of F.

Proposition 2.26. *Let (x_0, y_0) be a fixed point of a C^1 vector field F on $\mathbb{R}^2$ with $\det(JF(x_0, y_0)) \neq 0$. Then:*

(i) if (x_0, y_0) is a node, a star, an improper node, a focus or a centre, then $I_F(x_0, y_0) = +1$;

(ii) if (x_0, y_0) is a saddle, then $I_F(x_0, y_0) = -1$.

Putting together Propositions 2.25 and 2.26, we obtain information on regions of a phase space where a periodic orbit may exist or not. For example, it may not exist a periodic orbit encircling only a saddle. Each periodic orbit has to encircle sets of isolated fixed points for which the sum of their Poincaré indices is +1.

Example 2.10. Let us consider the system

$$\begin{cases} \dot{x} = x \\ \dot{y} = y^2 \end{cases}$$

then $I_F(0, 0) = 0$.

Example 2.11. Let us consider the system

$$\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = 2xy \end{cases}$$

then $I_F(0, 0) = 2$.

Proposition 2.27 (Curl method)**.** *Let $U \subset \mathbb{R}^2$ be a simply connected open set and assume that the vector field $F(x,y) = (f(x,y), g(x,y))$ satisfies*

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial g}{\partial x}(x,y)\,, \quad \forall\,(x,y) \in U$$

Then in U there exist no periodic orbits for the vector field F.

Proof. Let $\Gamma \subset U$ be a periodic orbit of period T parametrised by the solution $\gamma(t)$ of the Cauchy problem

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \\ (x(0), y(0)) = \gamma(0) \end{cases}$$

Then $\gamma(T) = \gamma(0)$ and $\gamma'(t) = F(\gamma(t))$ for all $t \in \mathbb{R}$.

By assumption and Poincaré's lemma, the vector field F is conservative in U, that is there exists a C^1 function $h : U \to \mathbb{R}$ such that $F = \nabla h$. Then

$$0 = h(\gamma(T)) - h(\gamma(0)) = \int_0^T \frac{d}{dt}\,(h\circ\gamma)(t)\,dt = \int_0^T \big\langle \nabla h(\gamma(t)), \gamma'(t)\big\rangle\; dt =$$

$$= \int_0^T \langle F(\gamma(t)), F(\gamma(t))\rangle\; dt = \int_0^T \|F(\gamma(t))\|^2\,dt$$

which is a contradiction because $\|F(\gamma(t))\| \neq 0$ for all t. □

Remark 2.28. The curl method can be easily extended to a differential equation in $\mathbb{R}^n$ with vector field F. By repeating the last part of the proof of Proposition 2.27 one can show that

Proposition 2.29 (Gradient systems)*. If there exists $h : \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla h$, then there are no periodic orbits for the differential equations $\dot{\underline{x}} = F(\underline{x})$.*

Proposition 2.30 (Bendixson-Dulac method)**.** *Let $U \subset \mathbb{R}^2$ be a simply connected open set and assume that there exists a C^1 function $\rho : U \to \mathbb{R}$ such that for the vector field $F(x,y) = (f(x,y), g(x,y))$ it holds*

$$\frac{\partial(\rho\cdot f)}{\partial x}(x,y) + \frac{\partial(\rho\cdot g)}{\partial y}(x,y) > 0 \;(\text{or} < 0)\,, \quad \forall\,(x,y) \in U$$

Then in U there exist no periodic orbits for the vector field F.

Proof. Let $\Gamma \subset U$ be a periodic orbit of period T and let A be the region enclosed by Γ. Then applying Gauss-Green Theorem

$$0 < \iint_A \left(\frac{\partial(\rho \cdot f)}{\partial x}(x,y) + \frac{\partial(\rho \cdot g)}{\partial y}(x,y) \right) dxdy = \int_\Gamma (-\rho\, g\, dx + \rho\, f\, dy) =$$

$$= \int_0^T \rho(x(t),y(t))\, (-g(x(t),y(t))\, \dot{x}(t) + f(x(t),y(t))\, \dot{y}(t)) = 0$$

where we have used that $\Gamma = (x(t),y(t))$ for $t \in [0,T]$ and $(x(t),y(t))$ is a solution of the differential equation associated to the vector field F. □

Example 2.12 (Species in competition). In Example 2.8 we have shown that predator-prey Lotka-Volterra models admit periodic orbits. Now we show that there are no periodic orbits in a Lotka-Volterra model for species in competition. Let $x, y \in \mathbb{R}_0^+$ denote the population of two species in competition for the same resources on a finite environment. The system of differential equations for x and y is of the form

$$\begin{cases} \dot{x} = x\,(A - a_1\, x - b_1\, y) = f(x,y) \\ \dot{y} = y\,(B - b_2\, x - a_2\, y) = g(x,y) \end{cases} \tag{2.13}$$

with $A, B, a_1, a_2, b_1, b_2 > 0$.

The axes are invariant sets, hence the simply connected set

$$U = \{(x,y) \in \mathbb{R}^2 \,:\, x > 0,\, y > 0\}$$

is also invariant. Consider the C^1 function $\rho(x,y) = 1/(xy)$ on U. We have

$$\frac{\partial(\rho \cdot f)}{\partial x}(x,y) + \frac{\partial(\rho \cdot g)}{\partial y}(x,y) = -\frac{a_1}{y} - \frac{a_2}{x} < 0\,, \quad \forall\,(x,y) \in U.$$

Hence, by Proposition 2.30, there are no periodic orbits in U.

Remark 2.31. The Bendixson-Dulac method uses the divergence of a vector field. For differential equations in $\mathbb{R}^n$, $n \geq 3$, it gives different information.

Proposition 2.32. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field such that there exists a constant $k > 0$ for which $div(F)(\underline{x}) \leq -k$ for all $\underline{x} \in \mathbb{R}^n$. Then the flow associated to F contracts the volumes.*

Proof. For $A \subset \mathbb{R}^n$ let $\phi_t(A)$ be the evolution of the set at time t, and let m be the n-dimensional Lebesgue measure. By applying the same ideas in the proof of Liouville Theorem 2.17, we obtain

$$\mathrm{vol}(\phi_t(A)) = \int_A \exp\left(\int_0^t \mathrm{div}(F)(\phi_s(\underline{x}))\, ds \right) dm.$$

If $\mathrm{div}(F)(\underline{x}) \leq -k$ for all $\underline{x} \in \mathbb{R}^n$ then

$$\mathrm{vol}(\phi_t(A)) \leq e^{-kt}\,\mathrm{vol}(A)\,, \quad \forall\, t \geq 0,$$

and the proof is finished. $\square$

Periodic orbits: existence in general

Theorem 2.33 (Poincaré - Bendixson)**.** *Let F be a C^1 vector field in $\mathbb{R}^2$, and assume that there exists a non-empty region $D \subset \mathbb{R}^2$ which is compact and does not contain fixed points of F. If for some $\underline{x}_0$ there exists t_0 such that $\phi_t(\underline{x}_0) \in D$ for all $t \geq t_0$, then there exists a periodic orbit $\Gamma \subset D$ and $\Gamma = \omega(\underline{x}_0)$.*

For the proof we need some preliminaries. Given the differential equation $\dot{\underline{x}} = F(\underline{x})$ in $\mathbb{R}^2$ with $F \in C^1$ and any non-fixed point $\underline{y}$ of F, we call *transversal line at $\underline{y}$* the line $\ell(\underline{y})$ which is the image of the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $\gamma(u) = \underline{y} + u\underline{v}$, where $\underline{v}$ is a vector applied at $\underline{y}$ which satisfies $\big\langle \underline{v}, F(\underline{y}) \big\rangle = 0$.

Definition 2.16. Given a non-fixed point $\underline{y}$ of F and a constant $k \in [0, 1)$, we call *k-wide local section at $\underline{y}$* the set $S_k(\underline{y})$ obtained by taking the connected component containing $\underline{y}$ of the set of points $\underline{z} \in \ell(\underline{y})$ for which $|\sin(\widehat{\underline{v}F(\underline{z})})| > k$.

The k-wide local section at $\underline{y}$ is non-empty since $\underline{y} \in S_k(\underline{y})$, and there exists $\varepsilon > 0$ such that $\gamma(-\varepsilon, \varepsilon) \subseteq S_k(\underline{y})$.

Proposition 2.34 (Local rectifiability of a vector field)**.** *Given a C^1 vector field F in $\mathbb{R}^2$, a non-fixed point $\underline{y}$ of F, and a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, there exists a diffeomorphisms $\psi : U(\underline{0}) \to V(\underline{y})$ which maps horizontal lines into the orbits of $\dot{\underline{x}} = F(\underline{x})$ passing through $S_k(\underline{y})$. That is $\psi(s, u) = \phi_s(\gamma(u))$ for all $(s, u) \in U(\underline{0})$.*

Applying Proposition 2.34, let $\sigma > 0$ and $N_\sigma := \{(s, u) \in U(\underline{0}) \,:\, |s| < \sigma\}$. Then we call *$\sigma$-rectangle of flux in $\underline{y}$* the set $\mathcal{N}_\sigma := \psi(N_\sigma)$. Then for each $\underline{z} \in \mathcal{N}_\sigma$ there exists a unique $s \in (-\sigma, \sigma)$ such that $\phi_s(\underline{z}) \in S_k(\underline{y})$.

Proposition 2.35. *Given a C^1 vector field F in $\mathbb{R}^2$, a non-fixed point $\underline{y}$ of F, and a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, let $\underline{z}$ be a point such that $\underline{y} = \phi_{t_0}(\underline{z})$ for some t_0. Then there exist $\varepsilon > 0$ and a continuous function $\tau : B_\varepsilon(\underline{z}) \to \mathbb{R}$ such that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{z})$.*

Proof. Let us define the function $p : \mathbb{R}^2 \to \mathbb{R}$ by $p(\underline{x}) = \langle \underline{x}, F(\underline{y}) \rangle$. We notice that $p(\underline{x}) = p(\underline{y})$ if and only if $\underline{x} \in \ell(\underline{y})$, in fact if $\underline{x} = \underline{y} + \underline{w}$ then

$$p(\underline{x}) = p(\underline{y}) + p(\underline{w}) = p(\underline{y}) \quad \Leftrightarrow \quad \langle \underline{w}, F(\underline{y}) \rangle = 0$$

Let then consider the regular function $G : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by $G(\underline{x}, t) = p(\phi_t(\underline{x}))$. Then by definition $G(\underline{z}, t_0) = p(\phi_{t_0}(\underline{z})) = p(\underline{y})$ and

$$\frac{\partial G}{\partial t}(\underline{z}, t_0) = p(\dot{\phi}_t(\underline{x}))\Big|_{\underline{x}=\underline{z}, t=t_0} = p(F(\phi_{t_0}(\underline{z}))) = p(F(\underline{y})) = \|F(\underline{y})\|^2 \neq 0$$

Hence we can apply the Implicit Function Theorem to G at $(\underline{z}, t_0)$ and prove the existence of $\varepsilon > 0$ and $\delta > 0$, and of a continuous function $\tau : B_\varepsilon(\underline{z}) \to (t_0 - \delta, t_0 + \delta)$ such that

$$p(\underline{y}) = G(\underline{x}, \tau(\underline{x})) = p(\phi_{\tau(\underline{x})}(\underline{x})) \qquad \forall\, \underline{x} \in B_\varepsilon(\underline{z})$$

It follows that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{z})$. □

We are now ready to prove Poincaré-Bendixson Theorem.

Proof of Theorem 2.33. Choose $\underline{x}_0 \in \mathbb{R}^2$ such that there exists t_0 for which $\phi_t(\underline{x}_0) \in D$ for all $t \geq t_0$. By Proposition 1.1, the set $\omega(\underline{x}_0) \subset D$ is non-empty, compact, and invariant. For any $\underline{x} \in \omega(\underline{x}_0)$ we show that $\mathcal{O}(\underline{x})$ is a periodic orbit Γ, and that $\Gamma = \omega(\underline{x}_0)$.

Fix $\underline{x} \in \omega(\underline{x}_0)$, and let $\underline{y} \in \omega(\underline{x}) \subset \omega(\underline{x}_0)$, which is not a fixed point by assumption. Consider a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, and a σ-rectangle of flux $\mathcal{N}_\sigma$ in $\underline{y}$.

Lemma 2.36. *The forward orbit of $\underline{x}$ intersects $S_k(\underline{y})$ exactly once.*

Proof. Since $\underline{y} \in \omega(\underline{x})$, there exists a point of $\mathcal{O}^+(\underline{x})$ in $\mathcal{N}_\sigma$, hence $\mathcal{O}^+(\underline{x}) \cap S_k(\underline{y}) \neq \emptyset$. Let's assume by contradiction that there exist $\underline{x}_1, \underline{x}_2 \in \mathcal{O}^+(\underline{x}) \cap S_k(\underline{y})$ with $\underline{x}_1 \neq \underline{x}_2$. Since $\underline{x} \in \omega(\underline{x}_0)$, also $\underline{x}_1, \underline{x}_2 \in \omega(\underline{x}_0)$ by invariance of the omega limit. Hence if $\mathcal{N}_\sigma(\underline{x}_1)$ and $\mathcal{N}_\sigma(\underline{x}_2)$ are disjoint σ-rectangles of flux, the forward orbit of $\underline{x}_0$ has countable points both in $\mathcal{N}_\sigma(\underline{x}_1)$ and in $\mathcal{N}_\sigma(\underline{x}_2)$. By the properties of the rectangles of flux, this implies that $\mathcal{O}^+(\underline{x}_0)$ intersects $S_k(\underline{y})$ countable many times, alternatively close to $\underline{x}_1$ and to $\underline{x}_2$. We now show that this is not possible.

Let us denote by $\{\underline{z}_1, \underline{z}_2, \dots\}$ the points in $\mathcal{O}^+(\underline{x}_0) \cap S_k(\underline{y})$ cronologically ordered, that is $\underline{z}_1 = \phi_{t_1}(\underline{x}_0)$, $\underline{z}_2 = \phi_{t_2}(\underline{x}_0)$, and so on, with $t_1 < t_2 < \dots$. Given three points $\underline{z}_{n-1}, \underline{z}_n, \underline{z}_{n+1}$ and an ordering on $S_k(\underline{y})$ it must hold $\underline{z}_{n-1} < \underline{z}_n < \underline{z}_{n+1}$ or $\underline{z}_{n+1} < \underline{z}_n < \underline{z}_{n-1}$. Indeed let Σ denotes the

Jordan curve given by the segment $\overline{\underline{z}_{n-1}\underline{z}_n}$ and the orbit $\cup_{t_{n-1}\le t\le t_n}\phi_t(\underline{x}_0)$, and let R be the region bounded by Σ. Then $\phi_t(\underline{x}_0) \in R$ for all $t > t_n$, because it cannot intersect any part of ∂R. It cannot intersect the orbit $\cup_{t_{n-1}\le t\le t_n}\phi_t(\underline{x}_0)$ by the uniqueness of solutions of a differential equation, and it cannot intersect the segment $\overline{\underline{z}_{n-1}\underline{z}_n}$ which is in $S_k(\underline{y})$, because the vector field points in the same direction in all the points of a local section. It follows that $\underline{z}_{n+1} \in R$ and it lies on the other side of $\underline{z}_{n-1}$ with respect to $\underline{z}_n$. It follows that the countable intersections of $\mathcal{O}^+(\underline{x}_0)$ with $S_k(\underline{y})$ must be ordered, so cannot be alternatively close to $\underline{x}_1$ and to $\underline{x}_2$. This shows that the forward orbit of $\underline{x}$ intersects $S_k(\underline{y})$ exactly once. $\square$

We have thus proved that $\mathcal{O}^+(\underline{x}) \cap S_k(\underline{y}) = \{\phi_{\bar{t}}(\underline{x})\}$. Since $\underline{y} \in \omega(\underline{x})$ there is a sequence $\{t_m\}$ such that $\phi_{t_m}(\underline{x}) \to \underline{y}$, hence for m big enough $\phi_{t_m}(\underline{x}) \in \mathcal{N}_\sigma$. It follows that for m big enough, there exist $\tau_m \in \mathbb{R}$ such that $\phi_{t_m+\tau_m}(\underline{x}) \in S_k(\underline{y})$ for all m, hence $\phi_{t_m+\tau_m}(\underline{x}) = \phi_{\bar{t}}(\underline{x})$ for all m. It follows that there exists $T > 0$ such that $\phi_T(\underline{x}) = \underline{x}$. We have thus proved that $\mathcal{O}(\underline{x})$ is a periodic orbit Γ.

It remains to show that $\Gamma = \omega(\underline{x}_0)$. By invariance of the omega limit $\Gamma \subset \omega(\underline{x}_0)$. Let now $\underline{y} \in \Gamma$ and consider a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, and a σ-rectangle of flux $\mathcal{N}_\sigma$ in $\underline{y}$. As discussed above, there exists a sequence $\{t_m\}$ such that $\phi_{t_m}(\underline{x}_0) \to \underline{y}$ and $\phi_{t_m}(\underline{x}_0) \in S_k(\underline{y})$, with $\phi_t(\underline{x}_0) \notin S_k(\underline{y})$ for $t \in (t_m, t_{m+1})$ for all m. Since $\phi_T(\underline{y}) = \underline{y}$, we can apply Proposition 2.35 and find $\varepsilon > 0$, $\delta > 0$, and a continuous function $\tau : B_\varepsilon(\underline{y}) \to (T-\delta, T+\delta)$ such that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{y})$, and $\tau(\underline{y}) = T$. Hence, choosing $\tilde{\varepsilon} < \varepsilon$ if necessary, we have $\phi_T(\underline{x}) \in \mathcal{N}_\sigma$ for all $\underline{x} \in B_{\tilde{\varepsilon}}(\underline{y})$. Since for m big enough $\phi_{t_m}(\underline{x}_0) \in B_{\tilde{\varepsilon}}(\underline{y})$, it follows that $\phi_T(\phi_{t_m}(\underline{x}_0)) = \phi_{T+t_m}(\underline{x}_0) \in \mathcal{N}_\sigma$, and there exists $s_m \in (-\sigma, \sigma)$ such that $\phi_{T+t_m+s_m}(\underline{x}_0) \in S_k(\underline{y})$. Since $\phi_t(\underline{x}_0) \notin S_k(\underline{y})$ for $t \in (t_m, t_{m+1})$, it must hold $t_{m+1} = T + t_m + s_m$, hence $t_{m+1} - t_m \le T + \sigma$ for all m big enough.

We now consider a fixed $\eta > 0$. By continuity of the flux ϕ_t, there exists $\delta > 0$ such that if $d(\underline{z}_1, \underline{z}_2) < \delta$ then $d(\phi_t(\underline{z}_1), \phi_t(\underline{z}_2)) < \eta$ for all $t \in (-T-\sigma, T+\sigma)$. Hence, for m big enough such that $d(\phi_{t_m}(\underline{x}_0), \underline{y}) < \delta$, we have

$$d\Big(\phi_t(\phi_{t_m}(\underline{x}_0)), \phi_t(\underline{y})\Big) < \eta \qquad \forall\, t \in (-T-\sigma, T+\sigma)$$

Since $\underline{y} \in \Gamma$, so that $\mathcal{O}(\underline{y}) = \Gamma$, and $t_{m+1} - t_m \le T + \sigma$ for all m big enough, we have that

$$d\Big(\phi_t(\underline{x}_0), \Gamma\Big) < \eta \qquad \forall\, t \in (t_m, t_{m+1})$$

for m big enough. We can conclude that $d(\phi_t(\underline{x}_0), \Gamma) \to 0$ as $t \to +\infty$. Hence $\omega(\underline{x}_0) \subset \Gamma$. This shows that $\Gamma = \omega(\underline{x}_0)$, and concludes the proof of the theorem. □

Example 2.13. Let us consider the following system in polar coordinates

$$\begin{cases} \dot{\rho} = \rho(1-\rho^2) + \varepsilon\, f(\rho, \theta) \\ \dot{\theta} = 1 + \varepsilon\, g(\rho, \theta) \end{cases}$$

with $f, g \in C^1(\mathbb{R}^2)$. For $\varepsilon = 0$ the system admits the orbitally asymptotically stable periodic orbit $\Gamma = \{\rho = 1\}$. We now show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a periodic orbit Γ_ε. Let

$$L = \max_{\rho \le 5} \left(|f| + |g|\right)$$

and $\varepsilon_0 = \frac{1}{4L}$. We now prove that if $\varepsilon < \varepsilon_0$ the set $D = \left\{\frac{1}{2} \le \rho \le 2\right\}$ satisfies the assumptions of Poincaré-Bendixson Theorem 2.33.

First of all for all $(\rho, \theta) \in D$

$$1 + \varepsilon\, g(\rho, \theta) \ge 1 - \varepsilon L > 1 - \varepsilon_0 L = \frac{3}{4}$$

so that D contains no fixed points of the system. Moreover

$$\dot{\rho}|_{\rho=2} = -6 + \varepsilon\, f(2, \theta) < -6 + \varepsilon\, L < -6 + \varepsilon_0\, L = -6 + \frac{1}{4} < 0$$

and

$$\dot{\rho}|_{\rho=\frac{1}{2}} = \frac{3}{8} + \varepsilon\, f\left(\frac{1}{2}, \theta\right) > \frac{3}{8} - \varepsilon\, L > \frac{3}{8} - \varepsilon_0\, L = \frac{1}{8} > 0$$

so that on ∂D the vector field is always directed towards the inside of D. This implies that for all $\underline{x} \in \partial D$ and for all $t > 0$ it holds $\phi_t(\underline{x}) \in D$, and completes the proof.

Finally, we state a result which extends Theorem 2.33 to the case of regions with fixed points.

Theorem 2.37. *Let F be a C^1 vector field in $\mathbb{R}^2$, and let $D \subset \mathbb{R}^2$ be a non-empty bounded positively invariant region containing at most a finite number of fixed points for F. Then, for all $\underline{x} \in D$, the set $\omega(\underline{x})$ is non-empty and one of the following possibilities holds:*

- *$\omega(\underline{x})$ is a fixed point;*
- *$\omega(\underline{x})$ is a periodic orbit;*
- *$\omega(\underline{x})$ consists of a finite number of fixed points and heteroclinic orbits connecting them.*

2.5 Exercises

2.1. Draw the phase portrait of the linear system $\underline{\dot{x}} = A\,\underline{x}$ in $\mathbb{R}^2$ and find the stable, unstable, and central eigenspace of $\underline{0}$, with A given by:

(a) $A = \begin{pmatrix} 5 & 4 \\ 2 & 7 \end{pmatrix}$ (b) $A = \begin{pmatrix} -8 & 0 \\ 1 & -6 \end{pmatrix}$ (c) $A = \begin{pmatrix} -8 & 6 \\ -9 & 13 \end{pmatrix}$

(d) $A = \begin{pmatrix} -8 & 4 \\ -1 & -4 \end{pmatrix}$ (e) $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$ (f) $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

(g) $A = \begin{pmatrix} -7 & -5 \\ 1 & -5 \end{pmatrix}$ (h) $A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$ (i) $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$

2.2. For the following systems, find the critical points and study their linear stability.

$$(a) \begin{cases} \dot{x} = -2x(x-1)(2x-1) \\ \dot{y} = -2y \end{cases} \qquad (b) \begin{cases} \dot{x} = x(4-2x-y) \\ \dot{y} = y(3-x-y) \end{cases}$$

$$(c) \begin{cases} \dot{x} = -y + x^3 \\ \dot{y} = x + y^3 \end{cases} \qquad (d) \begin{cases} \dot{x} = e^{(x+y)} + y \\ \dot{y} = y - xy \end{cases}$$

$$(e) \begin{cases} \dot{x} = 2xy \\ \dot{y} = y^2 - x^2 \end{cases} \qquad (f) \begin{cases} \dot{x} = x(60-4x-3y) \\ \dot{y} = y(42-3x-2y) \end{cases}$$

2.3. Find a Lyapunov function to study the stability of the fixed point $(0,0)$ for the following systems:

$$(a) \begin{cases} \dot{x} = y - 3x^3 \\ \dot{y} = -x - 7y^3 \end{cases} \qquad (b) \begin{cases} \dot{x} = -xy^4 \\ \dot{y} = yx^4 \end{cases}$$

$$(c) \begin{cases} \dot{x} = x - xy^4 \\ \dot{y} = y - y^3x^2 \end{cases} \qquad (d) \begin{cases} \dot{x} = x^2 - xy - x \\ \dot{y} = y^2 + 2xy - 7y \end{cases}$$

2.4. Determine the stability of the fixed point $(0,0)$ varying $\mu \in \mathbb{R}$ for the system

$$\begin{cases} \dot{x} = (\mu x + 2y)(z+1) \\ \dot{y} = (-x + \mu y)(z+1) \\ \dot{z} = -z^3 \end{cases}$$

2.5. Find the fixed points and study their stability varying $\mu \in \mathbb{R}$, $\mu \neq 4$, for the system

$$\begin{cases} \dot{x} = \mu\, x^3 - x^5 \\ \dot{y} = (2\mu y + z)(x-2) \\ \dot{z} = (-2y + \mu z)(x-2) \end{cases}$$

2.6. Draw the phase portrait for a mechanical Hamiltonian system with $H(x,y)$ of the form (2.4) with $m = 1$ and potential energy W given by:

(a) $W(x) = \frac{1}{3}x^2 + \frac{1}{9}x^3 - \frac{1}{4}x^4$;

(b) $W(x) = x \log(1 + x^2)$;

(c) $W(x) = \begin{cases} e^{-x^2}, & x \leq 0 \\ \cos(\sqrt{2}\,x), & x \geq 0 \end{cases}$;

(d) $W(x) = -\frac{\sin x}{x}$.

2.7. Consider the system

$$\begin{cases} \dot{x} = \frac{1}{2}y \\ \dot{y} = -(1+\mu)x + \mu x^2 + x^3 \end{cases}$$

varying $\mu \in \mathbb{R}$. Show that it is a mechanical Hamiltonian system writing down the Hamiltonian function. Let denote by $(x_\mu(t,0), y_\mu(t,y_0))$ the solution to the system with initial condition $(x(0), y(0)) = (0, y_0)$, then find

$$y^*(\mu) := \inf\{y_0 > 0 \,:\, \lim_{t\to+\infty} x_\mu(t) = +\infty\}\,.$$

2.8. Draw the phase portrait for the following systems:

$$(a) \begin{cases} \dot{x} = y - x^2 \\ \dot{y} = x - 2 \end{cases} \qquad (b) \begin{cases} \dot{x} = \sin x\,(-0.1\cos x - \cos y) \\ \dot{y} = \sin y\,(\cos x - 0.1\cos y) \end{cases} \text{ on } [0,\pi]^2$$

$$(c)\ \begin{cases} \dot{x} = x^2 - 1 \\ \dot{y} = -xy + x^2 - 1 \end{cases} \qquad (d)\ \begin{cases} \dot{x} = y\ \cos x \\ \dot{y} = \sin x \end{cases}$$

$$(e)\ \begin{cases} \dot{x} = y \\ \dot{y} = x^3 - x \end{cases} \qquad (f)\ \begin{cases} \dot{x} = y \\ \dot{y} = x^3 - x + \frac{1}{2}y \end{cases}$$

2.9. For the following systems, study the existence of a periodic orbit entirely contained in $\{x^2 + y^2 \geq 2\}$:

$$(a)\ \begin{cases} \dot{x} = x^3 - x + y^2 \\ \dot{y} = -2y \end{cases} \qquad (b)\ \begin{cases} \dot{x} = \frac{x^3}{1+x^4+y^4} \\ \dot{y} = \frac{y^3}{1+x^4+y^4} \end{cases}$$

2.10. Study the existence of a periodic orbit for the system

$$\begin{cases} \dot{x} = x\ \sqrt{x^2 + y^2} - 3x\,(x^2 + y^2) + \frac{1}{10}y^5 \\ \dot{y} = y\ \sqrt{x^2 + y^2} - 3y\,(x^2 + y^2) - \frac{1}{10}x^5 \end{cases}$$

Chapter 3

Discrete-time dynamical systems

In this chapter we consider discrete-time dynamical systems as defined in Definition 1.2. Hence we need to specify a set X and a map $T : X \to X$. The properties of X and T may vary and give rise to different areas of research. Here we assume that X is a locally compact connected metric space and T is a continuous map, and call (X,T) a *discrete-time continuous dynamical system*. Many interesting phenomena already occur for maps on an interval of the real line, which are studied in the first part of the chapter. In the second part, we consider higher dimensional systems.

We start with simple definitions.

Definition 3.1. Let (X,T) and $(\tilde{X},\tilde{T})$ be two discrete-time continuous dynamical systems. We say that $(\tilde{X},\tilde{T})$ is a *topological factor* of (X,T) if there exists a continuous map $h : X \to \tilde{X}$ that is surjective and satisfies

$$\tilde{T} \circ h = h \circ T\,. \tag{3.1}$$

If the map $h : X \to \tilde{X}$ is a homeomorphism and satisfies (3.1) then we say that (X,T) and $(\tilde{X},\tilde{T})$ are *topologically conjugate* and h is a *topological conjugacy*.

Example 3.1. Let's consider the full shift $(\Omega_{\mathcal{A}},\mathbb{N}_0,\sigma)$ on two symbols $\mathcal{A} = \{0,1\}$ of Example 1.8, and the Bernoulli map T_2 on S^1 of Example 1.7. Let $J_0 = [0,1/2)$ and $J_1 = [1/2,1)$ be a partition of S^1, and let the map $h : \Omega_{\{0,1\}} \to S^1$ be defined by

$$\omega = (\omega_i)_{i\in\mathbb{N}_0} \mapsto h(\omega) = \bigcap_{i\in\mathbb{N}_0} T_2^{-i}(J_{\omega_i})\,.$$

The map h is continuous and surjective, and satisfies $T_2 \circ h = h \circ \sigma$. Then the Bernoulli map is a topological factor of the full shift on two symbols.

Example 3.2. Let's consider the Tent map T_s with $s = 2$ of Example 1.5, and the logistic map T_λ with $\lambda = 4$ of Example 1.6. Let the map $h : [0,1] \to [0,1]$ be defined by

$$[0,1] \ni x \mapsto h(x) = \sin^2\left(\frac{\pi}{2}x\right) .$$

The map h is a homeomorphism, and satisfies $T_4 \circ h = h \circ T_2$. Hence the Tent map T_s with $s = 2$ is topologically conjugate to the logistic map T_λ with $\lambda = 4$.

Remark 3.1. In some situations it is interesting to study the regularity of a conjugacy. For example, if T and $\tilde{T}$ are C^k maps, with $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$, a natural question is whether there exists a conjugacy h between the systems (X,T) and $(\tilde{X},\tilde{T})$ which is of class C^k. If it exists we say that (X,T) and $(\tilde{X},\tilde{T})$ are C^k *conjugate*. This question is studied in Section 4.2 in the analytic case for homeomorphisms of the circle.

3.1 Stability in one dimension

Let $T : X \to X$ be a continuous map of a one-dimensional space $X = [a,b], (a,b), [a,+\infty), (a,+\infty), (-\infty,b], (-\infty,b), \mathbb{R}, S^1$.

Definition 3.2. A fixed point $x_0 \in X$ of T is called *attractive* if there exists a neighborhood U of x_0 such that, for all $x \in U$, one has $T^n(x) \in U$ for all $n \geq 0$, and $T^n(x) \to x_0$ as $n \to +\infty$.
A fixed point $x_0 \in X$ is called *repulsive* if there exists $\delta > 0$ such that, for all $x \in B_\delta(x_0)$, $x \neq x_0$, there exists $\bar{n} \in \mathbb{N}$ for which $T^{\bar{n}}(x) \notin B_\delta(x_0)$.

To study the dynamics in a neighbourhod of a fixed point x_0, first it is useful to try the linearization approach. Let T be differentiable at x_0. Then, there exists $\varepsilon > 0$ such that for all $x \in B_\varepsilon(x_0)$

$$T(x) = T(x_0)+T'(x_0)(x-x_0)+o(|x-x_0|) = x_0+T'(x_0)(x-x_0)+o(|x-x_0|) .$$

Hence,

$$|T(x) - x_0| = |T'(x_0)|\,|x - x_0| + o(|x - x_0|) . \tag{3.2}$$

We deduce that, at the first order, it is the derivative $T'(x_0)$ which may determine whether the orbit of a point $x \in B_\varepsilon(x_0)$ gets closer or further from the fixed point x_0. This justifies the following definition.

Definition 3.3. Let T be differentiable at a fixed point x_0. The fixed point $x_0 \in X$ is called *hyperbolic* if $|T'(x_0)| \neq 1$.

Theorem 3.2. *Let x_0 be a hyperbolic fixed point for a map T with $T \in C^1(B_\varepsilon(x_0))$ for some $\varepsilon > 0$. If $|T'(x_0)| < 1$ then the point is attractive, if $|T'(x_0)| > 1$ then the point is repulsive.*

Proof. Let $|T'(x_0)| < 1$ and fix $c \in (|T'(x_0)|, 1)$. There exists $\delta > 0$ such that $|T'(x)| \leq c$ for all $x \in B_\delta(x_0)$. We show that for all $n \geq 1$

$$|T^n(x) - x_0| \leq c^n \, |x - x_0| \, , \quad \forall \, x \in B_\delta(x_0) \, . \tag{3.3}$$

From (3.3) and $c \in (0,1)$, it follows that $T^n(x) \in B_\delta(x_0)$ for all $n \geq 0$ and $T^n(x) \to x_0$ as $n \to +\infty$.

We now prove (3.3) by induction. For $n = 1$, for all $x \in B_\delta(x_0)$ there exists ξ_1 between x and x_0 such that

$$|T(x) - x_0| = |T(x) - T(x_0)| = |T'(\xi_1)| \, |x - x_0| \leq c \, |x - x_0| \, ,$$

where $|T'(\xi_1)| \leq c$ since $\xi_1 \in B_\delta(x_0)$. Then, let's assume that (3.3) holds for a given n, and show that it holds for $n+1$. There exists ξ_n between $T^n(x)$ and x_0 such that

$$|T^{n+1}(x) - x_0| = |T(T^n(x)) - T(x_0)| = |T(\xi_n)| \, |T^n(x) - x_0| \leq$$

$$\leq c \cdot c^n \, |x - x_0| = c^{n+1} \, |x - x_0| \, ,$$

since $\xi_n \in B_\delta(x_0)$.

Let now $|T'(x_0)| > 1$, and first consider the case $T'(x_0) > 1$. Then we fix $c \in (1, T'(x_0))$ and choose $\delta > 0$ such that $T'(x) \geq c$ for all $x \in B_\delta(x_0)$. We now argue by contradiction and assume that there exists $x \in B_\delta(x_0)$, $x \neq x_0$, such that $T^n(x) \in B_\delta(x_0)$ for all $n \geq 1$. Then, we can repeat the argument above to show that

$$|T^n(x) - x_0| \geq c^n \, |x - x_0| \, , \quad \forall \, n \geq 1 \, ,$$

from which we find that $|T^n(x) - x_0| \to +\infty$ as $n \to +\infty$ since $c > 1$. This gives the contradiction with the assumption $T^n(x) \in B_\delta(x_0)$ for all $n \geq 1$.

A similar argument works in the case $|T'(x_0)| > 1$ and $T'(x_0) < -1$. $\square$

When the fixed point is not hyperbolic, the approach in (3.2) suggests that the higher derivatives of T at x_0 may give some information.

Definition 3.4. A fixed point $x_0 \in X$ is called *semi-attractive from the left* if there exists $\delta > 0$ such that it is attractive for points on $(x_0 - \delta, x_0)$ and repulsive for points on $(x_0, x_0 + \delta)$. A fixed point $x_0 \in X$ is called *semi-attractive from the right* if there exists $\delta > 0$ such that it is attractive for points on $(x_0, x_0 + \delta)$ and repulsive for points on $(x_0 - \delta, x_0)$.

Proposition 3.3. *Let x_0 be a fixed point for a map T which is differentiable at x_0 with $|T'(x_0)| = 1$. The following possibilities hold:*

(i) Let $T'(x_0) = 1$ and assume that $T \in C^2(B_\varepsilon(x_0))$ for some $\varepsilon > 0$, and $T''(x_0) \neq 0$. Then,

- *If $T''(x_0) > 0$, then x_0 is semi-attractive from the left;*
- *If $T''(x_0) < 0$, then x_0 is semi-attractive from the right;*

(ii) Let $T'(x_0) = 1$ and assume that $T \in C^3(B_\varepsilon(x_0))$ for some $\varepsilon > 0$, that $T''(x_0) = 0$, and $T'''(x_0) \neq 0$. Then,

- *If $T'''(x_0) > 0$, then x_0 is repulsive;*
- *If $T'''(x_0) < 0$, then x_0 is attractive;*

(iii) Let $T'(x_0) = -1$ and assume that $T \in C^3(B_\varepsilon(x_0))$ for some $\varepsilon > 0$. Then, we look at $ST(x_0)$, the Schwarzian derivative of T at x_0, defined as

$$ST(x) := \frac{T'''(x)}{T'(x)} - \frac{3}{2}\left(\frac{T''(x)}{T'(x)}\right)^2 . \tag{3.4}$$

Then,

- *If $ST(x_0) > 0$, then x_0 is repulsive;*
- *If $ST(x_0) < 0$, then x_0 is attractive.*

Proof. (i) and (ii). They follow immediately from the graphical approach.

(iii). Since $T'(x_0) = -1$, in a neighborhood of x_0 the map T is order-reversing. We look at $G := T^2$ for which $G(x_0) = x_0$, and use that x_0 has the same stability for G and T. We have

$$\begin{aligned} G'(x) &= T'(T(x))\, T'(x) \quad \Rightarrow \quad G'(x_0) = (T'(x_0))^2 = 1\,, \\ G''(x) &= T''(T(x))\, (T'(x))^2 + T'(T(x))\, T''(x) \\ &\qquad \Rightarrow \quad G''(x_0) = T''(x_0)\left((T'(x_0))^2 - T'(x_0)\right) = 0\,. \end{aligned}$$

Moreover $G \in C^3(B_\varepsilon(x_0))$, hence we can compute $G'''(x_0)$. It holds

$$
\begin{aligned}
G'''(x) = {} & T'''(T(x))\,(T'(x))^3 + 3\,T''(T(x))\,T'(x)\,T''(x) + T'(T(x))\,T'''(x) \\
& \Rightarrow \quad G'''(x_0) = T'''(x_0)\left((T'(x_0))^3 + T'(x_0)\right) + 3\,(T''(x_0))^2\,T'(x_0) \\
& \Rightarrow \quad G'''(x_0) = 2\,ST(x_0)\,.
\end{aligned}
$$

The result follows from (ii). □

We conclude this section by studying the stability for periodic orbits.

Definition 3.5. Let x_0 be a periodic point for T with minimal period p. The orbit $\mathcal{O}(x_0)$ is called *attractive* (respectively *repulsive*) if x_0 is an attractive (respectively repulsive) fixed point for T^p.

Remark 3.4. Let x_0 be a periodic point for T with minimal period p. If $T \in C^1$, it is a straightforward corollary of the chain rule that the derivative of T^p is the same on all the points of the orbit of x_0, i.e. $(T^p)'(T^i(x_0)) = (T^p)'(x_0)$ for all $i = 0, \dots, p-1$, since

$$
(T^p)'(x_0) = \prod_{j=0}^{n-1} T'(T^j(x_0))\,.
$$

3.2 Existence of periodic orbits in one dimension

In this section $[a, b]$ denotes a compact interval of the real line. Given a finite number of points $\{x_k\}_{k=0,\dots,n}$ such that

$$
a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\,,
$$

we consider the partition $\mathcal{J}$ of $[a, b]$ into the closed intervals $J_k = [x_{i-1}, x_i]$, $k = 1, \dots, n$.

Definition 3.6. Given a partition $\mathcal{J} = \{J_\ell\}$ of $[a, b]$ and two not necessarily distinct sets J_k and J_h of the partition, we say that J_k *T-covers* J_h *m-times*, with $m \in \mathbb{N} \cup \{\infty\}$, if there exist m open sub-intervals $K_1, \dots, K_m$ of J_k such that $K_i \cap K_j = \emptyset$ for $i \neq j$, and $T(\overline{K_i}) = J_h$ for all $i = 1, \dots, m$.

Definition 3.7. Given a partition $\mathcal{J} = \{J_\ell\}_{\ell=1,\dots,n}$ of $[a, b]$, the *T-graph of $\mathcal{J}$* is a graph with nodes given by the indices $\{1, \dots, n\}$, and such that there are m-arcs from k to h if J_k T-covers J_h m-times. An *admissible path* *of length* $s \in \mathbb{N}$ on the T-graph of $\mathcal{J}$ is a sequence $J_{p(1)}J_{p(2)}\dots J_{p(s)}$ with $p(j) \in \{1,\dots,n\}$ and such that there is at least one arc from $p(j)$ to $p(j+1)$ for all $j = 1,\dots,s-1$. An admissible path of length $s \in \mathbb{N}$ is called *closed* if $p(s) = p(1)$. An admissible closed path is called a *loop*.

Lemma 3.5. *If $J_{p(1)}J_{p(2)}\dots J_{p(s)}J_{p(s+1)}$ is a loop on the T-graph of a partition $\mathcal{J}$ with $s \in \mathbb{N}_0$, then there exists a point $x \in J_{p(1)}$ which is periodic for T with period s and such that $T^j(x) \in J_{p(j+1)}$ for all $j = 0,\dots,s$.*

Proof. Let us fix $K_{s+1} = \mathring{J}_{p(s+1)}$. Since the path $J_{p(1)}J_{p(2)}\dots J_{p(s)}J_{p(s+1)}$ is admissible, there exists a family $K_j \subset J_{p(j)}$, $j = 1,\dots,s$, of open intervals such that $T(K_j) = K_{j+1}$. Hence there exists an interval $K_1 \subset J_{p(1)}$ such that $T^s(K_1) = K_{s+1} \supseteq K_1$. The fixed-point theorem implies that there exists $x \in \overline{K_1}$ such that $T^s(x) = x$, moreover by construction $T^j(x) \in \overline{K_{j+1}} \subseteq J_{p(j+1)}$ for all $j = 0,\dots,s$. □

Remark 3.6. It is important to notice that Lemma 3.5 does not prove the existence of a periodic point with minimal period s. That the period s is minimal may be obtained by looking at the path used in the proof of the lemma.

Proposition 3.7. *Let $T : [a,b] \to [a,b]$ be a continuous map for which there exists a periodic orbit of odd period $m > 1$. Then T admits periodic orbits of minimal period n for all $n > m$, for all even $n < m$, and for $n = 1$.*

Proof. Let's assume that m is the smallest odd number greater than 1 for which T has a periodic orbit of period m[1]. In particular, m is the minimal period of the orbit. Let us denote by $p_1, p_2, \dots, p_m$ the points of the periodic orbit ordered in $[a,b]$, so that $T(p_1) > p_1$ and $T(p_m) < p_m$. It follows that there exists $\bar{h}$ such that $T(p_{\bar{h}}) > p_{\bar{h}}$ and $T(p_k) < p_k$ for all $k = \bar{h}+1,\dots,m$. Finally let $\mathcal{J}$ be the partition given by the points a, b and the points of the periodic orbit $p_1, p_2, \dots, p_m$, and let $J_0 := [a, p_1]$, $J_m := [p_m, b]$, and $J_k := [p_k, p_{k+1}]$ for $k \in \mathcal{N} := \{1,\dots,m-1\}$. By construction and the fact that $m > 2$ we have that one of the inequalities $T(p_{\bar{h}+1}) \le p_{\bar{h}}$ and $T(p_{\bar{h}}) \ge p_{\bar{h}+1}$ is strict, hence $J_{\bar{h}}$ T-covers itself at least once. By Lemma 3.5, this gives the result for $n = 1$.

We now proceed by proving intermediate statements.

Step 1. There exists an admissible path on the T-graph of the partition $\mathcal{J}$ from $J_{\bar{h}}$ to any set J_k of the partition with $k \in \mathcal{N}$.

[1] If not, we prove the result for such smallest odd number greater than 1 and obtain the proposition.

Let us define by recurrence the following subsets of the nodes $\mathcal{N}$ of the T-graph. We put $N_1 := \{\bar{h}\}$,

$$N_2 := \{r \in \mathcal{N} \,:\, J_{\bar{h}} \ T\text{-covers } J_r\} \ ,$$

and for $i \geq 3$

$$N_i := \{r \in \mathcal{N} \,:\, \exists\, s \in N_{i-1} \text{ such that } J_s \ T\text{-covers } J_r\} \ .$$

Since $m > 2$, each J_s with $s \in \mathcal{N}$, T-covers at least one set J_r with $r \neq s$. Moreover the fact that $J_{\bar{h}}$ T-covers itself implies that $\bar{h} \in N_i$ for all $i \geq 1$, hence $\{N_i\}$ is a non-decreasing sequence of sets. We conclude that there exists ℓ such that $N_\ell = N_{\ell+1} = \mathcal{N}$, since $N_\ell \neq \mathcal{N}$ implies that m is not the minimal period of the periodic orbit. This finishes the proof of this step.

Step 2. There exists $k \in \mathcal{N}$ such that J_k T-covers $J_{\bar{h}}$.
We argue by contradiction. If the thesis of this step is false, all points p_j of the periodic orbit with $j \leq \bar{h}$ have distinct images in the set $\{p_{\bar{h}+1}, \ldots, p_m\}$, and analogously all points p_j of the periodic orbit with $j \geq \bar{h}+1$ have distinct images in the set $\{p_1, \ldots, p_{\bar{h}}\}$. Since m is odd we get the contradiction.

Step 3. On the T-graph of the partition $\mathcal{J}$ there exists a loop starting from $J_{\bar{h}}$ and passing through all the sets J_k with $k \in \mathcal{N}$. This loop contains $J_{\bar{h}}$ only at the beginning and at the end.
We first show that the shortest loop from $J_{\bar{h}}$ to itself is of length m. Let $J_{\bar{h}}J_{p(2)} \ldots J_{p(s)}J_{\bar{h}}$ be such loop of length $s+1 < m$, there are two cases. If s is odd, by Lemma 3.5 there exists $x \in J_{\bar{h}}$ such that $T^s(x) = x$, but $s < m-1$ and we have a contradiction by the choice of m. If s is even, we can consider the loop $J_{\bar{h}}J_{p(2)} \ldots J_{p(s)}J_{\bar{h}}J_{\bar{h}}$ which is of length $s+2$ and gives, by Lemma 3.5, the existence of a periodic point of period $s+1 < m$. Again we have a contradiction by the choice of m.

Let $J_{\bar{h}}J_{p(2)} \ldots J_{p(m-1)}J_{\bar{h}}$ be the shortest loop from $J_{\bar{h}}$ to itself. All J_k appear at most once in this path, indeed if one J_k appears twice, we can construct a shorter loop from $J_{\bar{h}}$ to itself. Hence, $J_{\bar{h}}J_{p(2)} \ldots J_{p(m-1)}J_{\bar{h}}$ contains all the sets J_k with $k \in \mathcal{N}$. The same argument shows that the T-graph of the partition $\mathcal{J}$ contains one single arc from an index $k \in \mathcal{N}$ to $\bar{h}$.

Let us now relabel the sets of the partition $\mathcal{J}$ by letting $I_1 := J_{\bar{h}}$ and $I_2, \ldots, I_{m-1}$ be chosen so that there exists an arc from k to $k+1$ for all $k \in \mathcal{N}$.

Step 4. The map T admits periodic orbits of minimal period n for all $n > m$.
This follows by applying Lemma 3.5 to the loop $I_1 I_2 \ldots I_{m-1} I_1 \ldots I_1$ of length $n+1$.

Step 5. For each odd $k \in \mathcal{N}$ there exists an arc from $m-1$ to k.
The statement is clearly true for $m = 3$. If $m > 3$ we show that the sets I_k are ordered in $[a, b]$ in a precise way. From step 3 we know that I_1 T-covers itself and I_2, and no other set. So $T(p_{\bar{h}}) = p_{\bar{h}+2}$ and $T(p_{\bar{h}+1}) = p_{\bar{h}}$, or $T(p_{\bar{h}}) = p_{\bar{h}+1}$ and $T(p_{\bar{h}+1}) = p_{\bar{h}-1}$. In the first case $I_2 = [p_{\bar{h}+1}, p_{\bar{h}+2}]$, and since I_2 T-covers only I_3 we have $I_3 = [p_{\bar{h}-1}, p_{\bar{h}}]$. We can continue repeating the argument to conclude that $I_{m-1} = [p_{m-1}, p_m]$, and $T(p_{m-1}) = p_1$, $T(p_1) = p_m$ and $T(p_m) = p_{\bar{h}}$. Since I_k with k odd are of the form $[p_h, p_{h+1}]$ with $h < \bar{h}$, the thesis of the step follows.

Step 6. The map T admits periodic orbits of minimal period n for all even $n < m$.
This follows from step 5 by applying Lemma 3.5 to the loop of length $n+1$ from I_{m-1} to itself of the form $I_{m-1}I_jI_{j+1}\dots I_{m-1}$ where $j = m - n$ is odd. $\square$

Theorem 3.8 (Sharkovsky)**.** *Let $T : [a, b] \to [a, b]$ be a continuous map and consider the following ordering on $\mathbb{N}$*

$$\begin{aligned} &1 \prec 2 \prec 4 \prec 8 \prec \dots \prec 2^n \prec 2^{n+1} \prec \dots 2^{n+1}5 \prec 2^{n+1}3 \prec \dots \\ &\dots \prec 2^n5 \prec 2^n3 \prec \dots \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \dots 7 \prec 5 \prec 3 \end{aligned} \tag{3.5}$$

If T admits a periodic orbit of minimal period m then it admits a periodic orbit of minimal period n for all $n \prec m$ in the ordering (3.5)*.*

Proof. If m is odd, the thesis follows from Proposition 3.7.
If $m = 2 \cdot \tilde{m}$ with $\tilde{m}$ odd and T admits no periodic orbits with odd period, then we can repeat the same argument of the proof of Proposition 3.7 up to step 2. This shows that $\bar{h} = \tilde{m}$ and, in the T-graph of the partition including the sets J_k with $k \in \mathcal{N}$, there exists an admissible path from the set $[p_{\tilde{m}}, p_{\tilde{m}+1}]$ to all the sets J_k with $k \in \mathcal{N}$. This implies that T admits a fixed point. However there is no arc to $[p_{\tilde{m}}, p_{\tilde{m}+1}]$ from a different set, since otherwise by Lemma 3.5 we could find a periodic orbit of T with odd period. It follows that $T(p_j) \geq p_{\tilde{m}+1}$ for all $j \leq \tilde{m}$ and $T(p_j) \leq p_{\tilde{m}}$ for all $j \geq \tilde{m}+1$, so the points $p_1, \dots, p_{\tilde{m}}$ give a periodic orbit of period $\tilde{m}$ for T^2. We can then repeat the argument for T^2 and find periodic orbits of T^2 with period $\tilde{n}$ for all $\tilde{n} \prec \tilde{m}$ in the ordering (3.5). The thesis for T follows.
If $m = 2^r \cdot \tilde{m}$ with $r > 1$, $\tilde{m}$ odd and T admits no periodic orbits with odd period, then we do one step as in the previous case, and we are reduced to the case $m = 2^{r-1} \cdot \tilde{m}$. So we can repeat the argument and obtain the thesis. We remark that when $\tilde{m} = 1$, we only obtain periodic orbits with period powers of 2. $\square$

3.3 Topological chaos

Definition 3.8. Let $T : X \to X$ be a continuous map on a metric space X. We say that T is *chaotic in the sense of Devaney* if there exists a compact forward invariant set $\Lambda \subset X$ such that:

(i) the set of periodic orbits is dense in Λ;

(ii) T is topologically transitive on Λ, that is for all open sets $U, V \subset X$ with non-empty intersection with Λ, there exists $n \in \mathbb{N}$ such that $T^n(U \cap \Lambda) \cap (V \cap \Lambda) \neq \emptyset$;

(iii) T has sensitive dependence on initial conditions on Λ, that is there exists $c > 0$ such that for all $x \in \Lambda$ and all $\varepsilon > 0$ one can find $y \in B_\varepsilon(x) \cap \Lambda$ for which there exists $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) > c$.

Example 3.3. Show that the Symbolic dynamics of Example 1.8 is chaotic in the sense of Devaney.

Remark 3.9. Conditions (i) and (ii) in Definition 3.8 imply (iii) if Λ is infinite (see [Ru17, Theorem 7.4]). For interval maps, condition (ii) with $\Lambda = X$ implies (i) and (iii) (see [Ru17, Proposition 7.2]).

Definition 3.9. Let $T : X \to X$ be a continuous map on a compact metric space X. For $n \in \mathbb{N}$ and $\varepsilon > 0$, a set $S \subset X$ is called (n, ε)*-separated* if for all $x, y \in S$ there exists $k = 0, \dots, n$ such that $d(T^k(x), T^k(y)) > \varepsilon$. Then the quantity

$$h_{top}(T) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \Big(\max \{ \#S \,:\, S \text{ is } (n, \varepsilon)\text{-separated} \} \Big)$$

is well-defined and is called *topological entropy of* T.

It's not difficult to show the following result.

Proposition 3.10. *Let* (X, T) *and* $(\tilde{X}, \tilde{T})$ *be two discrete-time continuous dynamical systems on compact metric spaces, and assume that* $(\tilde{X}, \tilde{T})$ *is a topological factor of* (X, T)*. Then* $h_{top}(T) \geq h_{top}(\tilde{T})$*. In particular, topological entropy is invariant under topological conjugacy.*

Example 3.4. Using Definition 3.9 and Proposition 3.10, show that: The Symbolic dynamics has positive topological entropy; The Tent map of Example 1.5 with $s = 2$, the Bernoulli map of Example 1.7, and the Logistic map of Example 1.6 with $\lambda = 4$ have topological entropy $\log 2$; The rotations of the circle of Example 1.4 have null topological entropy.

We now move to the case of maps of the interval. First, we give a simple criterion to compute the topological entropy in a special case (see [Ru17, Section 4.4]).

Proposition 3.11. *Let $T : [a,b] \to [a,b]$ be a piecewise continuous monotone map with respect to a partition $\mathcal{J} = \{J_1, \dots, J_N\}$ of the compact interval $[a,b]$ into closed subintervals. Assume that $T(J_i) = [a,b]$ for all $i = 1, \dots, N$. Then*

$$h_{top}(T) = \lim_{k\to\infty} \frac{1}{k} \log \left(\# Fix(T^k) \right) = \log N \,.$$

We now introduce another notion of chaotic behaviour.

Definition 3.10. Let $T : X \to X$ be a continuous map on a compact interval $X = [a,b]$. We say that T has a *horseshoe* if there exists a closed sub-interval $J \subseteq X$ which T-covers itself 2-times.

Proposition 3.12. *Let $T : X \to X$ be a continuous map on a compact interval $X = [a,b]$. Then:*

(i) if T has a horseshoe then has periodic orbits with minimal period n for all $n \geq 1$;

(ii) if T has a periodic point with minimal odd period $m > 1$, then T^2 has a horseshoe.

Proof. (i) Let $J \subseteq [a,b]$ be the closed interval which covers itself 2-times, and let K_1 and K_2 be the open sub-intervals of J such that $K_1 \cap K_2 = \emptyset$ and $T(\overline{K_1}) = T(\overline{K_2}) = J$. We consider the T-graph of K_1, K_2, which is a full graph on the indices $\{1, 2\}$.

Let $K_1 = (\alpha, \beta)$ and $K_2 = (\beta + \varepsilon, \gamma)$, there are two cases. If $\varepsilon > 0$ or $\varepsilon = 0$ and β is not a fixed point, we apply Lemma 3.5 to the admissible path $K_1K_2K_2K_1$ to find a periodic point of period 3 which is not fixed, so it has minimal period 3 and we can apply Sharkovsky Theorem 3.8. If $\varepsilon = 0$ and β is a fixed point, then it follows that there exists $\delta \in (\beta, \gamma)$ such that $T([\delta, \gamma]) = J$, so we can repeat the argument with $K_1 = (\alpha, \beta)$ and $K_3 = (\delta, \gamma)$.

(ii) Let m be the smallest odd number for which T has a periodic orbit of minimal period m, and let $\{p_1, \dots, p_{m-1}, p_m\}$ be the points of the periodic orbit in dynamical order, that is $T(p_i) = p_{i+1}$ for all $i = 1, \dots, m-1$, and $T(p_m) = p_1$. By Step 5 in the proof of Proposition 3.7, the point of the periodic orbit are ordered in $[a,b]$ as

$$a \leq p_m < p_{m-2} < \cdots < p_5 < p_3 < p_1 < p_2 < p_4 < \cdots < p_{m-3} < p_{m-1} \leq b$$

or specularly. In the first case, we find $T(p_1, p_2) = (p_3, p_2)$ so that there exists $\delta \in (p_1, p_2)$ such that $T(\delta) = p_1$, and hence $T^2(\delta) = p_2$. We now show that $J = [p_m, p_2]$ T^2-covers itself 2-times. Let $K_1 = (p_m, p_{m-2})$, then $T^2(p_m) = p_2$ and $T^2(p_{m-2}) = p_m$, hence $T^2(\overline{K_1}) = J$. If we also let $K_2 = (p_{m-2}, \delta)$, then as shown before again $T^2(\overline{K_2}) = J$. Since $K_1 \cap K_2 = \emptyset$, we are done. □

Definition 3.11. Let $T : X \to X$ be a continuous map on a compact interval $X = [a, b]$. We say that T is *chaotic in the horseshoe sense* if there exists $n \in \mathbb{N}$ such that T^n has a horseshoe.

For interval maps, all the definitions of topological chaos coincide (the next result is proved putting together different statements in [Ru17]).

Theorem 3.13. *Let $T : X \to X$ be a continuous map on a compact interval $X = [a, b]$. Then the following are equivalent:*

(i) T is chaotic in the sense of Devaney;

(ii) $h_{top}(T) > 0$;

(iii) T is chaotic in the horseshoe sense;

(iv) T has a periodic point with minimal period not a power of 2.

Example 3.5. The Tent map T_s of Example 1.5 is chaotic for all $s > 1$. If $s \geq \sqrt{2}$ one shows that T_s^2 has a horseshoe by using the interval $J_s = [\frac{1}{s+1}, \frac{s}{s+1}]$, since $\frac{1}{2} \in J_s$ and $T^2(\frac{1}{2}) \leq \frac{1}{s+1}$, whereas $T^2(\frac{1}{s+1}) = T^2(\frac{s}{s+1}) = \frac{s}{s+1}$. If $s \in (1, \sqrt{2})$, the result follows by observing that there exist intervals J_1 and J_2 on which T_s^2 is equal to T_{s^2} after rescaling.

Remark 3.14. For a $C^{1,\alpha}$ diffeomorphism of a manifold, positive topological entropy is equivalent to existence of a *Smale horseshoe*, defined in Section 3.5 (see [Ka80]).

3.4 Systems in dimension greater than one

In this section, we recall the basic theory for discrete-time dynamical systems analogously to the approach to continuous-time systems in Chapter 2.

Linear systems

The simplest case is described by the action of a matrix A on a Euclidean space. In particular, let $X = \mathbb{R}^d$ and $T_A : X \to X$ be given by $T_A(\underline{x}) = A\underline{x}$ for $A \in M(d \times d, \mathbb{R})$. For simplicity, we assume that 0 is not an eigenvalue of A, so that the map T_A is invertible and the dynamical system given by T_A is the action of $\mathbb{Z}$ on X. We have the analogous result of Theorem 2.1 for matrices with simple eigenvalues.

Theorem 3.15. *Let $A \in M(d \times d, \mathbb{R})$ be a real $d \times d$ matrix with k distinct real eigenvalues $\lambda_1, \ldots, \lambda_k$, and $m = \frac{1}{2}(d-k)$ distinct couples of conjugate complex eigenvalues written in polar form as $\rho_j e^{\pm i\theta_j}$, with $\rho_j > 0$ and $\theta_j \in [0, 2\pi)$. Then there exists an invertible matrix $P \in M(d \times d, \mathbb{R})$ such that*

$$P^{-1}\, A\, P = \Lambda := diag\Big(\lambda_1, \ldots, \lambda_k, B_1, \ldots, B_m\Big)$$

where

$$B_j = \rho_j \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix} , \quad \forall\, j = 1, \ldots, m,$$

and the orbits of the map $T_A(\underline{x}) = A\underline{x}$ are given by

$$T_A^n(\underline{x}) = P\, \Lambda^n\, P^{-1}\, \underline{x}$$

where

$$\Lambda^n = diag\Big(\lambda_1^n, \ldots, \lambda_k^n, B_1^n, \ldots, B_m^n\Big)$$

and

$$B_j^n = \rho_j^n \begin{pmatrix} \cos(n\theta_j) & -\sin(n\theta_j) \\ \sin(n\theta_j) & \cos(n\theta_j) \end{pmatrix} , \quad \forall\, j = 1, \ldots, m.$$

Remark 3.16. Let us consider the case of matrices with multiple roots in the simple case $d = 2$. The matrix A can only have multiple real roots, and the possible Jordan normal form of a matrix A with a double real eigenvalue λ are

$$\Lambda = \mathrm{diag}\Big(\lambda, \lambda\Big) \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} .$$

In the non-diagonal case, one writes $\Lambda = \lambda I + N$, where N is the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for which $N^2 = 0$. So that[2]

$$\Lambda^n = \mathrm{diag}\Big(\lambda^n, \lambda^n\Big) \quad \text{or} \quad \lambda^{n-1}\begin{pmatrix} \lambda & n \\ 0 & \lambda \end{pmatrix} .$$

The analogous of Theorem 3.15 can be stated for a general matrix A, and the first dynamical properties of the map T_A are simple to prove.

First, using Definition 1.5 we find

Proposition 3.17. *The fixed points of the map T_A are the points in the kernel of $A - I$.*

In particular, the origin $\underline{x}_0 = \underline{0}$ is a fixed point for all A, and the other fixed points come in linear subspaces of $\mathbb{R}^d$. We'll see that the origin plays a special role in characterizing the dynamics of all the non-trivial orbits.

Concerning periodic orbits, it is straightforward from Theorem 3.15 that they can exist only if there is a couple of conjugate complex eigenvalues which are roots of 1. If this holds, all orbits within the relative eigenspace are periodic.

In general, the space $\mathbb{R}^d$ can be written as the direct sum of generalised eigenspaces of A, and according to the asymptotic behaviour of the orbits, it makes sense to consider the following decomposition.

Definition 3.12. Let $A \in M(d \times d, \mathbb{R})$ be a real $d \times d$ matrix and let E_λ denote the generalised eigenspace of an eigenvalue λ. We call:
Stable eigenspace of $\underline{0}$ the linear space $E^s(\underline{0})$ defined as

$$E^s(\underline{0}) := \mathrm{Span}\left\{v \in E_\lambda \,:\, |\lambda| < 1\right\} ;$$

Central eigenspace of $\underline{0}$ the linear space $E^c(\underline{0})$ defined as

$$E^c(\underline{0}) := \mathrm{Span}\left\{v \in E_\lambda \,:\, |\lambda| = 1\right\} ;$$

Unstable eigenspace of $\underline{0}$ the linear space $E^u(\underline{0})$ defined as

$$E^u(\underline{0}) := \mathrm{Span}\left\{v \in E_\lambda \,:\, |\lambda| > 1\right\} .$$

Theorem 3.18. *Let $A \in M(d \times d, \mathbb{R})$ be a real $d \times d$ matrix and consider the map $T_A(\underline{x}) = A\underline{x}$. Then:*

(i) $d = \dim E^s(\underline{0}) + \dim E^c(\underline{0}) + \dim E^u(\underline{0})$;

[2] Here we use the fact that the matrices I and N commute.

(ii) the eigenspaces $E^s(\underline{0}), E^c(\underline{0}), E^u(\underline{0})$ *are completely invariant;*

(iii) the following dynamical characterisation holds:

$$E^s(\underline{0}) = \{\underline{x} \in \mathbb{R}^n \, : \, T_A^n(\underline{x}) \to \underline{0} \text{ as } n \to +\infty\} \, ;$$

$$E^u(\underline{0}) = \{\underline{x} \in \mathbb{R}^n \, : \, T_A^n(\underline{x}) \to \underline{0} \text{ as } n \to -\infty\} \, .$$

Proof. It is a simple application of Theorem 3.15. □

As in the continuous case (see Remark 2.5), it's impossible to give a dynamical characterization of the central eigenspace $E^c(\underline{0})$ (see Exercise 3.1).

Remark 3.19. In the case $d = 2$, the dynamical properties of the orbits follow from the dimensions of the eigenspaces in Definition 3.12. As in the continuous case, we can identify three different main situations.
Case 1. A has two, non-necessarily distinct, eigenvalues $\lambda, \mu \in \mathbb{R}$ *of modulus not equal to 1.* The phase space $\mathbb{R}^2$ either coincides with the stable or unstable eigenspace of $\underline{0}$, or is the direct sum of the one-dimensional stable and unstable eigenspace. In the first case, the orbits converge towards $\underline{0}$ as $n \to +\infty$ or as $n \to -\infty$, and they lie on generalised parabolas through $\underline{0}$. The fixed point $\underline{0}$ is then a *stable* or *unstable (proper or improper) node (or a star)*. In the latter, the orbits outside the stable and unstable eigenspaces do not converge to $\underline{0}$, neither at $+\infty$ or $-\infty$, and lie on generalised hyperbolas. The fixed point $\underline{0}$ is then a *saddle*.
Case 2. A has two complex conjugate eigenvalues $\lambda, \bar{\lambda}$ *of modulus 1 with non-zero imaginary parts.* The phase space $\mathbb{R}^2$ coincides with the central eigenspace of $\underline{0}$. All orbits lie on closed curves, and are periodic if the eigenvalues are a root of unity. The fixed point $\underline{0}$ is then a *center*.
Case 3. A has one or both eigenvalues equal to ± 1. This is a degenerate case. If both eigenvalues are in $\{\pm 1\}$, the fixed point $\underline{0}$ is called *parabolic*.

Stability of fixed points

We can now adapt the study of the stability properties of a fixed point given in Section 3.1 to the higher dimensional case.

Let $X \subseteq \mathbb{R}^d$ be a Euclidean domain, $T : X \to X$ be a differentiable invertible map, and $\underline{x}_0$ be a fixed point. Then, there exists $\varepsilon > 0$ such that for all $\underline{x} \in B_\varepsilon(\underline{x}_0)$,

$$T(\underline{x}) - T(\underline{x}_0) = T(\underline{x}) - \underline{x}_0 = JT(\underline{x}_0)(\underline{x} - \underline{x}_0) + o(\|\underline{x} - \underline{x}_0\|) \, .$$

Hence,

$$\|T(\underline{x}) - \underline{x}_0\| = \|JT(\underline{x}_0)(\underline{x} - \underline{x}_0)\| + o(\|\underline{x} - \underline{x}_0\|)\,. \tag{3.6}$$

We deduce that, at the first order, it is the term $JT(\underline{x}_0)$ which may determine whether the orbit of a point $\underline{x} \in B_\varepsilon(\underline{x}_0)$ gets closer or further from the fixed point $\underline{x}_0$. This justifies the following definition.

Definition 3.13. Let T be differentiable at a fixed point $\underline{x}_0$. The fixed point $\underline{x}_0 \in X$ is called *hyperbolic* if all the eigenvalues of the matrix $JT(\underline{x}_0)$ have modulus not equal to 1.

Theorem 3.20. *Let $\underline{x}_0$ be a hyperbolic fixed point for a map T such that $T \in C^1(B_\varepsilon(\underline{x}_0))$ for some $\varepsilon > 0$. If all the eigenvalues of $JT(\underline{x}_0)$ have modulus less than 1, then the point is attractive. If all the eigenvalues of $JT(\underline{x}_0)$ have modulus greater than 1, then the point is repulsive.*

Proof. Argue as in the proof of Theorem 3.2. □

Finally, for hyperbolic fixed points we can state the analogous of two fundamental theorems of the qualitative theory of continuous-time dynamical systems. The first is the Hartman-Grobman Theorem.

Theorem 3.21 (Hartman-Grobman)**.** *Let $\underline{x}_0$ be a hyperbolic fixed point of a C^1 map $T : X \to X$. Then there exists a neighbourhood $U(\underline{x}_0)$ and a homeomorphism $h : U(\underline{x}_0) \to \mathbb{R}^d$ which sends orbits $\{T^n(\underline{x})\}_n$ into orbits of the linear map $\mathbb{R}^d \ni \underline{y} \mapsto T_{JF}(\underline{y}) := JF(\underline{x}_0)\underline{y} \in \mathbb{R}^d$, without changing their direction of time parametrisation*[3]*. In particular the homeomorphism h leaves invariant the stability properties of the fixed point $\underline{y}_0 = \underline{0}$.*

The second is the result on the local stable and unstable manifolds. We first recall the definitions.

Definition 3.14. Let $\underline{x}_0$ be a fixed point of an invertible map $T : X \to X$, and let U be a neighbourhood of $\underline{x}_0$. The *local stable manifold $W^s_{loc}(\underline{x}_0)$ of $\underline{x}_0$ in U* is the set

$$W^s_{loc}(\underline{x}_0) := \{\underline{x} \in U \;:\; T^n(\underline{x}) \in U \text{ for all } n \geq 0,\; T^n(\underline{x}) \to \underline{x}_0 \text{ as } nto + \infty\}$$

Analogously, the *local unstable manifold $W^u_{loc}(\underline{x}_0)$ of $\underline{x}_0$ in U* is the set

$$W^u_{loc}(\underline{x}_0) := \{\underline{x} \in U \;:\; T^n(\underline{x}) \in U \text{ for all } n \leq 0,\; T^n(\underline{x}) \to \underline{x}_0 \text{ as } n \to -\infty\}$$

[3] A formal statement is that for all $\underline{x} \in U(\underline{x}_0)$ we have $h(T^n(\underline{x})) = T^n_{JF}(h(\underline{x}))$ for all $n \in \mathbb{N}$ such that $T^n(\underline{x}) \in U(\underline{x}_0)$.

Theorem 3.22 (Stable and unstable manifolds)**.** *Let $\underline{x}_0$ be a fixed point of a C^k, $k \geq 1$, invertible map $T : X \to X$. Let's assume that $\underline{x}_0$ is hyperbolic and let $E^s(\underline{0})$ and $E^u(\underline{0})$ be the stable and unstable eigenspaces associated to the linear map $\underline{y} \mapsto T_{JF}(\underline{y}) := JF(\underline{x}_0)\underline{y}$. Then there exists $\varepsilon > 0$ such that there exist local stable and unstable manifolds, $W^s_{loc}(\underline{x}_0)$ and $W^u_{loc}(\underline{x}_0)$, of $\underline{x}_0$ in $B_\varepsilon(\underline{x}_0)$ with the following properties:*

(i) $W^s_{loc}(\underline{x}_0)$ *and* $W^u_{loc}(\underline{x}_0)$ *are unique in* $B_\varepsilon(\underline{x}_0)$*;*

(ii) $W^s_{loc}(\underline{x}_0)$ *is forward invariant, and* $W^u_{loc}(\underline{x}_0)$ *is backward invariant;*

(iii) $W^s_{loc}(\underline{x}_0)$ *and* $W^u_{loc}(\underline{x}_0)$ *are* C^k *manifolds,* $\dim W^s_{loc}(\underline{x}_0) = \dim E^s(\underline{0})$ *and* $\dim W^u_{loc}(\underline{x}_0) = \dim E^u(\underline{0})$*;*

(iv) $W^s_{loc}(\underline{x}_0)$ *is tangential to* $\underline{x}_0 + E^s(\underline{0})$ *at* $\underline{x}_0$*, and* $W^u_{loc}(\underline{x}_0)$ *is tangential to* $\underline{x}_0 + E^u(\underline{0})$ *at* $\underline{x}_0$*.*

Finally, we recall the definitions of the *global* manifolds.

Definition 3.15. Let $\underline{x}_0$ be a hyperbolic fixed point of a C^k, $k \geq 1$, invertible map $T : X \to X$. The *global stable* and *unstable manifolds* of $\underline{x}_0$ are defined as

$$W^s(\underline{x}_0) := \bigcup_{n \leq 0} T^n(W^s_{loc}(\underline{x}_0)) \,, \qquad W^u(\underline{x}_0) := \bigcup_{n \geq 0} T^n(W^u_{loc}(\underline{x}_0)), \tag{3.7}$$

where $W^{s,u}_{loc}(\underline{x}_0)$ are the local manifolds in $B_\varepsilon(\underline{x}_0)$ for some $\varepsilon > 0$.

In Section 3.5, we show that the intersections of the stable and unstable manifolds of fixed points create chaotic behaviour. First, we study the chaotic behavior of the Toral automorphisms introduced in Example 1.10.

Hyperbolic toral automorphisms

For this section we refer to [KH95]. This class of dynamical systems shows that a linear map may produce chaotic behavior when the phase space gets folded into itself. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2 \times 2, \mathbb{Z}) \,, \quad \text{with } \det(A) = 1,$$

and consider the map $T_A : \mathbb{T}^2 \to \mathbb{T}^2$, given by

$$T_A(x, y) = (ax + by \,, \, cx + dy) \, (\mathrm{mod} \; \mathbb{Z}^2).$$

The properties of A imply that T_A is a continuous invertible transformation. A first simple dynamical property of T_A is the following.

Proposition 3.23. *The periodic points of T_A are all and only the points in $\mathbb{T}^2$ with rational coordinates.*

Proof. If $(x, y) \in \mathbb{T}^2 \cap \mathbb{Q}^2$, then we can write $x = p/q$ and $y = r/q$ for some $p, r \in \mathbb{N}_0$ and $q \in \mathbb{N}$ such that $(p, q, r) = 1$. For all $n \in \mathbb{N}$, the matrix A^n has integer coefficients, implying that $T_A^n(x, y) \in \mathbb{T}^2 \cap \mathbb{Q}^2$ and both coordinates have q as denominator. Therefore, since there exist at most q^2 distinct rational couples in $\mathbb{T}^2$ with q as common denominator, there exists $m \in \mathbb{N}$ such that $T_A^m(x, y) = (x, y)$.

In the other direction, if $(x, y) \in \mathbb{T}^2$ satisfies $T_A^m(x, y) = (x, y)$ for some $m \geq 1$, there exist $k_1, k_2 \in \mathbb{Z}$ such that

$$A^m \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_m & b_m \\ c_m & d_m \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + k_1 \\ y + k_2 \end{pmatrix}$$

with $a_m, b_m, c_m, d_m \in \mathbb{Z}$. Inverting this relation, we obtain that $x, y \in \mathbb{Q}$. $\square$

Using what we have seen for linear systems, it is clear that the asymptotic properties of the non-periodic orbits depend on the spectrum of A. We say that a matrix A is *hyperbolic* if it does not have eigenvalues of modulus 1. If A is hyperbolic, T_A is called a *hyperbolic toral automorphism.*

Proposition 3.24. *Let T_A be a hyperbolic toral automorphism. Then, there exist two vectors $v_s, v_u \in \mathbb{R}^2$, such that for all $(x, y) \in \mathbb{T}^2$ the lines*

$$\ell_s(x, y) := \{(x, y) + tv_s \,:\, t \in \mathbb{R}\} \quad \text{and} \quad \ell_u(x, y) := \{(x, y) + tv_u \,:\, t \in \mathbb{R}\} \tag{3.8}$$

are both dense when projected into $\mathbb{T}^2$, and there exist $\theta \in (0, 1)$ such that

$$\lim_{n\to+\infty} \theta^{-n}\, d(T_A^n(x, y), T_A^n((x, y) + tv_s)) = 0\,, \quad \forall\, t \in \mathbb{R}\,;$$
$$\lim_{n\to-\infty} \theta^{-|n|}\, d(T_A^n(x, y), T_A^n((x, y) + tv_u)) = 0\,, \quad \forall\, t \in \mathbb{R}\,.$$

Proof. The existence of the vectors $v_s, v_u \in \mathbb{R}^2$ and the density of the lines ℓ_s, ℓ_v follow from the spectral properties of A. If A is hyperbolic, then it has two irrational distinct eigenvalues λ_s, λ_u, both positive or both negative. Moreover we set $|\lambda_u| = 1/|\lambda_s| > 1$. Then, we choose v_u to be an eigenvector of A with eigenvalue λ_u, and v_s to be an eigenvector of A with eigenvalue λ_s. It follows immediately that the lines ℓ_s, ℓ_v are dense when projected into $\mathbb{T}^2$.

In addition, if $\theta \in (|\lambda_s|, 1)$, then for all fixed $t \in \mathbb{R}$

$$T_A^n((x, y) + tv_s) = T_A^n(x, y) + t\,\lambda_s^n v_s \,(\text{mod } \mathbb{Z}^2)\,, \quad \forall\, n \in \mathbb{Z}\,.$$

Hence, for n big enough,

$$\theta^{-n}\, d(T_A^n(x,y), T_A^n((x,y)+tv_s)) = |t|\,|v_s|\,\theta^{-n}\,|\lambda_s|^n$$

and the result follows for v_s. The same computation for v_u concludes the proof. □

The lines $\ell_s(x,y)$ and $\ell_u(x,y)$ defined in (3.8) are the *stable* and *unstable manifold* of the point (x,y). The linearity of T_A implies that the vectors v_s, v_u do not depend on the point, and the density of the lines shows that, for all choice of points $(x,y),(x',y') \in \mathbb{T}^2$, the line $\ell_s(x,y)$ intersects $\ell_u(x',y')$ countably many times.

Proposition 3.25. *Hyperbolic toral automorphisms are topologically transitive and topologically mixing (that is, for all open sets $U,V \subset \mathbb{T}^2$ there exists $N = N(U,V)$ such that for all $n \geq N$ we have $U \cap T^n(V) \neq \emptyset$).*

Proof. First, we show that T_A with A hyperbolic is topologically transitive. Given two open sets $U,V \subset \mathbb{T}^2$, let $P \in U$ and $Q \in V$ be periodic points for T_A, which exist because periodic points are dense thanks to Proposition 3.23. Let n_P and n_Q be the minimal periods of P and Q, respectively, and let $m := LCM(n_p, n_Q)$. Then, P and Q are fixed points of T_A^m.

We now apply Proposition 3.24 to T_A^m and A^m. It follows that the lines $\ell_s(P)$ and $\ell_u(Q)$ intersect at least in a point R, which satisfies

$$\lim_{n\to+\infty} (T_A^m)^n(R) = P \quad \text{and} \quad \lim_{n\to-\infty} (T_A^m)^n(R) = Q\,.$$

Hence, there exists $N_0 \in \mathbb{N}$ such that $(T_A^m)^n(R) \in U$ and $(T_A^m)^{-n}(R) \in V$, for all $n \geq N_0$. Therefore, $T_A^{2mn}(T_A^{-mn}(R)) \in U \cap T_A^{2mn}(V) \neq \emptyset$ for all $n \geq N_0$.

Topological mixing of T_A follows similarly. We use the simple fact that, given a line ℓ in $\mathbb{R}$ which is dense when projected into $\mathbb{T}^2$, for all $\varepsilon > 0$ there exists $L = L(\varepsilon) > 0$ such that any segment of ℓ of length not less than L is ε-dense when projected into $\mathbb{T}^2$ (that is, all points of $\mathbb{T}^2$ are at a distance less than ε from the projected segment). Consider two open sets $U,V \subset \mathbb{T}^2$ and points $P \in U$ and $Q \in V$. Let $\varepsilon > 0$ such that $B_\varepsilon(P) \subset U$. Recalling Proposition 3.24, given $L = L(\varepsilon)$ as in the fact above, we define $N \in \mathbb{N}$ as the smallest $k \in \mathbb{N}$ such that $T_A^k(\ell_u(Q) \cap V)$ has length not less than L. Such N exists because $\ell_u(Q) \cap V$ is a segment of the unstable manifold of Q, and this segment is stretched by a factor $|\lambda_u| > 1$ after each iteration of T_A, where λ_u is one of the eigenvalues of A. Hence, for all $n \geq N$ we have $B_\varepsilon(P) \cap T_A^n(\ell_u(Q) \cap V) \neq \emptyset$. Otherwise, P would be at a distance greater than ε from $T_A^n(\ell_u(Q) \cap V)$. In particular, $U \cap T^n(V) \neq \emptyset$ for all $n \geq N$. □

We have thus proved that the hyperbolic toral automorphisms are chaotic in the sense of Devaney (see Definition 3.8) thanks to Remark 3.9. We also state the result that the topological entropy of these maps is positive. For a proof we refer to [KH95].

Proposition 3.26. *Let T_A be a hyperbolic toral automorphism and let λ_s, λ_u be the eigenvalues of A, with $|\lambda_u| = 1/|\lambda_s| > 1$. Then,*

$$h_{top}(T_A) = \log |\lambda_u| > 0.$$

3.5 Smale-Birkhoff theory

In this section, we describe the Smale horseshoe, one of the most famous example of a chaotic dynamical system, and show how this example has generated one of the most used methods to prove chaotic behavior in other systems. For this section we refer to [Wi03].

The Smale horseshoe

Let $D = [0,1] \times [0,1]$ be the unit square in $\mathbb{R}^2$, and $a \in (0, 1/2)$ and $b > 2$ be fixed parameters. Consider the following subsets of D, two *horizontal rectangles*

$$H_0 := \left\{ (x,y) \in D \,:\, 0 \le y \le \frac{1}{b} \right\}, \quad H_1 := \left\{ (x,y) \in D \,:\, 1 - \frac{1}{b} \le y \le 1 \right\},$$

and two *vertical rectangles*

$$V_0 := \{ (x,y) \in D \,:\, 0 \le x \le a \}, \quad V_1 := \{ (x,y) \in D \,:\, 1 - a \le x \le 1 \}.$$

The *Smale horseshoe* is the map $T : D \to \mathbb{R}^2$ defined as follows. First,

$$\begin{aligned} T(x,y) &= (ax, by), \quad \text{if } (x,y) \in H_0, \\ T(x,y) &= (-ax + 1, -by + b), \quad \text{if } (x,y) \in H_1. \end{aligned} \tag{3.9}$$

That is, T maps H_i into V_i homeomorphically and linearly for $i = 0, 1$. Then, T is extended continuously on $D \setminus (H_0 \cap H_1)$ in such a way that $T(D \setminus (H_0 \cap H_1)) \cap D = \emptyset$.

In the same way, the inverse $T^{-1} : D \to \mathbb{R}^2$ is defined such that $T^{-1}|_{V_i} = (T|_{H_i})^{-1}$, for $i = 0, 1$, and is extended continuously on $D \setminus (V_0 \cap V_1)$ with image outside D.

By its definition in (3.9), the Smale horseshoe T on the horizontal rectangles H_i has a similar behavior to the hyperbolic toral automorphisms. In particular, orbits are stretched far away by a factor b along the y direction, and get closer by a factor a along the x direction. Similarly to what happens along the stable and unstable lines for a hyperbolic toral automorphism. However, in this case, there are orbits which are sent outside D. Let's describe the fully invariant set

$$\Lambda := \bigcap_{n\in\mathbb{Z}} T^n(D)\,. \tag{3.10}$$

We begin with $\Lambda_+ := \cap_{n\geq 0} T^n(D)$. First,

$$\Lambda_+^1 := D \cap T(D) = V_0 \cup V_1$$

as obtained from (3.9). Then, write

$$\Lambda_+^2 := D \cap T(D) \cap T^2(D) = D \cap T(D \cap T(D)) = D \cap T(V_0 \cup V_1)\,.$$

Since $T(V_0 \cup V_1) = T(V_0) \cup T(V_1)$, and the image of T in D is given by $V_0 \cup V_1$, we can write

$$\Lambda_+^2 = \bigcup_{s_{-1},s_{-2}\in\{0,1\}} V_{s_{-1}} \cap T(V_{s_{-2}})\,.$$

In addition, the sets $T(V_i)$'s are vertical sub-rectangles of V_j of height 1 and width λ^2. Hence, Λ_+^2 is the union of four vertical rectangles of height 1 and width λ^2.

Then, we repeat this argument for all $k \in \mathbb{N}$. So that

$$\Lambda_+^k := \bigcap_{n=0}^{k} T^n(D) = \bigcup_{s_{-1},\dots,s_{-k}\in\{0,1\}} V_{s_{-1}} \cap T(V_{s_{-2}}) \cap \cdots \cap T^{k-1}(V_{s_{-k}})\,,$$

the union of 2^k vertical rectangles of height 1 and width λ^k.

In the limit,

$$\Lambda_+ = \lim_{k\to+\infty} \Lambda_+^k = \bigcap_{j=1}^{+\infty} \bigcup_{s_{-j}\in\{0,1\}} T^{j-1}(V_{s_{-j}}) = \bigcap_{j=-\infty}^{-1} \bigcup_{s_j\in\{0,1\}} T^{-j}(H_{s_j})\,,$$

where in the last equality we have used that $V_i = T(H_i)$, $i = 0, 1$. Hence, Λ_+ is the union of uncountably many vertical lines, each denoted by an infinite binary string.

The, we can repeat the argument for $\Lambda_- := \cap_{n\geq 0} T^{-n}(D)$, using the properties of T^{-1}. It follows as above that

$$\Lambda_- = \bigcap_{j=0}^{+\infty} \bigcup_{s_j\in\{0,1\}} T^{-j}(H_{s_j}),$$

the union of uncountably many horizontal lines of length 1, each denoted by an infinite binary string.

In conclusion,

$$\Lambda := \bigcap_{n\in\mathbb{Z}} T^n(D) = \Lambda_+ \cap \Lambda_- = \bigcap_{j\in\mathbb{Z}} \bigcup_{s_j\in\{0,1\}} T^{-j}(H_{s_j}) \tag{3.11}$$

is the compact set of points in D obtained by the intersections of the vertical lines of Λ_+ with the horizontal lines of Λ_-. Each point in Λ is identified by a bi-infinite binary string, which specifies the two lines associated to the point. Furthermore, the bi-infinite binary string describes the orbit of the the point, by specifying in which of the two horizontal rectangles H_i the iterates of the orbit lie.

The fundamental result on the dynamics of the Smale horseshoe is the following result. Recall the symbolic dynamics described in Example 1.8.

Theorem 3.27 (Smale)**.** *The map T restricted to the compact fully invariant set Λ is topologically conjugate to the two-sided full shift σ on the bi-infinite strings $\{0,1\}^{\mathbb{Z}}$.*

Proof. The result follows by constructing the conjugacy. Using (3.11), for any $(x, y) \in \Lambda$ there exists a unique $s = (s_j) \in \{0,1\}^{\mathbb{Z}}$ such that $T^j(x, y) \in H_{s_j}$ for all $j \in \mathbb{Z}$. This fact defines a map $h : \Lambda \to \{0,1\}^{\mathbb{Z}}$.

The map h is injective. This follows immediately from the fact that each bi-infinite string determines a vertical and a horizontal line, whose intersection is unique. Hence, if $h(x, y) = h(x', y')$, then (x, y) and (x', y') correspond to one of the unique intersection points, se they coincide.

The map h is surjective. Given $s = (s_j) \in \{0,1\}^{\mathbb{Z}}$, we find a vertical line in Λ_+ specified by $(s_j)_{j\leq -1}$ and a horizontal line in Λ_- specified by $(s_j)_{j\geq 0}$. Here, we are using that all the finite binary words occur in the intermediate sets $\{\Lambda^k_\pm\}_k$. Hence, the string $s = (s_j)_{j\in\mathbb{Z}}$ is $h(x, y)$ where (x, y) is the intersection point of the correspondent vertical and horizontal lines.

The map h is continuous. We show this property at any $(x, y) \in \Lambda$. Let's fix $\varepsilon > 0$, the aim is to show the existence of $\delta > 0$ such that if $(x', y') \in B_\delta(x, y) \cap \Lambda$ then $d_\theta(h(x, y), h(x', y')) < \varepsilon$, where we use the distance defined in (1.8) (as adapted to the two-sided case). Since $d_\theta(h(x,y),h(x',y'))<\varepsilon$ is equivalent to $h(x,y)_j=h(x',y')_j$ for all $j=-N_\varepsilon+1,\dots,-1,0,1,\dots,N_\varepsilon-1$ with $N_\varepsilon=\lfloor\log\varepsilon/\log\theta\rfloor$, it's enough to choose $\delta>0$ such that $B_\delta(x,y)$ is contained in

$$\Big(\bigcap_{j=-N_\varepsilon+1}^{-1} T^{-j}(H_{s_j})\Big)\cap\Big(\bigcap_{j=0}^{N_\varepsilon-1} T^{-j}(H_{s_j})\Big),$$

with $s=(s_j)=h(x,y)$, that is in the intersection of a vertical rectangle of height 1 and width $2^{-N_\varepsilon+1}$, and a horizontal rectangle of width 1 and height 2^{-N_ε}, respectively.

Finally, we show that $\sigma\circ h=h\circ T$. This follows from (3.11), shifting any $h(x,y)$ to the left we obtain a point (x',y') satisfying $T^j(x',y')\in H_{s_{j+1}}$ for all $j\in\mathbb{Z}$. This point is unique by the previous steps, and the same relations are satisfied by $T(x,y)$. Hence, $(x',y')=T(x,y)$. This completes the proof. □

Corollary 3.28. *The Smale horseshoe is topologically chaotic.*

Proof. An immediate consequence of the topological conjugacy exhibited in Theorem 3.27 is that the topological entropy of the Smale horseshoe is, thanks to Proposition 3.10, greater than or equal to $\log 2$. In particular, it is positive. □

Chaos generated by homoclinic points

We now generalize the construction of the Smale horseshoe to obtain sufficient conditions for a map to have positive topological entropy.

Let $D=[0,1]\times[0,1]$ and $T:D\to\mathbb{R}^2$. First, we introduce horizontal and vertical strips in D, which play the role of the horizontal and vertical rectangles.

Definition 3.16. Given $\mu_v\geq 0$, a *μ_v-vertical curve* is the graph of a function $[0,1]\ni y\mapsto\!\!\to v(y)\in[0,1]$ with Lipschitz constant μ_v. Analogously, given $\mu_h\geq 0$, a *μ_h-horizontal curve* is the graph of a function $[0,1]\ni x\mapsto\!\!\to h(x)\in[0,1]$ with Lipschitz constant μ_h.

Given two μ_v-vertical curves defined by functions v_1,v_2 such that $v_1(y)<v_2(y)$ for all $y\in[0,1]$, we call *μ_v-vertical strip* the set

$$V:=\{(x,y)\in D\,:\,v_1(y)\leq x\leq v_2(y)\}.$$

The *width* of a vertical strip is defined as

$$d(V) := \max_{y\in[0,1]} \Big(v_2(y) - v_1(y)\Big).$$

Analogously, given two μ_h-horizontal curves defined by functions h_1, h_2 such that $h_1(x) < h_2(x)$ for all $x \in [0,1]$, we call *μ_h-horizontal strip* the set

$$H := \{(x,y) \in D \,:\, h_1(x) \leq y \leq h_2(x)\}\,.$$

The *width* of a horizontal strip is defined as

$$d(H) := \max_{x\in[0,1]} \Big(h_2(x) - h_1(x)\Big).$$

Lemma 3.29. *(i) Let $\{V^k\}_{k\in\mathbb{N}}$ be a sequence of nested μ_v-vertical strips, that is $V^k \supset V^{k+1}$ for $k \in \mathbb{N}$, with the property that $d(V^k) \to 0$ as $k \to +\infty$. Then, $V^\infty := \cap_{k\in\mathbb{N}} V^k$ is a μ_v-vertical curve.*

(ii) Let $\{H^k\}_{k\in\mathbb{N}}$ be a sequence of nested μ_h-horizontal strips, that is $H^k \supset H^{k+1}$ for $k \in \mathbb{N}$, with the property that $d(H^k) \to 0$ as $k \to +\infty$. Then, $H^\infty := \cap_{k\in\mathbb{N}} H^k$ is a μ_h-horizontal curve.

(iii) If $0 \leq \mu_v\mu_h < 1$, then a μ_v-vertical curve and a μ_h-horizontal curve intersect in exactly one point.

Proof. (i). Let v_1^k and v_2^k the Lipschitz functions defining the μ_v-vertical strip V^k. Since $d(V^k) \to 0$ as $k \to +\infty$, the sequence $\{v_1^1, v_2^1, v_1^2, v_2^2, \dots\}$ is Cauchy in the sup-norm. Since the set of Lipschitz functions with constant μ_v on $[0,1]$ is a complete metric-space with the distance induced by the sup-norm, it follows that there exists a μ_v-Lipschitz function v^∞ which satisfies $v_j^k \to v^\infty$ as $k \to \infty$ for $j = 1,2$ in the sup-norm. Hence, V^∞ is the μ_v-vertical curve defined by v^∞.

(ii). It follows as in (i).

(iii). Let v and h be the μ_v and μ_h Lipschitz functions defining a μ_v-vertical curve and a μ_h-horizontal curve, respectively. Then, the composed function $h \circ v : [0,1] \to [0,1]$ is a contraction, since

$$\Big|h(v(y_2)) - h(v(y_1))\Big| \leq \mu_h \Big|v(y_2) - v(y_1)\Big| \leq \mu_h\,\mu_v\,|y_2 - y_1|,$$

and $\mu_v\mu_h < 1$. □

We are now ready to state the generalized version of Theorem 3.27. The proof is similar.

Theorem 3.30 (Conley-Moser)**.** *Let $T : D \to \mathbb{R}^2$ be a continuous invertible map. Assume that there exist a set of disjoint μ_v-vertical strips $\{V^1, \dots, V^N\}$ and a set of disjoint μ_h-horizontal strips $\{H^1, \dots, H^N\}$ satisfying the following conditions:*

(i) *It holds $0 \le \mu_v \mu_h < 1$ and T maps H^j homeomorphically onto V^j, for $j = 1, \dots, N$, sending the vertical (horizontal) boundaries of H^j onto the vertical (horizontal) boundaries of V^j.*

(ii) *There exists $\nu_h \in (0,1)$ such that for all μ_h-horizontal strips $H \subset \cup_j H^j$, the set $\tilde{H}^i := T^{-1}(H) \cap H^i$ is a μ_h-horizontal strip and $d(\tilde{H}^i) \le \nu_h d(H)$. Analogously, there exists $\nu_v \in (0,1)$ such that for all μ_v-vertical strips $V \subset \cup_j V^j$, the set $\tilde{V}^i := T(V) \cap V^i$ is a μ_v-vertical strip and $d(\tilde{V}^i) \le \nu_v d(V)$.*

Then, T has a fully invariant Cantor set Λ on which it is topologically conjugate to the two-sided full shift σ on the bi-infinite strings $\{1, \dots, N\}^{\mathbb{Z}}$.

Finally, we show how the assumptions of the Conley-Moser Theorem might hold in a neighborhood of a saddle point.

Theorem 3.31 (Moser, Smale)**.** *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^r, $r \ge 1$, diffeomorphism with a saddle fixed point at p, such that the global stable and unstable manifolds, $W^s(p)$ and $W^u(p)$, intersect transversally at a point q. Then, $h_{top}(T) > 0$.*

The proof follows by showing that we can apply Theorem 3.30 to an iterate of T. Here, we sketch the main steps of the proof.

First, without loss of generality, we can assume that $p = (0,0)$ and there exists a neighborhood U of p such that $W^s_{loc}(p) = \{y = 0\} \cap U$ and $W^u_{loc}(p) = \{x = 0\} \cap U$.

Then, let q be the *homoclinic transversal point* to p. We let R be a "rectangle" with a corner at q, sides along the stable and unstable manifolds of p and parallel to them. There exist $k_0, k_1 \in \mathbb{N}$ such that $R_0 := T^{k_0}(R) \subset U$ has a side along the x-axis, and $R_1 := T^{-k_1}(R) \subset U$ has a side along the y-axis.

We now look at the dynamics inside U, which is dominated by the saddle point. A crucial result is the following.

Lemma 3.32 (Lambda lemma). *Let C be a curve intersecting $W^s_{loc}(p)$ transversally in a point $u \in U$. Given $\varepsilon > 0$ and U sufficiently small, there exists $N_0 \in \mathbb{N}$ such that, for all $N \geq N_0$, the curve C^N, which is the connected component of $T^N(C) \cap U$ containing $T^N(u)$, is ε-close to $W^u_{loc}(p)$ in the C^1 norm.*

Lemma 3.32 tells us what happens to the positive iterates of R_0. In particular, we can define the set

$$\bar{R}_0 := \left\{(x, y) \in R_0 \, : \, \exists \, n(x, y) \in \mathbb{N} \; s.t. \; T^{n(x,y)}(x, y) \in R_1 \right\}.$$

In addition, the properties of T imply that we can find a set of disjoint μ_h-horizontal strips in $\bar{R}_0$ on which it is well defined the return map $T^r : \bar{R}_0 \to R_0$ given by $T^r(x, y) = T^{k_0+k_1+n(x,y)}(x, y)$, and such that the image of every horizontal strip is a μ_v-vertical strip, for μ_h and μ_v satisfying $0 \leq \mu_v \mu_h < 1$. This is assumption (i) in Theorem 3.30. Assumption (ii) is proved by using the notion of *stable* and *unstable cones*, for which we refer to [Wi03].

Exercises

3.1. Show the possible asymptotic behaviour of the orbits on the central eigenspace $E^c(\underline{0})$ of Definition 3.12.

Chapter 4

Circle homeomorphisms

For the material of this chapter, see [He79, KH95].

Let S^1 be represented by $[0,1]/(0 \sim 1)$. In this chapter we consider the discrete-time dynamical systems (S^1, T) where T is an *orientation preserving circle homeomorphism* (OPCH). It is clear that a homeomorphism of S^1 is either orientation preserving or orientation reversing, and if T reverses the orientation then it has a fixed point and T^2 preserves the orientation. Hence, the orientation reversing homeomorphisms are a small sub-class of the OPCH from the dynamical point of view.

Let $\pi : \mathbb{R} \to S^1$ be the standard projection given by $\pi(x) = \{x\}$, then we say that a homeomorphism $L : \mathbb{R} \to \mathbb{R}$ is a *lift* of a circle homeomorphism T if $\pi \circ L = T \circ \pi$. Since both T and L are invertible the dynamical systems are given by the action of the group $\mathbb{Z}$ on S^1 and $\mathbb{R}$ respectively.

Proposition 4.1. *The lift of a circle homeomorphism is uniquely determined up to an additive constant. For all lifts L it holds*

$$L^n(x+k) = L^n(x) + k\,, \quad \forall\, k, n \in \mathbb{Z}\,,\, x \in \mathbb{R}\,, \tag{4.1}$$

and

$$\big|L^n(x) - L^n(y)\big| < 1\,, \quad \forall\, n \in \mathbb{Z}\,,\, x, y \in \mathbb{R} \quad s.t. \quad |x-y| < 1\,. \tag{4.2}$$

If L_1 and L_2 are two different lifts with $L_1(x) - L_2(x) = c$ for all $x \in \mathbb{R}$, then

$$L_1^n(x) - L_2^n(x) = cn\,, \quad \forall\, n \in \mathbb{Z}\,,\, x \in \mathbb{R}\,. \tag{4.3}$$

Proof. By definition of a lift, for all $x \in [0,1)$ it holds $\pi(L(x)) = T(x)$ and $\pi(L(x+1)) = T(x)$. Hence, two different lifts L_1, L_2 satisfy $L_1(x) - L_2(x) \in$

$\mathbb{Z}$ for all $x \in [0,1)$, and all lifts L satisfy $L(x+1) - L(x) \in \mathbb{Z}$ for all $x \in [0,1)$. In addition, being all lifts homeomorphisms, it follows that for any two different lifts L_1, L_2 there exists $c \in \mathbb{Z}$ such that $L_1(x) - L_2(x) = c$ for all $x \in [0,1)$, and $L(x+1) = L(x) + 1$ for all $x \in [0,1)$ and all lifts L. We have thus proved that all lifts are defined up to an additive constant.

We now prove (4.1). We have shown that it holds for all $x \in \mathbb{R}$ with $n = k = 1$. For all $k \in \mathbb{N}$ we have

$$L(x+k) = L(x+k-1+1) = L(x+k-1) + 1 = \cdots = L(x) + k\,,$$

and

$$L(x-k) + k = L(x-k+1) + k - 1 = \cdots = L(x)\,,$$

hence (4.1) holds for all $x \in \mathbb{R}$ and all $k \in \mathbb{Z}$ with $n = 1$. Finally, by induction, let's assume that it holds for $n - 1 \geq 1$, and write

$$L^n(x+k) = L(L^{n-1}(x+k)) = L(L^{n-1}(x)+k) = L(L^{n-1}(x))+k = L^n(x)+k\,,$$

where we have used the case $n = 1$. Hence, (4.1) is proved also for all $n \in \mathbb{N}$. The case $n < 0$ follows in the same way starting with $n = -1$ for which, for all $k \in \mathbb{Z}$ and all $x \in \mathbb{R}$, given $y = L^{-1}(x)$, we can write

$$L^{-1}(x+k) = L^{-1}(L(y)+k) = L^{-1}(L(y+k)) = y + k = L^{-1}(x) + k\,.$$

To prove (4.2), it is enough to observe that L^n is a homeomorphism for all $n \in \mathbb{Z}$ and that the image $L^n(I)$ of any interval of length 1 is an interval of length 1.

We now prove (4.3). Again we have shown that it holds for all $x \in \mathbb{R}$ with $n = 1$. Let $n \geq 2$ and argue by induction. We assume that (4.3) holds for all $k = 1, \dots, n-1$, and use (4.1) to write

$$\begin{aligned} L_1^n(x) &= L_1(L_1^{n-1}(x)) = L_1(L_2^{n-1}(x) + c(n-1)) = L_1(L_2^{n-1}(x)) + c(n-1) = \\ &= L_2(L_2^{n-1}(x)) + c + c(n-1) = L_2^n(x) + cn\,. \end{aligned}$$

The case $n < 0$ follows in the same way, starting from the case $n = -1$. For a fixed $x \in \mathbb{R}$, let $y_1 = L_1^{-1}(x)$ and $y_2 = L_2^{-1}(x)$, then using (4.1)

$$L_2(y_1 + c) = L_2(y_1) + c = L_1(y_1) - c + c = L_1(y_1) = x\,,$$

hence $y_2 = y_1 + c$ which can be written as $L_1^{-1}(x) - L_2^{-1}(x) = -c$. $\square$

For a given circle homeomorphism T, we always consider the lift L for which $L(0) = T(0)$ and refer to it as the *principal lift*. In particular, let $\bar{x} \in [0, 1)$ such that $T(\bar{x}) = 0$, then

$$L(x) = \begin{cases} T(x)\,, & \text{if } x \in [0, \bar{x}), \\ T(x) + 1\,, & \text{if } x \in [\bar{x}, 1), \\ L(\{x\}) + \lfloor x \rfloor\,, & \text{otherwise.} \end{cases} \tag{4.4}$$

The typical example of an OPCH is given by the rotations of the circle R_α of Example 1.4, for which the principal lift is given by $R_\alpha(x) = x + \alpha$ (we use the same notation for the circle homeomorphism and the lift without fear of ambiguity).

4.1 Rotation number

An important notion for the OPCHs is the *rotation number*.

Definition 4.1. Let $T : S^1 \to S^1$ be an OPCH and let L be a lift of T. Let

$$\tau(L) := \lim_{n\to\infty} \frac{L^n(x) - x}{n} \tag{4.5}$$

for some $x \in \mathbb{R}$. We call *rotation number of* T the number $\tau(T) := \{\tau(L)\}$.

Proposition 4.2. *The rotation number $\tau(T)$ of an OPCH is well defined.*

In the proof, we need the following lemma.

Lemma 4.3. *Let (a_n) be a sequence in $\mathbb{R}$ for which there exists $c \in \mathbb{R}$ such that*

$$a_{n+m} \le a_n + a_m + c\,, \qquad \textit{for all } n, m \in \mathbb{N}. \tag{4.6}$$

Then the limit of a_n/n exists and

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_n \frac{a_n}{n}\,.$$

Proof. Let

$$\ell := \inf_n \frac{a_n}{n}.$$

Fix $k \in \mathbb{N}$. For $n = mk + r$ with $0 \le r \le k - 1$, repeated use of (4.6) gives

$$a_{mk+r} \le a_{mk} + a_r + c \le a_{(m-1)k} + a_k + c + a_r + c \le \cdots \le m a_k + a_r + (m+1)c.$$

Hence,

$$\limsup_{n\to\infty} \frac{a_n}{n} \le \limsup_{m\to\infty} \frac{ma_k + a_r + (m+1)c}{mk+r} = \frac{a_k}{k} + \frac{c}{k}.$$

Given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\frac{a_k}{k} < \ell + \frac{\varepsilon}{2} \qquad \text{and} \qquad \frac{c}{k} < \frac{\varepsilon}{2}\,.$$

Then.

$$\ell \le \liminf_{n\to\infty} \frac{a_n}{n} \le \limsup_{n\to\infty} \frac{a_n}{n} \le \frac{a_k}{k} + \frac{c}{k} < \ell + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the $\lim_{n\to\infty} a_n/n$ exists and it is equal to ℓ. □

Proof of Proposition 4.2.

We have to prove that in Definition 4.1 the limit exists, it does not depend on x, and that the number $\tau(T)$ does not depend on the choice of the lift L.

First, we apply Lemma 4.3 to prove that the limit in (4.5) exists for all $x \in [0,1)$. Fix $x \in [0,1)$ and set

$$a_n := L^n(x) - x.$$

It follows that

$$0 \le L^n(x) - x - \lfloor a_n \rfloor < 1. \tag{4.7}$$

Then, we write

$$\begin{aligned} a_{n+m} &= L^{n+m}(x) - x \\ &= L^m(L^n(x)) - L^n(x) + L^n(x) - x \\ &= \big(L^m(L^n(x)) - L^m(x + \lfloor a_n \rfloor)\big) + \big(L^m(x + \lfloor a_n \rfloor) - (x + \lfloor a_n \rfloor)\big) \\ &\quad + \big(x + \lfloor a_n \rfloor - L^n(x)\big) + \big(L^n(x) - x\big). \end{aligned}$$

Using (4.2) and (4.7), we obtain

$$L^m(L^n(x)) - L^m(x + \lfloor a_n \rfloor) < 1\,.$$

Using (4.1), we get

$$L^m(x + \lfloor a_n \rfloor) - (x + \lfloor a_n \rfloor) = L^m(x) + \lfloor a_n \rfloor - (x + \lfloor a_n \rfloor) = L^m(x) - x,$$

Therefore, using again (4.7) so that $x + \lfloor a_n \rfloor - L^n(x) < 0$,

$$a_{n+m} \le 1 + a_m + a_n \qquad \text{for all } n, m.$$

By Lemma 4.3, the limit in (4.5) exists for all $x \in \mathbb{R}$.

Now, let's prove that the limit does not depend on the choice of $x \in \mathbb{R}$. Assuming $|x - y| < 1$, using (4.2), we obtain

$$\left| \frac{L^n(x) - x}{n} - \frac{L^n(y) - y}{n} \right| \leq \frac{|L^n(x) - L^n(y)| + |x - y|}{n} \leq \frac{2}{n}.$$

Hence, the limit does not depend on $x \in [0, 1)$, and the convergence

$$\frac{L^n(x) - x}{n} \longrightarrow_{n \to \infty} \tau(L)$$

is uniform on $[0, 1]$. Using (4.1), the result follows on $\mathbb{R}$.

Finally, for lifts L_1 and L_2 differing by translations of a constant $c \in \mathbb{Z}$, using (4.3),

$$\frac{L_1^n(x) - x}{n} - \frac{L_2^n(x) - x}{n} = \frac{nc}{n}, \quad \forall\, x \in \mathbb{R}\,.$$

Therefore,

$$\tau(L_1) - \tau(L_2) = c.$$

Hence, $\tau(T)$ does not depend on the choice of the lift. □

It is immediate from Definition 4.1 that for a rotation of the circle R_α we have $\tau(R_\alpha) = \{\alpha\}$.

We now show that the rotation number of an OPCH is invariant under topological conjugacy (see Definition 3.1).

Proposition 4.4. *If T_1 and T_2 are topologically conjugate OPCHs, then $\tau(T_1) = \tau(T_2)$.*

Proof. Let L_2 be a lift of T_2 and T_2, let $h : S^1 \to S^1$ be a homeomorphism such that

$$h \circ T_1 = T_2 \circ h.$$

Then, given a lift H of h, we prove that

$$L_1 = H^{-1} \circ L_2 \circ H$$

is a lift of T_1. We have

$$\pi \circ L_1 = \pi \circ H^{-1} \circ L_2 \circ H = h^{-1} \circ \pi \circ L_2 \circ H = h^{-1} \circ T_2 \circ \pi \circ H$$

$$= (h^{-1} \circ T_2 \circ h) \circ \pi = T_1 \circ \pi.$$

Hence,

$$\tau(T_1) = \pi\left(\lim_{n\to+\infty}\frac{L_1^n(x)-x}{n}\right) = \pi\left(\lim_{n\to+\infty}\frac{(H^{-1}\circ L_2\circ H)^n(x)-x}{n}\right).$$

Since

$$(H^{-1}\circ L_2\circ H)^n = H^{-1}\circ L_2^n\circ H,$$

we get

$$\lim_{n\to+\infty}\frac{H^{-1}(L_2^n(H(x)))-x}{n} =$$

$$= \lim_{n\to+\infty}\Big(\frac{H^{-1}(L_2^n(H(x)))-L_2^n(H(x))}{n}+\frac{L_2^n(H(x))-H(x)}{n}+\frac{H(x)-x}{n}\Big).$$

Use that there exists $c>0$ such that

$$|H^{-1}(z)-z|<c\,, \qquad \forall z\in\mathbb{R}.$$

Hence

$$\left|H^{-1}(L_2^n(H(x)))-L_2^n(H(x))\right|<c\,,\ |H(x)-x|<c\,, \qquad \forall x,\ \forall n.$$

Therefore

$$\tau(T_1) = \pi\left(\lim_{n\to+\infty}\frac{L_2^n(H(x))-H(x)}{n}\right) = \tau(T_2).$$

□

In the following, we study the dynamics of an OPCH when the rotation number is rational or irrational.

Rational rotation numbers

Lemma 4.5. *Let $T: S^1\to S^1$ be an OPCH. Then, for all $k\in\mathbb{Z}$,*

$$\tau(T^k) = \{k\,\tau(T)\}.$$

Proof. For a lift L, we have for some $c\in\mathbb{Z}$,

$$\tau(T^k) = \{\tau(L^k)\} = \left\{k\lim_{n\to+\infty}\frac{L^{kn}(x)-x}{kn}\right\} = \{k\,(\tau(T)+c)\} = \{k\,\tau(T)\}.$$

□

Proposition 4.6. *Let $T : S^1 \to S^1$ be an OPCH with $\tau(T) \in \mathbb{Q}$. Then:*

(i) There exists at least one periodic orbit;

(ii) If x is a periodic point of minimal period $q \in \mathbb{N}$, then $\tau(T) = p/q$ for some $p \in N_0$;

(iii) If $\tau(T) = p/q$ with $p \in \mathbb{N}_0$, $q \in \mathbb{N}$, and $(p,q) = 1$, then all periodic orbits have minimal period q.

Proof. (i). Let $\tau(T) = p/q$ for $p \in \mathbb{N}_0$, $q \in \mathbb{N}$, and $(p,q) = 1$. Then, by Lemma 4.5

$$\tau(T^q) = 0.$$

Let $L : \mathbb{R} \to \mathbb{R}$ be a lift of T^q, so that $\tau(L) \in \mathbb{Z}$. We need to prove that there exists $x \in \mathbb{R}$ such that

$$L(x) - x \in \mathbb{Z}.$$

If $L(x) - x \notin \mathbb{Z}$ for all x, it means that for some $c \in \mathbb{Z}$

$$c < L(x) - x < c + 1\,, \quad \forall\, x \in \mathbb{R}.$$

Then, by continuity of L and (4.1) with $n = 1$, there exists $\delta > 0$ such that

$$c + \delta \leq L(x) - x \leq c + 1 - \delta, \qquad \forall x \in \mathbb{R}.$$

Hence

$$\tau(L) \in [c + \delta, c + 1 - \delta],$$

by writing

$$c + \delta \leq \frac{L^n(x) - x}{n} = \frac{1}{n} \sum_{j=1}^{n} \Big(L\big(L^{j-1}(x)\big) - L^{j-1}(x) \Big) \leq c + 1 - \delta.$$

This is a contradiction.

(ii). Given a lift L of T, if $T^q(x) = x$ then there exists $p \in \mathbb{N}_0$ such that $L^q(x) = x + p$. Then

$$\tau(L) = \lim_{n \to +\infty} \frac{L^{qn}(x) - x}{qn} = \lim_{n \to +\infty} \frac{np}{nq} = \frac{p}{q},$$

since

$$L^{qn}(x) = L^q\big(L^{q(n-1)}(x)\big) = L^{q(n-1)}(x) + p = \cdots = x + np.$$

(iii). Let $\tau(T) = p/q$ with $(p,q) = 1$. Then from (ii), if x is a periodic point of minimal period $s \in \mathbb{N}$, there exist $r \in \mathbb{N}_0$ such that $\tau(T) = r/s$. In particular, it must be

$$\frac{r}{s} = \frac{mp}{mq} \quad \text{for some } m \geq 1.$$

If $m \geq 2$, then $s > q$ and for the principal lift L

$$L^q(x) > x + p \qquad \text{or} \qquad L^q(x) < x + p.$$

In the first case, $L^q(x) - p > x$, and

$$\begin{aligned} L^{2q}(x) - 2p &= L^q\big(L^q(x)\big) - 2p = L^q\big(L^q(x) - p\big) + p - 2p \\ &= L^q\big(L^q(x) - p\big) - p > L^q(x) - p > x, \end{aligned}$$

since L^q is strictly increasing. Hence

$$L^{mq}(x) - mp > x,$$

which is a contradiction since $s = mq$ and $r = mp$, and x is s-periodic. Analogously, we obtain a similar contradiction if $L^q(x) < x + p$. Hence, the only possibility is $m = 1$, that is x is periodic of minimal period q. □

Remark 4.7. A consequence of Proposition 4.6 is that the orbits of an OPCH with rational rotation number are only of two types. There are periodic orbits of the same period and orbits which are asymptotic at $\pm\infty$ to two (different or coincident) periodic orbits. The second-type orbits exist only if the OPCH is not topologically conjugate to the rotation $R_{\tau(T)}$ of Example 1.4. Consequently, the only ergodic invariant measures (see Definitions **??** and **??**) of an OPCH with rational rotation number are the measures supported on the periodic orbits.

The orbits of an OPCH with rational rotation number are organized in the circle similarly to a rational rotation.

Proposition 4.8. *Let $T : S^1 \to S^1$ be an OPCH with $\tau(T) = p/q \in \mathbb{Q}$, with $(p,q) = 1$. Then, the periodic orbits are ordered in S^1 as the orbits of the rotation $R_{p/q}$.*

Proof. Let $p \in \mathbb{N}_0, q \in \mathbb{N}$, with $(p,q) = 1$. For $R_{p/q}$, there are q points in S^1 ordered as

$$\left\{0, \frac{kp}{q}, 2\,\frac{kp}{q}, \ldots, k\,\frac{(q-1)p}{q}\right\}$$

with $kp \equiv 1 \pmod{q}$.

Let $T : S^1 \to S^1$ be an OPCH with $\tau(T) = p/q$, and x a periodic point of T. By Proposition 4.6, the principal lift $L : \mathbb{R} \to \mathbb{R}$ satisfies

$$L^q(x) = x + p.$$

Let

$$A := \left\{ L^k(x) + m \;\middle|\; m, k \in \mathbb{Z} \right\} \cap [x, x+p] \,,$$

for which $\# A = qp+1$. The lift L sends $[L^i(x), L^{i+1}(x)]$ into $[L^{i+1}(x), L^{i+2}(x)]$ homeomorphically for $i = 0, \ldots, q-1$, and $L(A) \cap [x, x+p] \subset A$ by (4.1). Hence, in $[L^i(x), L^{i+1}(x)]$ there are p points of A for any $i = 0, \ldots, q-1$.

Let now y be the point in A which is the closest to x. So, $y = L^r(x) + s$ for $r, s \in \mathbb{Z}$. Next, we show that $L^r(y) + s$ is the closest point to the right of y in A. If there exists $z \in A \cap (y, L^r(y) + s)$, write $z = L^m(x) + n$ for some $m, n \in \mathbb{Z}$. Setting $\tilde{L}(\cdot) := L^r(\cdot) + s$, we have

$$\begin{aligned} \tilde{L}^{-1}(z) = L^{-r}(z) - s \in (\tilde{L}^{-1}(y), \tilde{L}^{-1}(L^r(y) + s)) &= (\tilde{L}^{-1}(\tilde{L}(x)), \tilde{L}^{-1}(\tilde{L}(y))) \\ &= (x, y). \end{aligned}$$

It follows that $\tilde{L}^{-1}(z)$ is in A, but $A \cap (x, y) = \emptyset$. We get a contradiction.

Hence $z = L^r(y) + s$, and by repeating the same argument we have

$$L(x) = \tilde{L}^p(x) = L^{rp}(x) + ps \,.$$

It follows that $rp \equiv 1 \pmod{q}$.

Thus $y = L^r(x) + s$ is the analogue of $R^k_{p/q}(0)$ with $kp \equiv 1 \pmod{q}$. $\square$

Irrational rotation numbers

Proposition 4.9. *Let $T : S^1 \to S^1$ be an orientation preserving circle homeomorphism with rotation number $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$. Then:*

(i) There are no periodic orbits.

(ii) For all $x, y \in S^1$ it holds $\omega(x) = \omega(y)$.

(iii) Let E be the ω-limit set of points in S^1, then either $E = S^1$ or it is a perfect set (closed with empty interior and no isolated points).

Proof. (i). This is an immediate consequence of Proposition 4.6-(i).

(ii). First, we show that for all $x \in S^1$ and all $n, m \in \mathbb{N}$, with $n < m$,

$$\bigcup_{r \geq 1} T^{-r(m-n)}\big([T^n(x), T^m(x)]\big) = S^1. \tag{4.8}$$

Observing that

$$T^{-(m-n)}\big([T^n(x), T^m(x)]\big) = [T^{2n-m}(x), T^n(x)],$$

and repeating the computation for $T^{-r(m-n)}$, we deduce that the sets

$$\{T^{-r(m-n)}[T^n(x), T^m(x)]\}_{r \geq 1}$$

are contiguous intervals. Therefore, if (4.8) is false, the length of these intervals must go to 0 as $r \to +\infty$. In particular, there exists $\xi \in S^1$ such that

$$\xi = \lim_{r\to\infty} T^{-r(m-n)}(T^m(x)) = \lim_{r\to\infty} T^{-r(m-n)}(T^n(x)).$$

But then,

$$\begin{aligned}\xi = & \lim_{r\to\infty} T^{-(r+1)(m-n)}(T^m(x)) = T^{-(m-n)}\Big(\lim_{r\to\infty} T^{-r(m-n)}(T^m(x))\Big) \\ & = T^{-(m-n)}(\xi).\end{aligned}$$

Thus ξ is periodic, which is a contradiction. This proves (4.8).

Now, let $x \in S^1$ and let $z \in \omega(x)$, hence there exists a sequence $\{k_n\}$ such that

$$k_n \to +\infty \quad \text{and} \quad T^{k_n}(x) \to z.$$

For $y \neq x$, let $\{k_n^1\}$ and $\{k_n^2\}$ be two subsequences of $\{k_n\}$ satisfying

$$k_n^1 \to +\infty, \quad k_n^2 \to +\infty, \quad k_n^2 - k_n^1 \to +\infty.$$

Then, by (4.8),

$$y \in \bigcup_{r\geq 1} T^{-r(k_n^2-k_n^1)}[T^{k_n^1}(x), T^{k_n^2}(x)] \quad \forall n.$$

It follows that for all n there exists $r_n \geq 1$ for which

$$T^{r_n(k_n^2-k_n^1)}(y) \in [T^{k_n^1}(x), T^{k_n^2}(x)].$$

Thus,

$$T^{r_n(k_n^2-k_n^1)}(y) \to z \quad \text{as } n \to \infty,$$

and $r_n(k_n^2-k_n^1) \to +\infty$. Therefore $z \in \omega(y)$, so $\omega(x) \subset \omega(y)$ for all $x, y \in S^1$. Hence, $\omega(x) = \omega(y)$ for all $x, y \in S^1$.

(iii). Let E be the ω-limit set, $E = \omega(x)$ for all $x \in S^1$. Then, by Proposition 1.2, E is non-empty, closed and $T(E) = T^{-1}(E) = E$.

First, we show that E is the minimal set with these properties. If A is any closed and completely invariant set, then for all $x \in A$, we have $\omega(x) \subseteq \overline{\mathcal{O}^+(x)} \subseteq A$. Hence, $E \subseteq A$.

Then, if $E = S^1$, we are done. On the contrary, if $E \subsetneq S^1$ then $\partial E \neq \varnothing$. But, ∂E is invariant by continuity of T, hence $E \subseteq \partial E$. This shows that $E = \partial E$, so E has empty interior.

Finally, if $x \in E = \omega(x)$ there exists $\{k_n\}$ such that

$$T^{k_n}(x) \to x \quad \text{and} \quad T^{k_n}(x) \in E \; \forall n.$$

Hence E has no isolated points. $\square$

We are now ready to prove the classical Poincaré classification of OPCHs (see Definition 3.1 for the definitions of topological conjugacy and factor).

Theorem 4.10 (Poincaré classification). *Let $T : S^1 \to S^1$ be an orientation preserving circle homeomorphism with rotation number $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$. If the ω-limit set E satisfies $E = S^1$ then T is topologically conjugate to $R_{\tau(T)}$, if on the contrary $E \subsetneq S^1$ then $R_{\tau(T)}$ is a topological factor of T.*

Proof. Let $L : \mathbb{R} \to \mathbb{R}$ be a lift of T, and $x \in \mathbb{R}$ be fixed. Then we consider the set

$$O_x := \{L^n(x) + m \,:\, n, m \in \mathbb{Z}\}$$

and the function $H : O_x \to \mathbb{R}$ given by $H(L^n(x) + m) = n\,\tau(T) + m$.

We first recall that for any $p, q, r, s \in \mathbb{Z}$ the function

$$\mathbb{R} \ni y \mapsto d_{q,s}^{p,r}(y) := L^p(y) + q - L^r(y) - s$$

is a continuous function which never vanishes. Indeed if there exists y_0 with $d_{q,s}^{p;r}(y_0) = 0$, then $L^p(y_0) - L^r(y_0) \in \mathbb{Z}$ and this implies that y_0 is a periodic point (which is absurd since $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$).

The proof of the theorem is divided into different steps.

Step 1. The function H is strictly monotone. Let us consider $n_1, n_2, m_1, m_2 \in \mathbb{Z}$ such that $L^{n_1}(x)+m_1 < L^{n_2}(x)+m_2$, hence $d^{n_1,n_2}_{m_1,m_2}(y) < 0$ for all y. Then with $y = 0$ we can write

$$L^{n_1-n_2}(L^{n_2}(0)) - L^{n_2}(0) < m_2 - m_1$$

which becomes $L^{n_1-n_2}(z) - z < m_2 - m_1$ for $z = L^{n_2}(0)$. This implies that $d^{n_1-n_2,0}_{m_1,m_2}(z) < 0$ for all z, which for $z = 0$ and $z = L^{n_1-n_2}(0)$ gives

$$L^{n_1-n_2}(0) < m_2 - m_1 \quad \text{and} \quad L^{2(n_1-n_2)}(0) - L^{n_1-n_2}(0) < m_2 - m_1$$

respectively. It follows $L^{2(n_1-n_2)}(0) < 2(m_2 - m_1)$ and, by repeating the argument $L^{k(n_1-n_2)}(0) < k(m_2 - m_1)$ for all $k \geq 1$. Let $n_1 > n_2$, then we can use the previous inequality in the definition of rotation number to obtain

$$\tau(L) = \lim_{k\to\infty} \frac{L^{k(n_1-n_2)}(0)}{k(n_1 - n_2)} < \frac{m_2 - m_1}{n_1 - n_2}$$

where the last inequality is strict since $\tau(L) \in \mathbb{R}\backslash\mathbb{Q}$. It follows $n_1\tau(T)+m_1 < n_2\tau(T)+m_2$. Analogously if $n_1 < n_2$. The case $n_1 = n_2$ follows immediately.

Step 2. The function H can be extended to a continuous monotone function $\tilde{H} : \mathbb{R} \to \mathbb{R}$. First we notice that by Proposition 1.3, the set $H(O_x)$ is dense in $\mathbb{R}$. This implies that H can be first extended to $\overline{O_x}$. Let $x \in \overline{O_x}$ and $\{y_n\}$ and $\{z_n\}$ sequences in O_x with $y_n \nearrow x$ and $z_n \searrow x$, if they exist. Then the limits

$$H^-(x) := \lim_{n\to\infty} H(y_n) \quad \text{and} \quad H^+(x) := \lim_{n\to\infty} H(z_n)$$

exist and they coincide. In fact, by monotonicity if $H^-(x) < H^+(x)$ then $(H^-(x), H^+(x)) \subset \mathbb{R} \setminus H(O_x)$, which is absurd because $H(O_x)$ is dense. Hence we set $H(x) = H^-(x) = H^+(x)$. If one of the sequences $\{y_n\}$ and $\{z_n\}$ does not exist, the definition for $H(x)$ is immediate.

We now let $\tilde{H}(x) = H(x)$ for all $x \in \overline{O_x}$. If $\overline{O_x} = \mathbb{R}$ we are done. If $\overline{O_x} \subsetneq \mathbb{R}$ then for $y \in \mathbb{R} \setminus \overline{O_x}$, let $x_1, x_2 \in \overline{O_x}$ with $x_1 < y < x_2$. Then we let $\tilde{H}(y) = H(x_1) = H(x_2)$, where the last equality follows again by the density of $H(O_x)$.

Step 3. $R_{\tau(T)} \circ \tilde{H} = \tilde{h} \circ L$, where $R_{\tau(T)} : \mathbb{R} \to \mathbb{R}$ is a lift of the rotation of angle $\tau(T)$. Let $y = L^n(x) + m \in O_x$, then

$$R_{\tau(T)} \circ \tilde{H}(y) = R_{\tau(T)}(n\tau(T) + m) = (n + 1)\tau(T) + m$$

$$\tilde{H} \circ L(y) = \tilde{H}(L^{n+1}(x) + m) = (n + 1)\tau(T) + m$$

The same relation holds for all $y \in \mathbb{R}$ by continuity.

Step 4. Let $h : S^1 \to S^1$ be the projection of $\tilde{H}$, then $R_{\tau(T)} \circ h = h \circ T$. First we remark that by definition $\tilde{H}$ satisfies $\tilde{H}(y+1) = \tilde{H}(y) + 1$ for all y. We can then consider the projection h which satisfies $h \circ \pi = \pi \circ \tilde{H}$. Then we can write

$$(R_{\tau(T)} \circ h) \circ \pi = R_{\tau(T)} \circ (h \circ \pi) = R_{\tau(T)} \circ (\pi \circ \tilde{H}) = (R_{\tau(T)} \circ \pi) \circ \tilde{H} =$$

$$= (\pi \circ R_{\tau(T)}) \circ \tilde{H} = \pi \circ (R_{\tau(T)} \circ \tilde{H}) = \pi \circ (\tilde{H} \circ L) = (\pi \circ \tilde{H}) \circ L =$$

$$= (h \circ \pi) \circ L = h \circ (\pi \circ L) = h \circ (T \circ \pi) = (h \circ T) \circ \pi$$

where we are using the notation $R_{\tau(T)}$ for both the rotation on S^1 and its lifts.

Final step. We show that $E = \pi(\overline{O_x})$. First $\pi(O_x) = \mathcal{O}(x)$, hence $E = \omega(x) \subset \overline{\mathcal{O}(x)} \subset \pi(\overline{O_x})$. On the other way round, we can assume that the fixed $x \in \mathbb{R}$ chosen to define B satisfies $x \in \pi^{-1}(E)$. Recall that by Proposition 1.2 the set E is compact and strongly invariant, since T is invertible. Hence $\pi(O_x) \subset E$, and then $\pi(\overline{O_x}) \subset \bar{E} = E$.

At this point, if $E = S^1$ then $\overline{O_x} = \mathbb{R}$, and the function h of Step 4 is a topological conjugacy between $R_{\tau(T)}$ and T. If instead $E \subsetneq S^1$ then $\overline{O_x} \subsetneq \mathbb{R}$, hence h is not injective, but by Step 4, $R_{\tau(T)}$ is a topological factor of T. $\square$

Remark 4.11. A consequence of Theorem 4.10 is that there are three possible types of orbits for OPCHs with irrational rotation number. There are orbits which are dense in S^1, orbits which are dense in a Cantor set strictly contained in S^1, and orbits which are homoclinic to a Cantor set. The first-type orbits exists only if the OPCH is topologically conjugate to the rotation $R_{\tau(T)}$, and the other two types of orbits both exists only if the OPCH is not topologically conjugate to the rotation $R_{\tau(T)}$. A simple argument, see [KH95, Section 11.2.c], shows that an OPCH with irrational rotation number has only one probability invariant measure (see Definition **??**) and is then metrically isomorphic to an irrational rotation with Lebesgue measure (see Definition **??**).

We now prove Denjoy Theorem, which gives a sufficient condition for an OPCH with irrational rotation number to be conjugate to the irrational rotation $R_{\tau(T)}$. The result is sharp, see Denjoy example in [KH95, Section 12.2]. For the proof, we follow [Ma88].

Theorem 4.12 (Denjoy)**.** *Let $T : S^1 \to S^1$ be an orientation preserving C^1 diffeomorphisms of the circle with derivative T' with bounded variation, and $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$. Then T is topologically conjugate to $R_{\tau(T)}$.*

Proof. By Theorem 4.10 it is enough to show that the set E, the ω-limit set of any point, is S^1, or equivalently that all orbits are dense. In particular it is enough to show that all points are positively recurrent, that is for all $x \in S^1$ there exists a sequence $k_n \to +\infty$ such that $T^{k_n}(x) \to x$. Indeed, if all points are positively recurrent then $x \in \omega(x)$ for all $x \in S^1$. But by Proposition 4.9 the ω-limit set E does not depend on x, hence if the orbit a point y is not dense, that is there exists $z \notin \overline{\mathcal{O}(y)}$, then $z \notin \omega(y) = E = \omega(z)$, and we have a contradiction.

Let us now prove that all orbits are positively recurrent. We argue by contradiction assuming that 0 is not positively recurrent. To obtain the contradiction we first construct a sequence of renormalised maps starting from T.

Let $L : \mathbb{R} \to \mathbb{R}$ be the principal lift of T for which $L(0) \notin \mathbb{Z}$ since $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$. Consider the intervals

$$S_0 := [L(-1), 0] \quad \text{and} \quad D_0 := [0, L(0)] \,,$$

and the maps

$$s_0 : S_0 \to S_0 \cup D_0 \,, \qquad s_0 := L|_{S_0}$$
$$d_0 : D_0 \to S_0 \cup D_0 \,, \qquad d_0 := (L \circ R_{-1})|_{D_0}$$

so that $s_0(S_0) = [L^2(-1), L(0)]$ and $d_0(D_0) = [L(-1), L^2(-1)]$. Notice that we can then write $S_0 = [d_0(0), 0]$ and $D_0 = [0, s_0(0)]$. Moreover $s_0(d_0(0)) = d_0(s_0(0)) = L^2(-1)$, and s_0 and d_0 together define a circle homeomorphism on $S_0 \cup D_0$.

We now make a new step in the procedure, defining intervals S_1, D_1 and maps s_1, d_1. Recall that $s_0(d_0(0)) = L^2(-1) \neq 0$ since $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$.

If $s_0(d_0(0)) = L^2(-1) < 0$, we set

$$S_1 := [s_0(d_0(0)), 0] \quad \text{and} \quad D_1 := D_0 \,,$$

and define the maps

$$s_1 := s_0|_{S_1} \quad \text{and} \quad d_1 := s_0 \circ d_0 \,.$$

If instead $s_0(d_0(0)) = L^2(-1) > 0$, we set

$$S_1 := S_0 \quad \text{and} \quad D_1 := [0, s_0(d_0(0))] \,,$$

and define the maps

$$s_1 := d_0 \circ s_0 \quad \text{and} \quad d_1 := d_0|_{D_1} .$$

In both cases $s_1(d_1(0)) = d_1(s_1(0)) = L^3(-1)$, and s_1 and d_1 together define a circle homeomorphism on $S_1 \cup D_1$. Moreover we can write $S_1 = [d_1(0), 0]$ and $D_1 = [0, s_1(0)]$.

Repeating this procedure, we obtain a sequence $(S_n, D_n, s_n, d_n)_{n\geq 0}$ of intervals and maps which is monotone in the sense that $s_n \leq s_{n-1}$ and $d_n \geq d_{n-1}$ for all $n \geq 0$. The maps s_n together with d_n define a circle homeomorphism on $S_n \cup D_n$, they satisfy $s_n(d_n(0)) = d_n(s_n(0))$, and the intervals can be written as $S_n = [d_n(0), 0]$ and $D_n = [0, s_n(0)]$. Moreover we can define a sequence $\{\sigma_n\}$ with $\sigma_n := \pm$ according to whether $s_n(d_n(0)) = d_n(s_n(0))$ is positive or negative, since it never vanishes because $\tau(T) \in \mathbb{R}\backslash\mathbb{Q}$.

Lemma 4.13. *Let $[a_n/b_n, c_n/d_n]$ be a sequence of nested intervals generated recursively as follows*

$$\left[\frac{a_0}{b_0}, \frac{c_0}{d_0}\right] = \left[\frac{0}{1}, \frac{1}{1}\right] , \quad \left[\frac{a_{n+1}}{b_{n+1}}, \frac{c_{n+1}}{d_{n+1}}\right] = \begin{cases} \left[\frac{a_n+c_n}{b_n+d_n}, \frac{c_n}{d_n}\right] , & \textit{if } \sigma_n = + \\ \left[\frac{a_n}{b_n}, \frac{a_n+c_n}{b_n+d_n}\right] , & \textit{if } \sigma_n = - \end{cases}$$

Then the intervals $[a_n/b_n, c_n/d_n]$ shrink to $\tau(T)$.

Proof of Lemma 4.13. We first show that for all $n \geq 0$

$$s_n = L^{b_n} \circ R_{-a_n} \quad \text{and} \quad d_n = L^{d_n} \circ R_{-c_n} . \tag{4.9}$$

We argue by induction. For $n = 0$, we have $s_0 = L$ and $d_0 = L \circ R_{-1}$, and $\frac{a_0}{b_0} = \frac{0}{1}$, $\frac{c_0}{d_0} = \frac{1}{1}$. The initial step of the induction is proved. Let's now assume (4.9) and consider s_{n+1} and d_{n+1}. By definition, if $\sigma_n = +$, then

$$s_{n+1} = d_n \circ s_n = \Big(L^{d_n} \circ R_{-c_n}\Big) \circ \Big(L^{b_n} \circ R_{-a_n}\Big) = L^{b_n+d_n} \circ R_{-(a_n+c_n)}$$

because L and R_p commute for all $p \in \mathbb{Z}$, and $d_{n+1} = d_n = L^{d_n} \circ R_{-c_n}$. The argument works similarly if $\sigma_n = -$. Hence the induction step is proved, and (4.9) holds for all $n \geq 0$.

Moreover at least one of the denominators b_n, d_n diverge as $n \to \infty$, and by induction it is easy to show that $c_n b_n - a_n d_n = 1$ for all $n \geq 0$ (also see Chapter A). It follows that the length of the interval $[a_n/b_n, c_n/d_n]$ vanishes as $n \to \infty$. Finally, let $b_n \to \infty$, since $L^{b_n}(0) = (R_{a_n} \circ s_n)(0) = s_n(0) + a_n$ again because L and R_{a_n} commute, we have

$$\tau(T) = \lim_{n\to\infty} \frac{L^{b_n}(0) - 0}{b_n} = \lim_{n\to\infty} \frac{s_n(0) + a_n}{b_n} = \lim_{n\to\infty} \frac{a_n}{b_n}$$

where in the last equality we have used that $s_n(0) \in [-1, 1]$. The same holds if $d_n \to \infty$. □

We now show how this construction leads to a contradiction if 0 is not positively recurrent. Indeed if this is the case, we find

$$\bigcap_{n\geq 0} (S_n \cup D_n) =: [\mathcal{S}, \mathcal{D}]$$

with $\mathcal{S} < 0 < \mathcal{D}$, since $\mathcal{S}$ and $\mathcal{D}$ are the accumulation points of the forward orbit of 0 which are closest to 0. We first show that

$$d_n(\mathcal{D}) \leq \mathcal{S} \quad \text{and} \quad s_n(\mathcal{S}) \geq \mathcal{D}\,, \quad \forall\, n \geq 0\,. \tag{4.10}$$

Let's assume that $s_N(\mathcal{S}) < \mathcal{D}$ for some N. Then $s_n(\mathcal{S}) < \mathcal{D}$ for all $n \geq N$ by monotonicity, and $s_n(d_n(0)) < \mathcal{D}$ for all $n \geq N$ because $d_n(0) < \mathcal{S}$, being the left boundary point of S_n, and the maps s_n and r_n are increasing in x. However this implies that in fact $s_n(d_n(0)) \leq \mathcal{S}$ for all $n \geq N$, since $s_n(d_n(0))$ is a boundary point of $S_{n+1} \cup D_{n+1}$. Hence for the sequence σ_n we have $\sigma_n = -$ for all $n \geq N$. This is absurd, since by Lemma 4.13 it implies $\tau(T) = \frac{a_N}{b_N} \in \mathbb{Q}$. In an analogous way we prove that $d_n(\mathcal{D}) \leq \mathcal{S}$ for all n. Hence (4.10) is proved.

A consequence of (4.10) together with $s_n(0)$ being the right boundary of D_n, is that $\mathcal{D} \leq s_n(\mathcal{S}) < s_n(0) \to \mathcal{D}$. Analogously $\mathcal{S} \geq d_n(\mathcal{D}) > d_n(0) \to \mathcal{S}$. It follows that necessarily

$$\inf_{S_n} s'_n \xrightarrow[n\to\infty]{} 0 \quad \text{and} \quad \inf_{D_n} d'_n \xrightarrow[n\to\infty]{} 0\,. \tag{4.11}$$

Moreover, since $s_n(d_n(0)) = d_n(s_n(0))$ we have that

$$d_n(\mathcal{D}) < d_n(s_n(0)) = s_n(d_n(0)) < s_n(\mathcal{S})\,,$$

and since $d_n(\mathcal{D})$ and $s_n(\mathcal{S})$ are different by (4.10), it follows that

$$\max\left\{\sup_{S_n} s'_n\,, \sup_{D_n} d'_n\right\} \xrightarrow[n\to\infty]{} +\infty\,. \tag{4.12}$$

Finally, we use that $\log s_0'$ and $\log d_0'$ have bounded variation since T is a C^1 diffeomorphism, and we show that for all $n \geq 1$

$$\mathrm{var}_{S_n}(\log s_n') + \mathrm{var}_{D_n}(\log d_n') \leq \mathrm{var}_{S_{n-1}}(\log s_{n-1}') + \mathrm{var}_{D_{n-1}}(\log d_{n-1}') \quad (4.13)$$

where $\mathrm{var}_I(f)$ denotes the variation of the function f on the interval I. Let's consider the case $\sigma_{n-1} = +$, that is $s_{n-1}(d_{n-1}(0)) > 0$. In this case by construction we have $S_n = S_{n-1}$ and $s_n = d_{n-1} \circ s_{n-1}$, hence

$$\mathrm{var}_{S_n}(\log s_n') \leq \mathrm{var}_{s_{n-1}(S_{n-1})}(\log d_{n-1}') + \mathrm{var}_{S_{n-1}}(\log s_{n-1}'),$$

and $D_n = [0, s_{n-1}(d_{n-1}(0))]$ and $d_n = d_{n-1}|_{D_n}$, hence

$$\mathrm{var}_{D_n}(\log d_n') = \mathrm{var}_{D_n}(\log d_{n-1}').$$

Since $s_{n-1}(S_{n-1}) = [s_{n-1}(d_{n-1}(0)), s_n(0)]$, it follows $D_{n-1} = D_n \cup s_{n-1}(S_{n-1})$ and

$$\mathrm{var}_{s_{n-1}(S_{n-1})}(\log d_{n-1}') + \mathrm{var}_{D_n}(\log d_{n-1}') \leq \mathrm{var}_{D_{n-1}}(\log d_{n-1}').$$

Putting together the previous inequalities (4.13) follows.

To finish the proof, (4.11) and (4.12) imply that

$$\mathrm{var}_{S_n}(\log s_n') + \mathrm{var}_{D_n}(\log d_n') \xrightarrow[n\to\infty]{} +\infty$$

but this in contradiction with (4.13). □

Monotone families of OPCHs

We now study the behaviour of the rotation number for continuous families of OPCHs.

Definition 4.2. Given two OPCHs T_1, T_2, with principal lifts L_1 and L_2 respectively, we say that $T_1 \prec T_2$ if $L_1(x) < L_2(x)$ $\forall x \in \mathbb{R}$.

Proposition 4.14. *The rotation number of the OPCHs has the following properties:*

(i) τ *is continuous as a function from* $(OPCH, \|\cdot\|_\infty)$ *to* $\mathbb{R}$.

(ii) If $T_1 \prec T_2$ *then* $\tau(T_1) \leq \tau(T_2)$.

(iii) If $\tau(T_1) \in \mathbb{R} \setminus \mathbb{Q}$ *then for all* $T_2 \succ T_1$, $\tau(T_2) > \tau(T_1)$, *and for all* $T_2 \prec T_1$, $\tau(T_2) < \tau(T_1)$

(iv) If $\tau(T_1) \in \mathbb{Q}$ and T_1 is not topologically conjugate to the rotation $R_{\tau(T_1)}$, then $\exists\, T_2$ either $T_2 \succ T_1$ or $T_2 \prec T_1$ with $\tau(T_2) = \tau(T_1)$.

Proof. (i). We need to show that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|L_1 - L_2\|_\infty < \delta$ then $|\tau(L_1) - \tau(L_2)| < \varepsilon$.

Fix $\varepsilon > 0$ and L_1. Let $p/q \in \mathbb{Q}$ with $p \in \mathbb{N}_0$, $q \in \mathbb{N}$, and $p/q \in (\tau(L_1), \tau(L_1) + \varepsilon)$. First, we prove that

$$\tau(L_1) < \frac{p}{q} \quad \Rightarrow \quad \exists\, \bar{x} \in \mathbb{R} \text{ s.t. } L_1^q(\bar{x}) \leq \bar{x} + p\,. \tag{4.14}$$

Assume that $L_1^q(x) > x + p$ for all $x \in \mathbb{R}$. Then,

$$\frac{L_1^{qk}(x) - x}{k} = \frac{1}{k} \sum_{j=1}^{k} \left[L_1^q(L_1^{q(j-1)}(x)) - L_1^{q(j-1)}(x) \right] > \frac{1}{k} \sum_{j=1}^{k} p = p$$

This is a contradiction, since it implies that

$$\tau(L_1) = \lim_{k \to +\infty} \frac{L_1^{qk}(x) - x}{qk} \geq \frac{p}{q}\,.$$

Then, we strengthen (4.14), by showing that

$$\tau(L_1) < \frac{p}{q} \quad \Rightarrow \quad L_1^q(x) < x + p\,, \quad \forall\, \bar{x} \in \mathbb{R}\,. \tag{4.15}$$

By (4.14) and the continuity of L_1, either the claim is true or there exists x_0 such that $L_1^q(x_0) = x_0 + p$. If such x_0 exists, then

$$L_1^{2q}(x_0) = L_1^q(L_1^q(x_0)) = L_1^q(x_0 + p) = L_1^q(x_0) + p = x_0 + 2p\,,$$

and, in general, $L_1^{kq}(x_0) = x_0 + kp$. Then,

$$\tau(L_1) = \lim_{k \to \infty} \frac{L_1^{kq}(x_0) - x_0}{kq} = \frac{p}{q}\,,$$

which is again a contradiction.

We now use that L_1 is the lift of an OPCH. In particular, (4.15) implies that there exists $\delta_0 > 0$ such that $L_1^q(x) \leq x + p - \delta_0$ for all $x \in \mathbb{R}$. Then, let L_2 be a lift for which $\|L_2^q - L_1^q\|_\infty < \delta_0$. In particular, $L_2^q(x) \leq L_1^q(x) + \delta_0 \leq x + p$ for all $x \in \mathbb{R}$. Arguing as above, we find that

$$\frac{L_2^{qk}(x) - x}{qk} = \frac{1}{kq} \sum_{j=1}^{k} \left[L_2^q(L_2^{q(j-1)}(x)) - L_2^{q(j-1)}(x) \right] \leq \frac{p}{q}$$

from which $\tau(L_2) \leq p/q < \tau(L_1) + \varepsilon$.

Now we choose $p'/q' \in (\tau(L_1) - \varepsilon, \tau(L_1))$ and, analogously, we prove that there exists $\delta_1 > 0$ such that if a lift L_2 satisfies $\|L_1^{q'} - L_2^{q'}\|_\infty < \delta_1$, then $\tau(L_2) \geq p'/q' > \tau(L_1) - \varepsilon$.

In conclusion, fixed $\varepsilon > 0$, we choose $\delta > 0$ such that $\|L_1 - L_2\|_\infty < \delta$ implies $\|L_2^q - L_1^q\|_\infty < \delta_0$ and $\|L_1^{q'} - L_2^{q'}\|_\infty < \delta_1$. Then,

$$\tau(L_2) \in (\tau(L_1) - \varepsilon, \tau(L_1) + \varepsilon)\,.$$

(ii). If $T_1 \prec T_2$, consider the principal lifts L_1 and L_2, for which $L_1(x) < L_2(x)$ for all $x \in \mathbb{R}$. Then,

$$\frac{L_1^n(x) - x}{n} \leq \frac{L_2^n(x) - x}{n}$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$. Taking the limits as $n \to +\infty$, we obtain $\tau(L_1) \leq \tau(L_2)$.

(iii). Let T_1 be an OPCH with $\tau(T_1) \in \mathbb{R} \setminus \mathbb{Q}$, and consider any OPCH T_2 with $T_2 \succ T_1$.

Given the principal lifts for which $L_2 > L_1$, let $\delta := \min(L_2(x) - L_1(x)) > 0$. Then, since $\tau(L_1) \in \mathbb{R} \setminus \mathbb{Q}$, we prove that

$$\exists\, p \in \mathbb{N}_0,\ q \in \mathbb{N} \text{ s.t. } (p,q) = 1 \text{ and } \frac{p - \delta}{q} < \tau(L_1) < \frac{p}{q}\,. \tag{4.16}$$

This follows choosing p/q as one of the convergents to $\tau(L_1)$ defined in Proposition A.2 which approximates $\tau(L_1)$ from above, and applying Proposition A.3 for $q > 1/\delta$.

Arguing as in (4.14), we obtain that there exists $\bar{x} \in \mathbb{R}$ such that $L_1^q(\bar{x}) \geq \bar{x} + p - \delta$. It follows that

$$\begin{aligned} L_2^q(\bar{x}) &= L_2(L_2^{q-1}(\bar{x})) \geq L_1(L_2^{q-1}(\bar{x})) + \delta \\ &> L_1(L_1^{q-1}(\bar{x})) + \delta = L_1^q(\bar{x}) + \delta \geq \bar{x} + p\,, \end{aligned}$$

where we have used that L_1 is strictly increasing. From this, either $L_2^q(x) > x + p$ for all $x \in \mathbb{R}$ or there exists $x_0 \in \mathbb{R}$ such that $L_2^q(x_0) = x_0 + p$. In both cases, arguing as in the proofs of (4.14)-(4.15), we obtain $\tau(L_2) \geq p/q > \tau(L_1)$.

(iv). Let $\tau(T_1) = p/q$ with $(p,q) = 1$. Then, by Lemma 4.5, $\tau(T_1^q) = 0$ and we prove the statement for T_1^q.

Since T_1 is not conjugate to $R_{p/q}$, there exists $\bar{x} \in S^1$ such that $T_1^q(\bar{x}) = \bar{x}$ and for all $\varepsilon > 0$ we can find $x \in (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ such that $T_1^q(x) \neq x$. This implies that we can find a small enough perturbation T_2 of T_1, with either $T_2 \succ T_1$ or $T_2 \prec T_1$, for which T_2^q has a fixed point close to $\bar{x}$. In particular, by Proposition 4.6-(ii), it follows that $\tau(T_2^q) = 0$. The same is true for all T between T_1 and T_2.

Finally, Lemma 4.5 implies that $\tau(T_2)$ is of the form p'/q for some $p' \in \mathbb{N}$. The continuity of τ gives that $p' = p$. □

Proposition 4.15. *Let $a, b \in \mathbb{R}$, $a < b$, and let $[a, b] \ni \lambda \mapsto T_\lambda \in (OPCH, \|\cdot\|_\infty)$ be a continuous family of OPCH which is strictly increasing ($\lambda_1 < \lambda_2 \Rightarrow T_{\lambda_1} \prec T_{\lambda_2}$). Then, $\lambda \mapsto \tau(T_\lambda)$ is an increasing continuous function.*

Moreover, if $\tau(T_a) \neq \tau(T_b)$ and there exists $S \subseteq \mathbb{Q}$ which is dense in $[0, 1]$ and such that $\forall s \in S$ there exists $\lambda(s) \in [a, b]$ for which $\tau(T_{\lambda(s)}) = s$ and $T_{\lambda(s)}$ is not conjugate to R_s, then $\lambda \mapsto \tau(T_\lambda)$ is a "devil staircase".

Proof. The first part of the statement is a consequence of Proposition 4.14-(ii). Let now assume that $\lambda \mapsto \tau(T_\lambda)$ is not constant and that a set $S \subseteq \mathbb{Q}$ exists with the properties above. We show that there exists a family of open intervals $\{J_s\}_{s \in S}$ which are disjoint and with dense union, such that $\tau(T_\lambda)$ is constant on each J_s.

Let $J_s \subset [a, b]$ be the set of λ's such that T_λ has rotation number $s \in S$. By assumption, there exists $\lambda(s) \in J_s$ such that $T_{\lambda(s)}$ is not conjugate to R_s. Hence, by Proposition 4.14-(iv), J_s is an open interval of positive length.

It remains to show that

$$\overline{\bigcup_{s \in S} J_s} = [a, b]\,.$$

Let's assume that there exists $\lambda \in [a, b] \setminus \overline{\cup_s J_s}$, and let $\varepsilon > 0$ such that $(\lambda - \varepsilon, \lambda + \varepsilon) \subset [a, b] \setminus \overline{\cup_s J_s}$. Then, $(\tau(T_{\lambda-\varepsilon}), \tau(T_{\lambda+\varepsilon})) \cap S = \emptyset$, and by Proposition 4.14-(iii), it is continuos and strictly increasing when taking irrational values. Hence, $(\tau(T_{\lambda-\varepsilon}), \tau(T_{\lambda+\varepsilon}))$ is an interval of positive length and, by density of S, must contain at least one $\bar{s} \in S$. This is a contradiction. □

Example 4.1 (Arnold tongues). A famous example of a family of OPCHs to which Proposition 4.15 applies is given by

$$T_{\alpha,\beta}(x) = \{x + \alpha - \beta \sin(2\pi x)\}$$

for $\alpha \in [0, 1]$ and $0 \leq |\beta| < \frac{1}{2\pi}$. In particular, we fix β and consider the continuous family $\alpha \mapsto T_{\alpha,\beta}$.

For $\beta = 0$, the family reduces to the rotations of the circle. For $0 < |\beta| < \frac{1}{2\pi}$ instead, the family $\alpha \mapsto T_{\alpha,\beta}$ is clearly continuous on $(OPCH, \|\cdot\|_\infty)$ and strictly increasing. Moreover, $\tau(T_{0,\beta}) = 0$, and there exists $\alpha \in (0,1)$ for which $T_{\alpha,\beta}(x) - x \notin \mathbb{Z}$ for all $x \in S^1$. Hence, $\alpha \mapsto \tau(T_{\alpha,\beta})$ is not constant. Finally, to apply Proposition 4.15, we need to show the existence of a set $S \subseteq \mathbb{Q}$ with the right properties.

In particular, we have that for all $p/q \in \mathbb{Q}$, if $\alpha \in [0,1]$ is such that $\tau(T_{\alpha,\beta}) = p/q$, then $T_{\alpha,\beta}$ is not conjugate to $R_{p/q}$. On the contrary, the principal lift $L_{\alpha,\beta}$ satisfies

$$(R_{-p} \circ L^q_{\alpha,\beta})(x) = x\,, \quad \forall\, x \in \mathbb{R}$$

which implies that $R_{-p} \circ L^{q-1}_{\alpha,\beta}$ is a left inverse of the entire function $L_{\alpha,\beta}(z) = z + \alpha - \beta \sin(2\pi z)$, which is the extension to $\mathbb{C}$ of the lift of $T_{\alpha,\beta}$. However, a corollary of Picard's Theorem implies that $L_{\alpha,\beta}(z)$ must be a polynomial of degree 1. This is a contradiction.

It follows that, for all β with $0 \le |\beta| < \frac{1}{2\pi}$, the function $[0,1] \ni \alpha \mapsto \tau(T_{\alpha,\beta})$ is a devil staircase. As a consequence, for all $p/q \in \mathbb{Q}$, the set

$$\left\{ (\alpha, \beta) \in [0,1] \times \Big(-\frac{1}{2\pi}, \frac{1}{2\pi} \Big) \,:\, \tau(T_{\alpha,\beta}) = \frac{p}{q} \right\}$$

has positive area. These sets are called *Arnold tongues*, since for $\beta = 0$ they are given by one point, $\alpha = p/q$, but are an interval of positive length for all $\beta \neq 0$.

4.2 Regularity of the conjugacy

In this section, we consider OPCHs T with an irrational rotation number $\tau(T)$ which are conjugate to the rotation $R_{\tau(T)}$. As observed in Remark 4.11, these OPCHs admit only one probability invariant measure μ. But this measure may be equivalent to the Lebesgue measure m or not. Since $\mu = h_* m$, where h is the conjugacy between T and $R_{\tau(T)}$, the regularity of μ follows from that of h. In particular, if h is Lipschitz continuous then $\mu \sim m$. Hence, we study the regularity of the conjugacy h. It follows that a sufficient condition to obtain regularity of h depends on the number theoretic properties of the irrational $\tau(T)$.

First, we consider the case of analytic regularity, proving Arnol'd theorem, see [Ar61].

Theorem 4.16 (Local analytic case - Arnol'd). *Let $T : S^1 \to S^1$ be the projection of the map $L : \mathbb{R} \to \mathbb{R}$ defined as $L(x) = x + \alpha + \eta(x)$, where $\eta(x)$ is the restriction to the real axis of a complex function η with the following properties: η is analytic on a strip $S_\sigma := \{|\Im(z)| < \sigma\}$ for some $\sigma > 0$; $\eta(x+1) = \eta(x)$ for all $x \in \mathbb{R}$; $\|\eta\|_\sigma := \sup_{S_\sigma} |\eta(z)| < +\infty$. Moreover we assume that $\tau(T) = \alpha$ and $\alpha \in D(c, \nu)$ (see Definition A.1). Then there exists $\varepsilon = \varepsilon(c, \nu, \sigma) > 0$ such that if $\|\eta\|_\sigma < \varepsilon$ then T is analytically conjugate to R_α.*

Proof. Idea: iterative construction of the conjugacy $H : \mathbb{R} \to \mathbb{R}$ for L and the lift R_α via successive approximation.

Let $H(x) = x + G(x)$ with G small and 1-periodic. Then if $L \circ H = H \circ R_\alpha$ we find

$$x + G(x) + \alpha + \eta(x + G(x)) = x + \alpha + G(x + \alpha)\,,$$

from which

$$G(x + \alpha) - G(x) = \eta(x + G(x))\,. \tag{4.17}$$

Step 1. Let us solve (4.17) with $\eta(x)$ on the right hand side instead of $\eta(x + G(x))$. However if $G(x)$ is 1-periodic, then $G(x + \alpha) - G(x)$ has vanishing integral on $(0, 1)$, hence we look for a solution G_0 of

$$G(x + \alpha) - G(x) = \eta(x) - \int_0^1 \eta(s)\,ds\,. \tag{4.18}$$

Using Fourier series expansion for G_0 and η, equation (4.18) becomes

$$\sum_{n\in\mathbb{Z}} \hat{g}_n e^{2\pi i n x}\left(e^{2\pi i n\alpha} - 1\right) = \sum_{n\in\mathbb{Z}} \hat{\eta}_n e^{2\pi i n x} - \hat{\eta}_0$$

from which it follows

$$\hat{g}_n = \frac{\hat{\eta}_n}{e^{2\pi i n\alpha} - 1} \quad \forall\, n \in \mathbb{Z},\ n \neq 0\,. \tag{4.19}$$

Lemma 4.17. *$\alpha \in D(c, \nu)$ implies $|e^{2\pi i n\alpha} - 1| \geq 4c|n|^{-(\nu+1)}$ for all $n \in \mathbb{Z}$, $n \neq 0$.*

proof. For all $m \in \mathbb{Z}$

$$|e^{2\pi i n\alpha} - 1| = |e^{2\pi i m}(e^{2\pi i(n\alpha - m)} - 1)| = 2|\sin(\pi(n\alpha - m))|\,.$$

Now for all $n \in \mathbb{Z}$ there exists $m \in \mathbb{Z}$ such that $|n\alpha - m| \leq 1/2$, and $|\sin(\pi x)| \geq 2|x|$ if $|x| \leq 1/2$. Then for all $n \in \mathbb{Z}$ there exists $m \in \mathbb{Z}$ such that

$$|e^{2\pi i n\alpha} - 1| \geq 4|n\alpha - m| \geq 4c|n|^{-(\nu+1)}$$

because $\alpha \in D(c, \nu)$. □

Lemma 4.18. $|\hat{\eta}_n| \leq \|\eta\|_\sigma \, e^{-2\pi\sigma|n|}$ *for all* $n \in \mathbb{Z}$.

proof. Let us consider the analytic function $\xi(w)$ defined on the domain $D := \{e^{-2\pi\sigma} < |w| < e^{2\pi\sigma}\}$ by the relation $\eta(z) = \xi(e^{2\pi iz})$. Then $\|\xi\|_D = \|\eta\|_\sigma$ and $\xi(e^{2\pi iz}) = \sum_{n\in\mathbb{Z}} \hat{\eta}_n \, e^{2\pi inz}$. Hence by the Cauchy integral formula, for any $\sigma' \in (0, \sigma)$ and $n > 0$

$$\hat{\eta}_n = \frac{1}{2\pi i} \oint_{|w|=e^{2\pi\sigma'}} \frac{\xi(w)}{w^{n+1}} \, dw \leq \|\xi\|_D \, e^{-2\pi n\sigma'} \, ,$$

and for $n < 0$

$$\hat{\eta}_n = \frac{1}{2\pi i} \oint_{|w|=e^{-2\pi\sigma'}} \frac{\xi(w)}{w^{n+1}} \, dw \leq \|\xi\|_D \, e^{2\pi n\sigma'} \, .$$

□

Lemma 4.19. *We assume here that* $\sigma < (\log 2)/(2\pi)$. *Let* $\delta_0 \in (0, \sigma/6)$ *be such that*

$$2\pi\Gamma(\nu+2) \, \|\eta\|_\sigma < c(2\pi\delta_0)^{\nu+3} < 2\pi\delta_0 \, \Gamma(\nu+2) \, .$$

Then G_0 *with Fourier coefficients given as in* (4.19) *is analytic on* $S_{\sigma-\delta_0}$, *satisfies*

$$\|G_0\|_{\sigma-\delta_0} \leq \frac{\Gamma(\nu+2)}{c(2\pi\delta_0)^{\nu+2}} \, \|\eta\|_\sigma < \delta_0 \, ,$$

and its derivative G_0' *is analytic on* $S_{\sigma-2\delta_0}$ *with*

$$\left\|G_0'\right\|_{\sigma-2\delta_0} \leq \frac{2\pi\Gamma(\nu+2)}{c(2\pi\delta_0)^{\nu+3}} \, \|\eta\|_\sigma < 1 \, .$$

proof. For $z \in S_{\sigma-\delta_0}$ using Lemmas 4.17 and 4.18 we find

$$|G_0(z)| = \left| \sum_{n\in\mathbb{Z}} \frac{\hat{\eta}_n \, e^{2\pi inz}}{e^{2\pi in\alpha} - 1} \right| \leq \sum_{n\neq 0} \frac{|n|^{\nu+1}}{4c} \, \|\eta\|_\sigma \, e^{-2\pi\sigma|n|} \, e^{2\pi|n|(\sigma-\delta_0)} =$$

$$= \frac{\|\eta\|_\sigma}{2c} \sum_{n\geq 1} n^{\nu+1} \, e^{-2\pi n\delta_0} = \frac{\|\eta\|_\sigma}{2c} \sum_{n\geq 1} \frac{(2\pi n\delta_0)^{\nu+1} \, e^{-2\pi n\delta_0}}{(2\pi\delta_0)^{\nu+1}} \leq$$

$$\leq \frac{\|\eta\|_\sigma}{2c} \, \frac{2}{(2\pi\delta_0)^{\nu+2}} \sum_{n\geq 1} \int_{2\pi n\delta_0}^{2\pi(n+1)\delta_0} t^{\nu+1} e^{-t} \, dt \leq$$

$$\leq \frac{\|\eta\|_\sigma}{c(2\pi\delta_0)^{\nu+2}} \int_0^{+\infty} t^{\nu+1} e^{-t} \, dt = \frac{\Gamma(\nu+2)}{c(2\pi\delta_0)^{\nu+2}} \, \|\eta\|_\sigma \, ,$$

where in the third line we have used that $\sigma < (\log 2)/(2\pi)$ to have

$$(2\pi n\delta_0)^{\nu+1}\, e^{-2\pi n\delta_0} < \frac{2}{2\pi\delta_0} \int_{2\pi n\delta_0}^{2\pi(n+1)\delta_0} t^{\nu+1} e^{-t}\, dt\,.$$

Using again Cauchy integral formula we write for all $z \in S_{\sigma-2\delta_0}$ and any $r < \delta_0$

$$|G_0'(z)| = \left| \frac{1}{2\pi i} \oint_{|w-z|=r} \frac{G_0(w)}{(w-z)^2}\, dw \right| \leq \frac{\|G_0\|_{\sigma-\delta_0}}{r}\,.$$

□

Proposition 4.20. *Let $H_0(x) := x + G_0(x)$, then H_0 has an analytic inverse function on $H_0(S_{\sigma-2\delta_0})$ and $S_{\sigma-3\delta_0} \subset H_0(S_{\sigma-2\delta_0})$. Moreover we can write $H_0^{-1}(z) = z - G_0(z) + F_0(z)$ with*

$$\|F_0\|_{\sigma-4\delta_0} \leq \frac{2\pi(\Gamma(\nu+2))^2}{c^2(2\pi\delta_0)^{2\nu+5}}\, \|\eta\|_\sigma^2$$

proof. Since $\|G_0'\|_{\sigma-2\delta_0} < 1$ by Lemma 4.19, the function H_0 is invertible on $H_0(S_{\sigma-2\delta_0})$. Moreover, again by Lemma 4.19 we have $\Im(z + G_0(z)) \geq \sigma - 2\delta_0 - \delta_0 = \sigma - 3\delta_0$ for all $z \in S_{\sigma-2\delta_0}$, hence $S_{\sigma-3\delta_0} \subset H_0(S_{\sigma-2\delta_0})$.

Finally

$$z = H_0^{-1}(H_0(z)) = H_0^{-1}(z+G_0(z)) = z+G_0(z)-G_0(z+G_0(z))+F_0(z+G_0(z))$$

implies

$$F_0(z+G_0(z)) = G_0(z+G_0(z)) - G_0(z) = \int_0^1 G_0'(z+sG_0(z))\, G_0(z)\, ds\,,$$

from which, using that $H_0^{-1}(w) \in S_{\sigma-3\delta_0}$ and $H_0^{-1}(w) + sG_0(H_0^{-1}(w)) \in S_{\sigma-2\delta_0}$ for all $w \in S_{\sigma-4\delta_0}$,

$$\|F_0\|_{\sigma-4\delta_0} \leq \left\|G_0'\right\|_{\sigma-2\delta_0} \|G_0\|_{\sigma-\delta_0}\,.$$

□

Step 2. Let $L_1 : \mathbb{R} \to \mathbb{R}$ be defined as $L_1 = H_0^{-1} \circ L \circ H_0$, where H_0 is given in Proposition 4.20, and write

$$L_1(x) = x + \alpha + \eta_1(x)$$

Proposition 4.21. *L_1 satisfies $\tau(L_1) = \alpha$ and η_1 is the restriction to the real axis of a complex function η_1 with the following properties: η_1 is analytic on a strip $S_{\sigma-6\delta_0}$; $\eta_1(x+1) = \eta_1(x)$ for all $x \in \mathbb{R}$;*

$$\|\eta_1\|_{\sigma-6\delta_0} \leq \frac{16\pi(\Gamma(\nu+2))^2}{c^2(2\pi\delta_0)^{2\nu+5}} \|\eta\|_\sigma^2$$

proof. By definition and Proposition 4.20 we write

$$L_1(x) = H_0^{-1}\Big(H_0(x)+\alpha+\eta(H_0(x))\Big) = H_0^{-1}\Big(x+G_0(x)+\alpha+\eta(x+G_0(x))\Big) =$$

$$= x + G_0(x) + \alpha + \eta(x+G_0(x)) - G_0\Big(x + G_0(x) + \alpha + \eta(x+G_0(x))\Big) +$$

$$+F_0\Big(x + G_0(x) + \alpha + \eta(x+G_0(x))\Big) =$$

$$= x + \alpha + \hat{\eta}_0 + I(x) + II(x) + III(x) =: x + \alpha + \eta_1(x)\,,$$

where we have used that

$$\hat{\eta}_0 = G_0(x) - G_0(x+\alpha) + \eta(x)$$

by definition of G_0 in (4.18), and we have denoted

$$I(x) := \eta(x+G_0(x)) - \eta(x)\,,$$

$$II(x) := G_0(x+\alpha) - G_0\Big(x + G_0(x) + \alpha + \eta(x+G_0(x))\Big)\,,$$

$$III(x) := F_0\Big(x + G_0(x) + \alpha + \eta(x+G_0(x))\Big)\,.$$

Now, since L_1 is topologically conjugate to L it holds $\tau(L_1) = \alpha$, hence there exists $x_0 \in \mathbb{R}$ such that $\eta_1(x_0) = 0$. In particular,

$$\hat{\eta}_0 = -I(x_0) - II(x_0) - III(x_0)\,,$$

so that we can estimate $\hat{\eta}_0$ by uniform estimates on the other terms.

Let us start with $I(x)$. We can write

$$|I(x)| = \left|\int_0^1 \eta'(x+sG_0(x))\, G_0(x)\, ds\right| \leq \left\|\eta'\right\|_{\sigma-\delta_0} \|G_0\|_{\sigma-\delta_0}\,,$$

hence by Lemma 4.19 and the conditions on δ_0,

$$\|I\|_{\sigma-6\delta_0} \leq \frac{2\pi\Gamma(\nu+2)}{c(2\pi\delta_0)^{\nu+3}} \|\eta\|_\sigma^2 \leq \frac{2\pi(\Gamma(\nu+2))^2}{c^2(2\pi\delta_0)^{2\nu+5}} \|\eta\|_\sigma^2\,.$$

Then, since $z + G_0(z) + \alpha + \eta(z + G_0(z)) \in S_{\sigma-4\delta_0}$ if $z \in S_{\sigma-6\delta_0}$,

$$II(x) = -\int_0^1 G_0'\Big(x+\alpha+s(G_0(x)+\eta(x+G_0(x)))\Big)\,(G_0(x)+\eta(x+G_0(x)))\,ds\,,$$

and using Lemma 4.19 and the conditions on δ_0 again

$$\|II\|_{\sigma-6\delta_0} \le \left\|G_0'\right\|_{\sigma-2\delta_0}\,\Big(\|G_0\|_{\sigma-\delta_0} + \|\eta\|_\sigma\Big) \le \frac{4\pi(\Gamma(\nu+2))^2}{c^2(2\pi\delta_0)^{2\nu+5}}\,\|\eta\|_\sigma^2\,.$$

Finally by Proposition 4.20

$$\|III\|_{\sigma-6\delta_0} \le \frac{2\pi(\Gamma(\nu+2))^2}{c^2(2\pi\delta_0)^{2\nu+5}}\,\|\eta\|_\sigma^2\,.$$

□

Step 3.

Proposition 4.22. *For $n \ge 0$ we can define sequences of transformations $H_n(x) := x+G_n(x)$ and $L_n(x) = x+\alpha+\eta_n(x)$, with $L_0 = L$, and sequences of constants $\delta_n = \delta_0/(1+n^2)$, with $c(2\pi\delta_0)^{\nu+3} < 2\pi\delta_0\,\Gamma(\nu+2)$ and $\delta_0 < \sigma/6$, $\sigma_{n+1} = \sigma_n - 6\delta_n$, with $\sigma_0 = \sigma$, $\varepsilon_n = \|\eta\|_\sigma^{(3/2)^n}$, such that if*

$$\|\eta\|_\sigma < \left(\frac{c^2\,(2\pi\delta_0)^{2\nu+5}}{16\pi\,(\Gamma(\nu+2))^2}\right)^2$$

then

- *(i)* $G_n(x+\alpha) - G_n(x) = \eta_n(x) - \int_0^1 \eta_n(s)\,ds$;
- *(ii)* G_n *is analytic on* $S_{\sigma_n-\delta_n}$ *and* $\|G_n\|_{\sigma_n-\delta_n} \le \frac{\Gamma(\nu+2)}{c(2\pi\delta_n)^{\nu+2}}\,\varepsilon_n$;
- *(iii)* $H_n^{-1}(x) = x - G_n(x) + F_n(x)$ *with* $\|F_n\|_{\sigma_n-4\delta_n} \le \frac{2\pi(\Gamma(\nu+2))^2}{c^2(2\pi\delta_n)^{2\nu+5}}\,\varepsilon_n^2$;
- *(iv)* $L_{n+1} = H_n^{-1}\circ L_n \circ H_n$ *and* $\|\eta_{n+1}\|_{\sigma_{n+1}} \le \varepsilon_{n+1}$;
- *(v)* $\lim_{n\to+\infty}\sigma_n = \sigma^* > \frac{\sigma}{2}$.

Step 4. We can now conclude the proof of the theorem. Let $\mathcal{H}_N := H_1 \circ H_2 \circ\cdots\circ H_N$, where the H_i are defined as in Proposition 4.22. Then we can write

$$\mathcal{H}_N(x) = x + G_N(x) + G_{N-1}(x+G_N(x)) + \dots$$
$$\cdots+G_0\Big(x+G_N(x)+G_{N-1}(x+G_N(x))+G_{N-2}(x+G_N(x)+G_{N-1}(x+G_N(x)))+\dots\Big)$$

with $\mathcal{H}_N$ analytic on $S_{\sigma_N-\delta_N}$. Hence by Proposition 4.22-(ii) for all $z \in S_{\sigma_N-\delta_N}$

$$|\mathcal{H}_N(z) - z| \leq \sum_{n=0}^{N} \|G_n\|_{\sigma_N-\delta_N} \leq \sum_{n=0}^{+\infty} \frac{\Gamma(\nu+2)}{c(2\pi\delta_n)^{\nu+2}}\,\varepsilon_n =: \triangle\,.$$

If we now write

$$\mathcal{H}_{N+1}(z)-\mathcal{H}_N(z) = \mathcal{H}_N(H_{N+1}(z))-\mathcal{H}_N(z) = \int_0^1 \mathcal{H}'_N(z+sG_{N+1}(z))\,G_{N+1}(z)\,ds\,,$$

it follows

$$\|\mathcal{H}_{N+1} - \mathcal{H}_N\|_{\sigma^*} \leq \left(1 + \frac{\triangle}{\delta_{N+1}}\right) \frac{\Gamma(\nu+2)}{c(2\pi\delta_{N+1})^{\nu+2}}\,\varepsilon_{N+1}\,,$$

and then

$$\sum_{k=0}^{+\infty} \|\mathcal{H}_{k+1} - \mathcal{H}_k\|_{\sigma^*} < +\infty\,.$$

We can then conclude that the sequence $\{\mathcal{H}_N\}$ converges uniformly to a function $\mathcal{H}^*$ which is analytic on S_{σ^*} and is invertible on a strip $S_{\sigma^{**}} \subset S_{\sigma^*}$.

Finally for all N it holds $L(\mathcal{H}_N(x)) = \mathcal{H}_N(L_N(x)) = \mathcal{H}_N(x+\alpha+\eta_N(x))$, hence passing to the limit $N \to +\infty$ one finds $L \circ \mathcal{H}^* = \mathcal{H}^* \circ R_\alpha$ since η_N vanishes. $\square$

Arnol'd Theorem 4.16 is a *local* result because it holds for OPCHs with Diophantine rotation number which are sufficiently close to the associate irrational rotation. On the contrary, a result is *global* if it holds without any closeness assumption. We conclude the section with a schematic account of regularity results in the local and global case. We assume T to be an OPCH with irrational rotation number α which is conjugate to R_α by the conjugacy h.

Local results ($T = R_\alpha + \eta$, with η "small enough").

- If T is analytic and α is Diophantine with parameters $c, \nu > 0$, then h is analytic (Arnol'd Theorem 4.16).

- If T is analytic and α is Brjuno, (see Definition A.2 for Brjuno numbers $\mathcal{B}$), then h is analytic. If α is not Brjuno, there exists T analytic for which h is not analytic (Yoccoz).

- There exists $T \in C^k$, with $k \geq 3$, and α Liouville for which h is not absolutely continuous (Herman).

Global results.

- There exists a space $\mathcal{H}$ with $\cup_{c,\nu} D(c,\nu) \subsetneq \mathcal{H} \subsetneq \mathcal{B}$, such that if T is analytic and $\alpha \in \mathcal{H}$, then h is analytic. If $\alpha \notin \mathcal{H}$, there exists T analytic for which h is not analytic (Yoccoz).
- If $T \in C^\infty$ and α is Diophantine with parameters $c, \nu > 0$, then $h \in C^\infty$ (Herman-Yoccoz).
- Let β, ν such that $0 \leq \nu < \beta \leq 1$ and $\beta - \nu \neq 1$, if $T \in C^{2,\beta}$ and $\alpha \in D(c,\nu)$ for some $c > 0$, then $h \in C^{1,\beta-\nu}$ (Khanin-Teplinsky).

Chapter 5

Twist maps on cylinders

For the material of this chapter, see [Ba88, Go01, He83, KH95, Si04].

We consider homeomorphisms of a finite or infinite cylinder with a *twist* property, as defined below. The class of maps we study include Examples 1.11, 1.12, 1.13. In the first part, we give results about the existence of periodic orbits. In the second part, we consider the problem of existence and non-existence of *rotationally invariant circles*. In both parts, we give an exposition of the classical results by Poincaré and Birkhoff and of the more recent variational approach by the Aubry-Mather theory.

Let S^1 be represented by $[0,1]/(0 \sim 1)$. Given $a, b \in \mathbb{R} \cup \{\pm\infty\}$ with $a < b$, we consider homeomorphisms on a cylinder

$$T : S^1 \times (a,b) \to S^1 \times (a,b)\,,$$

and their lifts on a strip

$$L : \mathbb{R} \times (a,b) \to \mathbb{R} \times (a,b)\,,$$

satisfying $T \circ \pi = \pi \circ L$, where $\pi : \mathbb{R} \times (a,b) \to S^1 \times (a,b)$ is defined by $\pi(x,y) = (\{x\}\,, y)$.

In the opposite direction, we may consider maps L on a strip which are homeomorphisms and commute with integer translations in the x-direction, that is $L(x+k,y) = L(x,y) + (k,0)$, for all $k \in \mathbb{Z}$, and their projections on a cylinder given by $T = \pi \circ L \circ \pi^{-1}$, which are well defined since $\pi(L(x+n,y)) = \pi(L(x,y)+(n,0)) = \pi(L(x,y))$ for all $x \in S^1$ and $n \in \mathbb{Z}$.

5.1 The Poincaré-Birkhoff Theorem

We begin with the classical result on the existence of fixed points for maps of the annulus with *twist on the boundary.*

Theorem 5.1 (Poincaré-Birkhoff). *Let L be an order-preserving homeomorphism of the closed finite strip $\mathbb{R} \times [a, b]$, with $a, b \in \mathbb{R}$, and use the notation $L(x, y) = (L_1(x, y), L_2(x, y))$. Assume that:*

(i) $L(x + 1, y) = L(x, y) + (1, 0)$ *for all* $(x, y) \in \mathbb{R} \times [a, b]$;

(ii) L *preserves the area;*

(iii) $L_2(x, a) = a$, $L_2(x, b) = b$, *for all* $x \in \mathbb{R}$;

(iv) $(L_1(x, a) - x)(L_1(x, b) - x) < 0$, *for all* $x \in \mathbb{R}$.

Then L admits at least two fixed points P, Q which are not equivalent, i.e.

$$\forall k \in \mathbb{Z}, \qquad P + (k, 0) \neq Q.$$

As discussed above, assumption (i) means that L may be projected to a homeomorphism of the cylinder, or of the annulus, $S^1 \times [a, b]$ by π. Assumption (iii) means that L *preserves the boundaries* of the strip. Finally, assumption (iv) is the *twist on the boundary* condition. Looking at the projected map on the cylinder, it means that the boundaries are rotated in opposite directions, hence the cylinder is twisted on the boundary.

Remark 5.2. We show that assumptions (ii) and (iv) are necessary. First, consider the map

$$L(x, y) = \left(x + y - \frac{1}{2},\ y^2\right) \quad \text{on } \mathbb{R} \times [0, 1]\,.$$

It satisfies (i), (iii), (iv) of Theorem 5.1, but not (ii). It is immediate to verify that L has no fixed points.

The second example is the map

$$L(x, y) = (x + c, y), \quad \text{on } \mathbb{R} \times [0, 1]\,.$$

for any $c > 0$. It satisfies (i), (ii), (iii) of Theorem 5.1, but not (iv). Again, it is clear that L has no fixed points.

Proof of Theorem 5.1. Assumption (iv) has two possible cases, depending on the choice of the signs. We denote by (iv+) the choice

$$L_1(x, a) - x > 0\,, \quad L_1(x, b) - x < 0\,, \quad \forall\, x \in \mathbb{R}\,. \tag{iv+}$$

Equivalently, (iv-) will indicate assumption (iv) with the opposite choice for the signs.

First, we prove the existence of at least one fixed point. By contradiction, let us assume that L satisfies (i)-(ii)-(iii)-(iv+) but has no fixed points. Since $P \neq L(P)$ for all $P \in \mathbb{R} \times [a,b]$, we may consider the function

$$\alpha_L : \mathbb{R} \times [a,b] \to \mathbb{R}, \qquad \alpha_L(P) = \text{angle}\big(\overrightarrow{P\,L(P)}, x\text{-axis}\big)$$

in the anti-clockwise direction. We initialize α_L by setting $\alpha_L(x,a) = 0$ for all $x \in \mathbb{R}$, and introduce the *index*

$$\jmath(L) := \alpha_L\big|_{\{y=b\}}. \tag{5.1}$$

By (iv+), we have $\jmath(L) = \pi \pmod{2\pi}$. We now show that $\jmath(L)$ can be computed by a line integral.

Lemma 5.3. *Given a simple curve $\mathcal{C}$ from $\{y = a\}$ to $\{y = b\}$, the integral*

$$\int_{\mathcal{C}} d\alpha_L$$

is well defined and does not depend on the choice of $\mathcal{C}$.

Proof. Let $\mathcal{C}_1, \mathcal{C}_2$ be two such curves, and let U be the region bounded by them and by the two horizontal arcs I and II on the boundaries of $\mathbb{R} \times [a,b]$. Since on the arcs I and II the angle α_L is constant, we obtain

$$\begin{aligned}\int_{\mathcal{C}_1} d\alpha_L - \int_{\mathcal{C}_2} d\alpha_L &= \int_{\mathcal{C}_1} d\alpha_L - \int_{\mathcal{C}_2} d\alpha_L - \int_I d\alpha_L - \int_{II} d\alpha_L \\ &= -\int_{\partial U} d\alpha_L = -\iint_U d^2\alpha_L = 0,\end{aligned}$$

by Stokes' theorem. □

By Lemma 5.3, we can compute the index $\jmath(L)$ using

$$\jmath(L) = \int_{\mathcal{C}} d\alpha_L \tag{5.2}$$

where $\mathcal{C}$ is any simple curve from $\{y = a\}$ to $\{y = b\}$.

We now use (5.2) to prove the following properties for $\jmath(L)$.

Lemma 5.4. *Let L be a homeomorphism of the strip $\mathbb{R} \times [a,b]$ satisfying assumptions (i)-(ii)-(iii)-(iv). Then:*

(a) L^{-1} satisfies (i)-(ii)-(iii), and (iv-) if L satisfies (iv+) and (iv+) if L satisfies (iv-). Morevoer, $\jmath(L^{-1}) = \jmath(L)$.

(b) *Let $\rho : X \to X$, $\rho(x, y) = (-x, y)$. Then $\rho^{-1}L\rho$ satisfies (i)-(ii)-(iii), and (iv-) if L satisfies (iv+) and (iv+) if L satisfies (iv-). Morevoer, $\rho^{-1}L\rho$ has no fixed points, and $\jmath(\rho^{-1}L\rho) = -\jmath(L)$.*

(c) *If L satisfies (iv+) then $\jmath(L) = \pi$.*

Before proving Lemma 5.4, we use it to conclude the proof of the existence of at least one fixed point. We are assuming that L satisfies (i)-(ii)-(iii)-(iv+) but has no fixed points. Then, using Lemma 5.4, the map $\rho^{-1}L^{-1}\rho$ satisfies (i)-(ii)-(iii)-(iv+) and has no fixed points. Hence,

$$\pi \overset{L.5.4-(c)}{=} \jmath(\rho^{-1}L^{-1}\rho) \overset{L.5.4-(b)}{=} -\jmath(L^{-1}) \overset{L.5.4-(a)}{=} -\jmath(L) \overset{L.5.4-(c)}{=} -\pi,$$

a contradiction. Therefore L has at least one fixed point.

Proof of Lemma 5.4. (a). Properties (i)-(ii)-(iii) for L^{-1} follow immediately. We now show that if L satisfies (iv+) then L^{-1} satisfies (iv-). Let $(L^{-1})_1$ denote the first component of L^{-1}. First, for all $x \in \mathbb{R}$ there exists $\tilde{x} \in \mathbb{R}$ such that $L(\tilde{x}, a) = (x, a)$, hence

$$(L^{-1})_1(x, a) - x = (L^{-1})_1(L_1(\tilde{x}, a), a) - L_1(\tilde{x}, a) = \tilde{x} - L_1(\tilde{x}, a) < 0$$

since L satisfies (iv+). Analogously, $(L^{-1})_1(x, a) - x > 0$ for all $x \in \mathbb{R}$. Hence, L^{-1} satisfies (iv-). In the same way, one can show that if L satisfies (iv-) then L^{-1} satisfies (iv+).

Let now P and Q be points in the strip such that $L^{-1}(Q) = P$. Then $L(P) = Q$ and

$$\alpha_{L^{-1}}(Q) = \alpha_L(P) + \pi.$$

Hence, given any simple curve $\mathcal{C}$ from $\{y = a\}$ to $\{y = b\}$,

$$\jmath(L^{-1}) = \int_{\mathcal{C}} d\alpha_{L^{-1}} = \int_{L^{-1}(\mathcal{C})} d\alpha_L = \jmath(L).$$

(b). Again, we leave to the reader to check (i)-(ii)-(iii) for $\rho^{-1}L\rho$. Let L satisfy (iv+), then for all $x \in \mathbb{R}$

$$\begin{aligned}(\rho^{-1}L\rho)_1(x, a) - x &= (\rho^{-1}L)_1(-x, a) - x = \rho_1(L_1(-x, a), a) - x \\ &= -L_1(-x, a) + (-x) < 0\,,\end{aligned}$$

where again the subscripts denote the first component of the maps. Moreover, if $\rho^{-1}L\rho$ has a fixed point P, then

$$(\rho^{-1}L\rho)(P) = P \quad \Leftrightarrow \quad L(\rho(P)) = \rho(P)\,.$$

Hence, $\rho(P)$ is a fixed point for L. It follows that $\rho^{-1}L\rho$ has no fixed points.

Finally, let us compute $\jmath(\rho^{-1}L\rho)$ by using Lemma 5.3 along the curve $\tilde{\mathcal{C}} = \{x = 0\}$. Since $\tilde{\mathcal{C}}$ is fixed by the function ρ, it follows that

$$\alpha_{\rho^{-1}L\rho}(P) = \pi - \alpha_L(P), \quad \forall P \in \tilde{\mathcal{C}}.$$

Hence,

$$\jmath(\rho^{-1}L\rho) = \int_{\tilde{\mathcal{C}}} d\alpha_{\rho^{-1}L\rho} = \int_{\tilde{\mathcal{C}}} d(\pi - \alpha_L) = -\int_{\tilde{\mathcal{C}}} d\alpha_L = -\jmath(L).$$

(c). Let us extend L to a homeomorphism on $\mathbb{R}^2$ by letting

$$L(x,y) = (L_1(x,a), y) \quad \forall y \le a\,, \qquad L(x,y) = (L_1(x,b), y) \quad \forall y \ge b\,.$$

Since L has no fixed points and using (i), there exists $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{2} \min_{\mathbb{R}\times[a,b]} d(P, L(P)).$$

We use ε to define the pertubation $L_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$L_\varepsilon(x,y) = L(x,y) + (0,\varepsilon).$$

Then L_ε has no fixed points and the function α_ε, a shortcut for α_{L_ε}, is well defined.

We now consider the sets

$$D_0 := \{(x,y) \in \mathbb{R}^2 \,:\, a \le y < a + \varepsilon\}, \qquad D_j := L_\varepsilon^j(D_0).$$

We have

$$L_\varepsilon^{-1}(D_0) = \{a - \varepsilon \le y < a\}, \qquad L_\varepsilon^j(D_0) = \{a + j\varepsilon \le y < a + (j+1)\varepsilon\} \quad \forall j < 0.$$

Thus the sets $\{D_j\}_{j\le 0}$ are disjoint. In particular, it follows that the sets $\{D_j\}_{j\in\mathbb{Z}}$ are disjoint, and since L_ε is area preserving, there exists $N > 0$ such that

$$D_0, D_1, D_2, \ldots, D_{N-1} \subset \mathbb{R}\times[a,b],$$

and D_N is not contained in $\mathbb{R}\times[a,b]$. Let

$$Q \in \overline{D}_N, \qquad Q \notin \mathbb{R}\times[a,b],$$

be the point with largest y-component in $\overline{D}_N$. Let $P_0 \in \{y = a\}$ be the point for which

$$P_j = L_\varepsilon^j(P_0) \in D_j, \quad j = 1, \ldots, N, \qquad Q = L_\varepsilon^{N+1}(P_0).$$

Let $\gamma : [0,\varepsilon] \to \mathbb{R}^2$ be the parametrization of the segment $P_0\,P_1$, hence

$$\gamma(t) = \left(1 - \frac{t}{\varepsilon}\right) P_0 + \frac{t}{\varepsilon} P_1,$$

and let

$$\gamma(t+\varepsilon j) = L_\varepsilon^j(\gamma(t)), \qquad t \in [0,\varepsilon], \quad j = 1,\dots,N,$$

so that

$$\gamma : [0,\varepsilon(N+1)] \to \mathbb{R}^2$$

is a curve joining P_0 to Q, with $\gamma(0) = P$ and $\gamma(\varepsilon(N+1)) = Q$. We now consider the variation of α_ε along the support of γ. For small ε, this variation is close to $\jmath(L)$, hence the thesis follows by the following estimates.

To estimate

$$\int_{\gamma([0,\varepsilon N])} d\alpha_\varepsilon$$

consider

$$\beta : [0,\varepsilon(N+1)] \times [0,\varepsilon(N+1)] \to \mathbb{R}$$

to be the angle between the vector joining two points on the support of γ and the positive x-axis, namely

$$\beta(u,v) := \operatorname{angle}\big(\overrightarrow{\gamma(u)\,\gamma(v)}, x\text{-axis}\big),$$

so that

$$\alpha_\varepsilon(\gamma(t)) = \operatorname{angle}\big(\overrightarrow{\gamma(t)\,L_\varepsilon(\gamma(t))}, x\text{-axis}\big) = \beta(t,t+\varepsilon)$$

for all $t \in [0,\varepsilon N]$. Hence

$$\int_{\gamma([0,\varepsilon N])} d\alpha_\varepsilon = \int_0^{\varepsilon(N+1)} d\beta(t,t+\varepsilon).$$

Since $d\beta$ is closed, we can move the line of integration in the plane (u,v) to the union of two segments,

$$I = \{(u,v) \,:\, u = 0\,,\, \varepsilon \le v \le \varepsilon(N+1)\}$$

and

$$II = \{(u,v) \,:\, 0 \le u \le \varepsilon N\,,\, v = \varepsilon(N+1)\}\,.$$

So that

$$\int_{\gamma([0,\varepsilon N])} d\alpha_\varepsilon = \int_I d\beta + \int_{II} d\beta.$$

Now
$$\beta|_I = \beta(0, v) = \text{angle}\big(\overrightarrow{P_0\,\gamma(v)}, x\text{-axis}\big),$$
and therefore, for ε small enough,
$$\int_I d\beta \in (0, \pi).$$
Similarly
$$\beta|_{II} = \beta(u, \varepsilon(N+1)) = \text{angle}\big(\overrightarrow{\gamma(u)\,Q}, x\text{-axis}\big),$$
and therefore, for ε small enough,
$$\int_{II} d\beta \in (0, \pi).$$
Thus, for ε small enough,
$$\int_{\gamma([0,\varepsilon N])} d\alpha_\varepsilon \in (0, 2\pi).$$
This implies that $\jmath(L) \in (0, 2\pi)$, hence $\jmath(L) = \pi$. □

This concludes the proof of the existence of at least one fixed point for L satisfying (i)-(ii)-(iii)-(iv+). The argument is the same if L satisfies (iv-).

We now sketch the proof of the existence of a second non-equivalent fixed point. Without loss of generality, let the first fixed point be on the line $\{x = 1/2\}$. If there are no fixed points for L in $(-1/2, 1/2) \times [a, b]$, we can define $\jmath(L)$ as in Lemma 5.3 by using a simple curve contained in $(-1/2, 1/2) \times [a, b]$. Hence, we can repeat the same argument as above, with the exception that in the proof of the analogous statement of Lemma 5.4-(c), we need to set
$$L_\varepsilon(x, y) = L(x, y) + (0, \varepsilon\chi(x)),$$
where χ is a continuous 1-periodic function satisfying
$$\int_0^1 \chi(x)\, dx > 0, \qquad \chi(x) \in [0, 1],$$
and
$$\chi(x) = 0 \quad \text{for } x \in [-1/8, 1/8].$$
Then, we consider
$$D_0 = \{-1/2 \le x \le 1/2, a \le y < a + \varepsilon\chi(x)\}$$
and the rest of the proof follows along the same lines. This gives the existence of the second non-equivalent fixed point. □

Corollary 5.5. *Let L be an order-preserving homeomorphism of the closed finite strip $\mathbb{R} \times [a,b]$, with $a, b \in \mathbb{R}$, satisfying assumptions (i)-(ii)-(iii) of the Poincaré-Birkhoff Theorem 5.1. Moreover, we assume the following twist condition on the boundaries: Let $\omega_-, \omega_+ \in \mathbb{R}$ with $\omega_- < \omega_+$ and such that*

$$L_1(x,b) - x \leq \omega_- < \omega_+ \leq L_1(x,a) - x \qquad \forall x \in \mathbb{R},$$

or equivalenty,

$$L_1(x,a) - x \leq \omega_- < \omega_+ \leq L_1(x,b) - x \qquad \forall x \in \mathbb{R}.$$

Then for every $p \in \mathbb{Z}$, $q \in \mathbb{N}$, with $(p,q) = 1$ and such that $p/q \in (\omega_-, \omega_+)$, L has at least two periodic points of type (p,q) which are not equivalent, where P is a periodic point of type (p,q) if

$$L^q(P) = P + (p, 0).$$

Proof. Let $R_k : \mathbb{R} \times [a,b] \to \mathbb{R} \times [a,b]$, $k \in \mathbb{Z}$, be the integer translation in the x-direction, namely

$$R_k(x,y) = (x+k, y).$$

Assume the first condition on the twist on the boundaries. Then, for every $p \in \mathbb{Z}$, $q \in \mathbb{N}$, with $(p,q) = 1$ and such that $p/q \in (\omega_-, \omega_+)$,

$$\widetilde{L} := R_{-p} \circ L^q$$

satisfies (i)-(iv) of Poincaré-Birkhoff Theorem 5.1. Indeed, using that L and R_1 commute by (i),

$$\begin{aligned}
\widetilde{L}(x+1,y) &= (R_{-p} \circ L^q)(x+1,y) = L^q(x+1,y) + (-p,0) \\
&= L^q(x,y) + (-p+1,0) = (R_{-p} \circ L^q)(x,y) + (1,0) \\
&= \widetilde{L}(x,y) + (1,0).
\end{aligned}$$

Conditions (ii) and (iii) are immediate. Moreover, for all $x \in \mathbb{R}$,

$$\begin{aligned}
\big(\widetilde{L}(x,a)\big)_1 - x &= \big(R_{-p} \circ L^q(x,a)\big)_1 - x = \big(L^q(x,a)\big)_1 - x - p \\
&= \sum_{j=1}^{q} \left[L_1\big(L^{j-1}(x,a)\big) - \big(L^{j-1}(x,a)\big)_1\right] - p \geq q\omega_+ - p > 0,
\end{aligned}$$

since by assumption $\omega_+ \leq L_1(Q) - Q_1$ with $Q = (Q_1, a)$. In the same way,

$$\big(\widetilde{L}(x,b)\big)_1 - x < 0, \quad \forall x \in \mathbb{R}.$$

Hence, we can apply Poincaré-Birkhoff Theorem 5.1 to $\widetilde{L}$, to find two non-equivalent fixed points for $\widetilde{L}$. If P is such a point, then

$$R_{-p} \circ L^q(P) = P \quad \Leftrightarrow \quad L^q(P) = P + (p, 0).$$

Finally, we show that given $p, p' \in \mathbb{Z}$, $q, q' \in \mathbb{N}$, with $(p, q) = (p', q') = 1$ and such that $p/q \neq p'/q' \in (\omega_-, \omega_+)$, the periodic points P and P' of type (p, q) and (p', q') are non-equivalent. We have

$$L^q(P) = R_p(P), \qquad L^{q'}(P') = R_{p'}(P').$$

If there exists $k \in \mathbb{Z}$ such that $P = R_k(P')$, then

$$\begin{aligned} &L^q(P') = L^q(R_{-k}(P)) = R_{-k}(L^q(P)) = R_{-k}(R_p(P)) = R_p(P'), \\ &L^{q'}(P') = R_{p'}(P'). \end{aligned}$$

This implies that

$$L^{qq'}(P') = \begin{cases} (L^q)^{q'}(P') = R_p^{q'}(P') = R_{pq'}(P'), \\ (L^{q'})^q(P') = R_{p'}^q(P') = R_{p'q}(P'), \end{cases}$$

so $pq' = p'q$, which is false. □

Example 5.1. Let us consider the map T of the Birkhoff billiards as defined in Example 1.13 in a domain Ω. The map T defines a homeomorphism of the closed finite cylinder $S^1 \times [0, \pi]$, with coordinates (s, ϑ) denoting the arc-length coordinate of the collision of the ball with $\partial\Omega$ and the angle between the trajectory of the ball after the collision and the oriented tangent vector to $\partial\Omega$ at the collision point. Here we show that, after a change of coordinates, T has a lift L which is a homeomorphism on a closed finite strip satisfying the assumptions of Corollary 5.5.

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be the arc-length parameterization of $\partial\Omega$ with components $\gamma(s) = (X(s), Y(s)) \in \mathbb{R}^2$, then $\gamma \in C^3$, $\gamma(0) = \gamma(1)$, and $|\gamma'(s)| = 1$ for all $s \in [0, 1]$. Using the Euclidean distance d in $\mathbb{R}^2$, we consider the function

$$h : [0, 1] \times [0, 1] \to [0, +\infty), \quad h(s_0, s_1) := d(\gamma(s_0), \gamma(s_1)). \tag{5.3}$$

Let now $s_0 \neq s_1$, introduce the notation $P_i = \gamma(s_i)$, $i = 0, 1$, and consider the segment $\overrightarrow{P_0 P_1}$. Clearly $|\overrightarrow{P_0 P_1}| = h(s_0, s_1)$ and it represents the ball trajectory from the collision at P_0 to a collision at P_1. The function h

is differentiable at points not on the diagonal of the square $[0,1] \times [0,1]$, therefore we can compute

$$\begin{aligned}\frac{\partial h}{\partial s_0}(s_0,s_1) &= \frac{1}{h(s_0,s_1)}\Big[-(X(s_1)-X(s_0))X'(s_0)-(Y(s_1)-Y(s_0))Y'(s_0)\Big]\\ &= -\left\langle \frac{\overrightarrow{P_0\,P_1}}{h(s_0,s_1)}\,,\,\gamma'(s_0)\right\rangle = -\cos\vartheta_0,\end{aligned}$$

where ϑ_0 is the angle between $\gamma'(s_0)$ and $\overrightarrow{P_0\,P_1}$. Similarly,

$$\frac{\partial h}{\partial s_1}(s_0,s_1) = \left\langle \frac{\overrightarrow{P_0\,P_1}}{h(s_0,s_1)}\,,\,\gamma'(s_1)\right\rangle = \cos\vartheta_1,$$

where ϑ_1 is the angle between $-\gamma'(s_1)$ and $\overrightarrow{P_1\,P_0}$. It follows that

$$dh(s_0,s_1) = -\cos\vartheta_0\,ds_0 + \cos\vartheta_1\,ds_1\,.$$

Since $T(s_0,\vartheta_0)=(s_1,\vartheta_1)$, we have proved that the measure μ on $S^1\times[0,\pi]$ with density $g(s,\vartheta)=\sin\vartheta$ is T-invariant.

We now introduce the *Birkhoff coordinates*. Let $x := s \in S^1$ and $y := -\cos\vartheta \in [-1,1]$, then T becomes a map on $S^1\times[-1,1]$ with a lift L on the strip $\mathbb{R}\times[-1,1]$. It is now elementary to check that L satisfies the assumptions of Corollary 5.5 with

$$L_1(x,-1)-x = 0 = \omega_- < \omega_+ = 1 = L_1(x,1)-x.$$

As a result, the billiard map has at least two periodic points of type (p,q) for every $p/q\in(0,1)$.

5.2 The variational approach

We now restrict the class of cylinder homeomorphisms by specifying more assumptions on their lifts. First, now we consider smooth diffeomorphisms.

Definition 5.1 (Exact symplectic twist maps)**.** Let $a,b\in\mathbb{R}\cup\{\pm\infty\}$ with $a<b$, and L be an order-preserving C^1 diffeomorphism of the strip $\mathbb{R}\times(a,b)$, and use the notation $L(x,y)=(L_1(x,y),L_2(x,y))$. We say that L is an *exact symplectic twist map* if:

(i) $L(x+1,y)=L(x,y)+(1,0)$ for all $(x,y)\in\mathbb{R}\times(a,b)$;

(ii) $\lim_{y\to a^+} L_2(x,y)=a$, $\lim_{y\to b^-} L_2(x,y)=b$, for all $x\in\mathbb{R}$;

(iii) There exist $\omega_-, \omega_+ \in \mathbb{R} \cup \{\pm\infty\}$, with $\omega_- < \omega_+$, such that

$$\lim_{y \to a^+} \Big(L_1(x,y) - x \Big) = \omega_- \,, \quad \lim_{y \to b^-} \Big(L_1(x,y) - x \Big) = \omega_+ \,, \qquad \forall x \in \mathbb{R};$$

(iv) There exists $K > 0$ such that

$$\frac{\partial L_1}{\partial y}(x,y) \geq K \,, \qquad \forall (x,y) \in \mathbb{R} \times (a,b);$$

(v) There exists a C^2 function

$$h : \Big\{ (u,v) \in \mathbb{R}^2 \,:\, \omega_- < v - u < \omega_+ \Big\} \to \mathbb{R}$$

called the *generating function*, such that $h(u+1, v+1) = h(u,v)$ for all (u,v) in its domain and[1]

$$\partial_1 h(x_0, x_1) = -y_0 \,, \quad \partial_2 h(x_0, x_1) = y_1$$

for all $(x_0, y_0) \mathbb{R} \times (a,b)$ with $(x_1, y_1) = L(x_0, y_0)$.

If L is defined on the closed strip $\mathbb{R} \times [a,b]$, we assume that conditions (ii) and (iii) hold on the boundaries $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$.

A cylinder map T on $S^1 \times (a,b)$ is called exact symplectic twist if its lift L satisfies the conditions in Definition 5.1.

Remark 5.6. Let's make some remarks on the conditions in Definition 5.1.

(i) This is the standard assumption which guarantees that L may be projected to a continuous cylinder map.

(ii) This condition implies that L preserves the boundaries of the strip.

(iii) This is the *twist on the boundaries* condition. The interval (ω_-, ω_+) is called the *interval of twist*.

(iv) This is the first new important condition with respect to the maps of the previous section. We assume that the *twist* occurs on the entire strip and it is strong, since the derivative is bounded away from 0[2]. This condition may be stated by saying that the image of a vertical segment is a curve tilted to the right.

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[1] The notation $\partial_i h$ indicates the partial derivative of h with respect to its i-th variable.

[2] Clearly, an analogous situation occurs when $K < 0$ and $\partial L_1 / \partial y \leq K$ (in this case, we need $\omega_- > \omega_+$ in (iii)).

The twist condition has the following consequences. Let $x_0 < x_1 \in \mathbb{R}$ with $\omega_- < x_1 - x_0 < \omega_+$. Then, by (iii) and the twist, there exists a unique $y_0 \in (a, b)$ such that $L_1(x_0, y_0) = x_1$. Therefore, we can write $y_0 = \tilde{y}_0(x_0, x_1)$. In addition, $y_1 := L_2(x_0, y_0)$ also may be written as a function $y_1 = \tilde{y}_1(x_0, x_1)$.

(v) Using the previous remarks, the condition on h may be written as

$$\tilde{y}_0(x_0, x_1) = -\partial_1 h(x_0, x_1)\,, \quad \tilde{y}_1(x_0, x_1) = \partial_2 h(x_0, x_1)\,.$$

Hence, h is a C^2 function with

$$dh(x_0, x_1) = -\tilde{y}_0\, dx_0 + \tilde{y}_1\, dx_1\,.$$

In particular,

$$0 = d^2 h(x_0, x_1) = -d\tilde{y}_0 \wedge dx_0 + d\tilde{y}_1 \wedge dx_1\,.$$

This implies that L preserves the area on $\mathbb{R} \times (a, b)$. Also note that

$$\partial_{12} h(x_0, x_1) = -\partial_{x_1} \tilde{y}_0(x_0, x_1) = -\Big(\partial_{y_0} x_1(x_0, y_0)\Big)^{-1} \tag{5.4}$$

which is negative by the twist condition.

Finally, we comment on the periodicity assumption on h, namely $h(u + 1, v + 1) = h(u, v)$. Let (x_0, y_0) be a point in $\mathbb{R} \times (a, b)$, and $(x_1, y_1) = L(x_0, y_0) = (L_1(x_0, y_0), L_2(x_0, y_0))$. As explained above, y_0 and y_1 may be written as functions of x_0 and x_1, denoted as $\tilde{y}_0$ and $\tilde{y}_1$. Consider the following identities

$$\tilde{y}_0(x_0, L_1(x_0, y_0)) = y_0, \tag{5.5}$$

$$\tilde{y}_1(x_0, x_1) = L_2(x_0, \tilde{y}_0(x_0, x_1)). \tag{5.6}$$

Then, differentiate (5.5) with respect to x_0 and y_0 to get

$$\frac{\partial \tilde{y}_0}{\partial x_0}(x_0, x_1) + \frac{\partial \tilde{y}_0}{\partial x_1}(x_0, x_1)\frac{\partial L_1}{\partial x_0}(x_0, x_1) = 0$$
$$\frac{\partial \tilde{y}_0}{\partial x_1}(x_0, x_1)\frac{\partial L_1}{\partial y_0}(x_0, x_1) = 1.$$

And differentiate (5.6) with respect to x_0 to get

$$\frac{\partial \tilde{y}_1}{\partial x_0}(x_0, x_1) = \frac{\partial L_2}{\partial x_0}(x_0, x_1) + \frac{\partial L_2}{\partial y_0}(x_0, \tilde{y}_0)\frac{\partial \tilde{y}_0}{\partial x_0}(x_0, x_1).$$

Putting together the three identities and condition (iv), straightforward manipulations give

$$\frac{\partial \tilde{y}_0}{\partial x_1}(x_0,x_1) + \frac{\partial \tilde{y}_1}{\partial x_0}(x_0,x_1) = \Big(\frac{\partial L_1}{\partial y_0}(x_0,x_1)\Big)^{-1}\Big[1 - \det JL(x_0,x_1)\Big],$$

with JL being the Jacobian matrix of L. If, we assume that L is area-preserving, hence $\det JL \equiv 1$, then the 1-form

$$\omega(x_0,x_1) = -\tilde{y}_0\, dx_0 + \tilde{y}_1\, dx_1$$

is closed, hence locally exact. Since $x_1 = L_1(x_0,y_0)$, we can look at ω as a 1-form on the strip $\mathbb{R}\times(a,b)$. This means that L is locally exact symplectic. The form ω is then exact on simply connected domains. However, we wish to have exactness when projecting the map and the form on the cylinder. Therefore, we need to check what happens along closed curves on the cylinder, in particular along the non-contractible ones.

Let $\mathcal{C}$ be one of these curves, in particular, we write it as the points of the form $(x,y(x))$ with $x\in[x_0,x_0+1]$, and $y(x)$ a smooth function with $y(x_0+1)=y(x_0)$. Then,

$$\int_{\mathcal{C}} \omega(x_0,y_0) = \int_{\tilde{\mathcal{C}}} \omega(x_0,x_1)$$

where $\tilde{\mathcal{C}}$ is the curve in the plane x_0,x_1 obtained by letting $y_0=\tilde{y}_0(x_0,x_1)$. The initial and final point of $\tilde{\mathcal{C}}$ are $(x_0,L_1(x_0,y_0))$ and

$$(x_0+1, L_1(x_0+1,y_0)) = (x_0+1, L_1(x_0,y_0)) + (0,1).$$

Since $\omega(x_0,x_1) = dh(x_0,x_1)$, the condition

$$0 = \int_{\tilde{\mathcal{C}}} \omega(x_0,x_1),$$

which is necessary for ω to be exact, is $h(x_0+1,x_1+1) = h(x_0,x_1)$.

In the following, we are interested on the existence of invariant curves for the map T which are not contractible on the cylinder $S^1\times(a,b)$.

Definition 5.2. A curve $\mathcal{C}$ in $S^1\times(a,b)$ is called *rotational* if it is closed, simple, and not contractible.

Definition 5.3. Let T be an area-preserving homeomorphism of the cylinder $S^1 \times (a, b)$. We call *net flux* of T, denoted by Φ_T, the quantity

$$\Phi_T := \int_{T(\mathcal{C})} y\, dx - \int_{\mathcal{C}} y\, dx\,,$$

with $\mathcal{C}$ a rotational curve.

For two rotational curves $\mathcal{C}_1$ and $\mathcal{C}_2$, we have

$$\int_{\mathcal{C}_2} y\, dx - \int_{\mathcal{C}_1} y\, dx = \iint_U dy \wedge dx,$$

where U is the domain bounded by $\mathcal{C}_1$, $\mathcal{C}_2$, and the vertical line connecting the initial points of the two curves. Hence, the difference between Φ_T computed using $\mathcal{C}_1$ or $\mathcal{C}_2$ is equal to the difference between the areas of U and $T(U)$. Since T is area-preserving, the difference vanishes. Hence, the definition of the net flux of T does not depend on the choice of the rotational curve chosen.

Proposition 5.7. *Let T be an exact symplectic twist map of the cylinder $S^1 \times (a, b)$. Then, $\Phi_T = 0$.*

Proof. Let $\mathcal{C}$ be a rotational curve, then

$$\int_{T(\mathcal{C})} y\, dx = \int_{\mathcal{C}} T^*(y\, dx),$$

where $T^*(ydx)$ denotes the push-forward of the one form ydx. Denoting by (x_0, y_0) the coordinates on $\mathcal{C}$, and by $(x_1, y_1) = T(x_0, y_0)$ the coordinates on $T(\mathcal{C})$, we obtain

$$\begin{aligned} \Phi_T &= \int_{\mathcal{C}} \Big[T^*(y_0\, dx_0) - y_0\, dx_0\Big] = \int_{\mathcal{C}} \Big[y_1\, dx_1 - y_0\, dx_0\Big] \\ &= \int_{\tilde{\mathcal{C}}} \Big[\tilde{y}_1\, dx_1 - \tilde{y}_0\, dx_0\Big] = \int_{\tilde{\mathcal{C}}} dh(x_0, x_1) = 0, \end{aligned}$$

where, as above, $\tilde{\mathcal{C}}$ is the curve in the plane x_0, x_1 obtained by letting $y_0 = \tilde{y}_0(x_0, x_1)$, and we have used the periodicity assumption on the generating function h. □

Corollary 5.8. *Let T be an exact symplectic twist map of the cylinder $S^1 \times (a, b)$. Then, for any rotational curve $\mathcal{C}$, we have $\mathcal{C} \cap T(\mathcal{C}) \neq \emptyset$. Moreover, if $\mathcal{C}$ is a rotational T-invariant curve (RIC), then the sets below and above $\mathcal{C}$ are invariant.*

Proof. The first statement follows by Proposition 5.7, since $\int_{\mathcal{C}} y dx$ can be interpreted as the area between $\mathcal{C}$ and the circle $\{y = a\}$[3]. Hence, being the net flux zero, the area of the cylinder below $\mathcal{C}$ and that below $T(\mathcal{C})$ coincide. Therefore, if $\mathcal{C} \cap T(\mathcal{C}) = \emptyset$ we obtain a contradiction.

The second statement follows using also that T is order-preserving. □

Example 5.2. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the order-preserving C^∞ diffeomorphism given by

$$L(x, y) = (x + y, y + 1) .$$

It is immediate that it verifies the conditions from (i) to (iv) of Definition 5.1, with $\omega_\pm = \pm\infty$ and $K = 1$. One can also verify that $h(u, v) = 1/2(v-u)^2+v$ satisfies $\partial_1 h(x_0, x_1) = -y_0$ and $\partial_2 h(x_0, x_1) = y_1$. However, $h(u + 1, v + 1) = h(u, v) + 1$, hence the periodicity condition in (v) is violated. This corresponds to the intuitive property of L of having net flux equal to 1.

Example 5.3. Let $g : S^1 \to \mathbb{R}$ be any C^1 function, and consider the map T on $S^1 \times \mathbb{R}$ defined as the projection of the map $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L(x, y) = (x + y + g(x), y + g(x)) .$$

When $g(x) = -k/(2\pi) \sin(2\pi x)$ we obtain the Standard map of Example 1.11. It is an exercise to verify that L is an exact symplectic twist map with generating function

$$h(u, v) = \frac{1}{2}(v - u)^2 + G(u),$$

with $G'(u) = g(u)$.

It is immediate to verify that also the Bouncing ball map of Example 1.12 is an exact symplectic twist map.

Example 5.4. We now continue Example 5.1. We have proved that the billiard map T on $S^1 \times [-1, 1]$, using Birkhoff coordinates, has a lift L which satisfies conditions (i), (ii), and (iii), of Definition 5.1, with $\omega_- = 0$ and $\omega_+ = 1$. Moreover, condition (v) is satisfied with $h(u, v) = -d(\gamma(u), \gamma(v))$, that is minus the Euclidean distance between two points on the boundary $\partial\Omega$. The periodicity condition is trivial, and the relations with y_0 and y_1 follows from the computations in Example 5.1 and the choice of the Birkhoff coordinates. Finally, we claim that also condition (iv) holds. Hence, the Birkhoff billiard map is an exact symplectic twist map.

[3] If $a = -\infty$, use $\{y = \tilde{a}\}$ for any $\tilde{a} \in \mathbb{R}$ such that $\mathcal{C}$ lies above it.

Dynamics near the fixed points

Before discussing the existence of fixed and periodic points for exact symplectic twist maps via the variational approach, we apply the classification of the fixed points introduced in Section 3.4 to this case.

In particular, we can apply Theorem 3.15 for $d = 2$ to JT under the assumption that $\det(JT) = 1$, since T is area-preserving. It follows that a fixed point (x_0, y_0) is of one of the following three categories.

Hyperbolic: If the eigenvalues of $JT(x_0, y_0)$ are λ, λ^{-1} with $\lambda \in (-\infty, -1) \cup (1, \infty)$.

Parabolic: If the eigenvalues of $JT(x_0, y_0)$ are $\lambda = \lambda^{-1} = \pm 1$.

Elliptic: If the eigenvalues of $JT(x_0, y_0)$ are $\lambda, \bar{\lambda} \in S^1 \setminus \{\pm 1\}$.

Note that a hyperbolic point (x_0, y_0) satisfies Definition 3.13. In particular, the Hartman-Grobman Theorem 3.21 and Theorem 3.22 apply, and there exist one-dimensional local stable and unstable manifolds for (x_0, y_0). Hence, it is a saddle point.

For elliptic fixed points, we cannot apply the results of Section 3.4, but since the map T is area-preserving, the fixed point is stable but not asymptotically stable. Hence, there is a neighborhood, called *island*, on which the dynamics is close to a rotation. The precise dynamics for an exact symplectic twist map is characterized by the result below.

First, we give the simple observation that the nature of a fixed point may be identified just by looking at the trace of JT.

Proposition 5.9. *Assume T is an area-preserving differentiable map of the cylinder $S^1 \times (a, b)$. Then:*

- *A fixed point (x_0, y_0) is hyperbolic if and only if $|tr(JT(x_0, y_0))| > 2$;*
- *A fixed point (x_0, y_0) is parabolic if and only if $|tr(JT(x_0, y_0))| = 2$;*
- *A fixed point (x_0, y_0) is elliptic if and only if $|tr(JT(x_0, y_0))| < 2$.*

Remark 5.10. The same characterization holds for periodic points. If (x_0, y_0) is periodic of period q, then it is a fixed point of T^q. Hence it is hyperbolic, parabolic, or elliptic if the conditions above are satisfied by $JT^q(x_0, y_0)$.

The next result is really important to understand the phenomenon of "islands around islands" in symplectic maps with mixed phase space.

Theorem 5.11 (Birkhoff normal form). *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism such that $L(0) = 0$, and suppose $\det(JL)(0) = 1$ with eigenvalues $\lambda = e^{2\pi i \alpha}$, with $\alpha \in \mathbb{R}$, and $\bar{\lambda}$. If there exists $\ell \in \mathbb{N}$, $\ell \geq 2$, such that $n\alpha \notin \mathbb{Q}$ for all $n = 1, \ldots, \ell - 1$, then there exist neighborhoods $U(0), V(0) \subset \mathbb{R}^2$, and a symplectic diffeomorphism $\phi : U(0) \to V(0)$ such that $\widetilde{T} := \phi \circ T \circ \phi^{-1}$ can be written in polar coordinates $(\theta, r) \in S^1 \times \mathbb{R}^+$ as*

$$\widetilde{T}(\theta, r) = \Big(\theta + \alpha + p(r^2) + O(r^{2m}),\ r + O(r^{2m})\Big),$$

with $2m + 2 \leq \ell - 1$, and

$$p(t) = a_1 t + a_2 t^2 + \cdots + a_m t^m.$$

In addition, if some $a_j \neq 0$,

$$\frac{\partial \widetilde{T}_1}{\partial r} = 2r\, p'(r^2) \neq 0$$

for $r = r_0$ sufficiently small. Hence, $\widetilde{T}$ has a twist in an annulus around $\{r = r_0\}$.

An immediate application of Theorem 5.11 tells us that, generically, in a neighborhood of an elliptic periodic point of an exact symplectic twist map, the dynamics is generated by a twist map. Therefore, all the results giving existence of periodic orbits and RICs hold, giving birth to structure of "second order" living around the periodic point inside its island. In particular, we expect other islands to exists, and so on for smaller and smaller scales.

Periodic points of twist maps as critical points

We now give another proof of the existence of periodic points for an exact symplectic twist map by introducing the variational approach.

Definition 5.4. Let L be an exact symplectic twist map on $\mathbb{R} \times (a, b)$ with generating function h. We call the *action* of a finite sequence $\{x_m, \ldots, x_n\}$ of real values the function

$$W(x_m, \ldots, x_n) := \sum_{k=m}^{n-1} h(x_k, x_{k+1}).$$

Proposition 5.12. *Let $\{x_m, \dots, x_n\}$ be the projections on $\mathbb{R}$ of points of an orbit of an exact symplectic twist map L. Then, the action*

$$(\xi_{m+1}, \dots, \xi_{n-1}) \mapsto W(x_m, \xi_{m+1}, \dots, \xi_{n-1}, x_n)$$

has a critical point at $(x_{m+1}, \dots, x_{n-1})$. Conversely, fixed x_m and x_n, every such critical point determines an orbit segment.

Proof. Write

$$W(x_m, \xi_{m+1}, \dots, \xi_{n-1}, x_n) = h(x_m, \xi_{m+1}) + \sum_{k=m+1}^{n-2} h(\xi_k, \xi_{k+1}) + h(\xi_{n-1}, x_n).$$

Then,

$$\frac{\partial W}{\partial \xi_k} = \partial_2 h(\xi_{k-1}, \xi_k) + \partial_1 h(\xi_k, \xi_{k+1}), \quad \forall\, k = m+1, \dots, n-1$$

where $\xi_m = x_m$ and $\xi_n = x_n$. Hence,

$$\frac{\partial W}{\partial \xi_k}(x_{m+1}, \dots, x_{n-1}) = 0, \quad \forall\, k = m+1, \dots, n-1$$

if and only if

$$\partial_2 h(x_{k-1}, x_k) + \partial_1 h(x_k, x_{k+1}) = 0, \quad \forall\, k = m+1, \dots, n-1. \tag{5.7}$$

It follows that, if $\{(x_k, y_k)\}$ for $k = m, \dots, n$ are the points of an orbit of L, then the conditions on the generating function imply that

$$y_k = \partial_2 h(x_{k-1}, x_k) = -\partial_1 h(x_k, x_{k+1}) \quad \forall\, k = m+1, \dots, n-1,$$

so (5.7) holds. Conversely, if (5.7) holds, we have an orbit by letting $y_m = -\partial_1 h(x_m, x_{m+1})$, $y_k = \partial_2 h(x_{k-1}, x_k)$ for $k = m+1, \dots, n-1$, and $y_n = \partial_2 h(x_{n-1}, x_n)$. □

Theorem 5.13 (General Poincaré–Birkhoff)**.** *Let L be an exact symplectic twist map on $\mathbb{R} \times (a, b)$ with interval of twist (ω_-, ω_+). Then, for every $p \in \mathbb{Z}$, $q \in \mathbb{N}$, with $(p, q) = 1$ and $p/q \in (\omega_-, \omega_+)$, there exist at least two periodic orbits for the projected map T of type (p, q), i.e. points (x_0, y_0) such that $L^q(x_0, y_0) = (x_0 + p, y_0)$.*

Proof. Fix $p \in \mathbb{Z}$, $q \in \mathbb{N}$, with $(p,q) = 1$ and $p/q \in (\omega_-, \omega_+)$, and consider the action

$$(x_0, \xi_1, \dots, \xi_{q-1}) \mapsto W(x_0, \xi_1, \dots, \xi_{q-1}, x_0 + p),$$

which is given by

$$W(x_0, \xi_1, \dots, \xi_{q-1}, x_0 + p) = h(x_0, \xi_1) + \sum_{k=1}^{q-2} h(\xi_k, \xi_{k+1}) + h(\xi_{q-1}, x_0 + p).$$

First, we show that critical points of this action produce periodic orbits of type (p,q). Let $(x_0, x_1, \dots, x_{q-1})$ be such a critical point, then we obtain an orbit

$$(x_0, y_0),\ (x_1, y_1),\ \dots,\ (x_{q-1}, y_{q-1}),\ (x_0 + p, y_q),$$

by defining $y_0 = -\partial_1 h(x_0, x_1)$,

$$y_k = -\partial_1 h(x_k, x_{k+1}) = \partial_2 h(x_{k-1}, x_k), \quad \forall k = 1, \dots, q-1,$$

using the computations in Proposition 5.12, and $y_q = \partial_2 h(x_{q-1}, x_0 + p)$.

Use now the periodicity of h to write the identity

$$W(x_0, \xi_1, \dots, \xi_{q-1}, x_0 + p) = W(\xi_1, \dots, \xi_{q-1}, x_0 + p, \xi_1 + p).$$

Since $(x_1, \dots, x_{q-1}, x_0)$ is a critical point of the right-hand side, differentiating with respect to x_0 we obtain

$$\partial_2 h(x_{q-1}, x_0 + p) + \partial_1 h(x_0 + p, x_1 + p) = 0.$$

Hence, $y_q = -\partial_1 h(x_0 + p, x_1 + p) = -\partial_1 h(x_0, x_1) = y_0$. Thus, $L^q(x_0, y_0) = (x_0 + p, y_0)$. Moreover, the period of (x_0, y_0) is minimal because $(p,q) = 1$.

We now need to show that the action

$$(x_0, \xi_1, \dots, \xi_{q-1}) \mapsto W(x_0, \xi_1, \dots, \xi_{q-1}, x_0 + p),$$

has at least one critical point. Restrict the domain to the set

$$D := \left\{(x_0, \xi_1, \dots, \xi_{q-1}) \in \mathbb{R}^q \,:\, 0 \le x_0 \le \xi_1 \le \xi_2 \le \cdots \le \xi_{q-1} \le x_0 + p \le 1 + p\right\}.$$

This set is compact, therefore W attains a minimum. We need to show that the minimum is attained in the interior of D.

Let $(x_0, x_1, \dots, x_{q-1})$ be a point of minimum such that for some k, $x_{k-2} < x_k = x_{k+1} < x_{k+2}$. Since the point is critical,

$$y_k = \partial_2 h(x_{k-1}, x_k) = -\partial_1 h(x_k, x_{k+1}),$$

and

$$y_{k+1} = \partial_2 h(x_k, x_{k+1}) = -\partial_1 h(x_{k+1}, x_{k+2}).$$

Subtracting gives

$$0 = \partial_2 h(x_{k-1}, x_k) - \partial_2 h(x_k, x_{k+1}) + \partial_1 h(x_k, x_{k+1}) - \partial_1 h(x_{k+1}, x_{k+2}).$$

Applying the mean value theorem, using $x_k = x_{k+1}$, gives

$$0 = \partial_{12} h(\eta_1, x_k)(x_{k-1} - x_k) + \partial_{21} h(x_{k+1}, \eta_2)(x_{k+1} - x_{k+2}),$$

for suitable intermediate points η_1, η_2. Since $\partial_{12} h < 0$ by (5.4), and

$$x_{k-1} - x_k < 0, \qquad x_{k+1} - x_{k+2} < 0,$$

we obtain a contradiction. Hence

$$x_0 < x_1 < \cdots < x_{q-1} < x_0 + p.$$

Finally, we prove the existence of at least a second critical point. If $(x_0, x_1, \ldots, x_{q-1})$ is a point of minimum for

$$W(\xi_0, \xi_1, \ldots, \xi_{q-1}, \xi_0 + p)$$

in D, then for every $k \in \{0, \ldots, q-1\}$ there exists $j_k \in \mathbb{Z}$ such that

$$x_k + j_k \in [0, 1]$$

and the point

$$P_k = (x_k + j_k,\ x_{k+1} + j_k,\ \ldots,\ x_{q-1} + j_k,\ x_0 + p + j_k,\ \ldots,\ x_{k-1} + p + j_k) \in D.$$

By periodicity of h, we have

$$W(P_k, x_k + p) = W(x_0, x_1, \ldots, x_{q-1}, x_0 + p),$$

hence W possesses at least one min-max critical point. □

Remark 5.14. The proof of Theorem 5.13 gives that the periodic orbits may be points of minimum or of min-max for the action functional. It is remarked in [Me92], that the minima give rise to hyperbolic periodic orbits, whereas the min-max are hyperbolic or elliptic orbits.

Example 5.5. Consider the map T of the Birkhoff billiards as described in Examples 5.1 and 5.4 with generating function given by minus the Euclidean distance between two points on the boundary of the domain of the billiard. Proposition 5.12 implies that if we want to send the billiard ball from the point P in the boundary to a point Q with one intermediate bounce, we should make the ball to hit the boundary in the point for which the length of the entire trajectory is a critical point. Also, by Theorem 5.13, the periodic orbits of the billiard map give rise to closed paths in the domain which are critical points for the length. An example is given by regular polygons inside the circular billiard.

5.3 Birkhoff theory and Mather sets

We now consider the rotational invariant curves (RIC) for a twist map. We have seen in Corollary 5.8 that the existence of a RIC for an exact symplectic twist map implies that the orbits with initial conditions below the curve remain below. However, the existence of RICs is a difficult problem. We begin with classical results about the properties of these curves, assuming that they exist.

Theorem 5.15 (Birkhoff). *Let $T : S^1 \times (a, b) \to S^1 \times (a, b)$ be an area-preserving diffeomorphism with the twist condition*[4] *and zero net flux, and let U be an open invariant set satisfying the following*

(i) there exist $r_0, r_1 \in (a, b)$ with $r_0 < r_1$ such that $S^1 \times \{a < y < r_0\} \subset U \subset S^1 \times \{a < y < r_1\}$;

(ii) U is homeomorphic to $S^1 \times \mathbb{R}$;

(iii) ∂U is homeomorphic to S^1.

Then ∂U is the graph of a continuous function $\psi : S^1 \to \mathbb{R}$.

Proof. Let's assume that T has the positive twist condition, the negative case is proved analogously.

Given a simple regular curve $\gamma : [0, 1] \to U$ with $\dot{y}(0) > 0$ and $\gamma(0)$ sufficiently below ∂U (for example $\gamma(0) \in \{y = a + \varepsilon\}$ for $\varepsilon > 0$ if $a \in \mathbb{R}$), let $\delta(t)$ be the angle measured in the counter-clockwise direction from the positive y-axis to the tangent vector $\gamma'(t)$, with the choice that $\delta(0) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

[4] If $T = (T_1, T_2)$ we assume that there exists $K > 0$ (or $K < 0$) such that $\partial T_1/\partial y \geq K$ (or $\partial T_1/\partial y \leq K$) everywhere on the cylinder.

and $t \mapsto \delta(t)$ is a continuous function. We say that a curve γ is *tilted to the right* if $\delta(t) \leq 0$ for all t. Since T has the positive twist property, if γ is tilted to the right, then $T(\gamma)$ is tilted to the right (let $v = (0, 1)$ be a vector applied to $\gamma(t)$, then $dT(v) = (\partial T_1/\partial y(\gamma(t)), \partial T_2/\partial y(\gamma(t)))$ is to the right of the vertical; moreover since T preserves the orientation $dT(\gamma'(t))$ makes a negative angle with $dT(v)$). Then we define the set of *right-accessible points*

$$W^R := \{P \in U \, : \, \exists \text{ a curve } \gamma \text{ tilted to the right such that } \gamma(1) = P\}$$

and by the previous argument $T(W^R) \subset W^R$. Analogously we define curves which are tilted to the left and the set of left-accessible points W^L. Then $T^{-1}(W^L) \subset W^L$.

If $W^R \subsetneq U$, then ∂W^R is made of vertical segments and parts of ∂U. In particular, this means that U has lobes to the left. Moreover, we obtain that $T^{-1}(W^R) \cap (W^R)^c$ is non-empty and close to ∂U. We now want to obtain a contradiction. By considering a segment $\{y = y_0\}$ far enough from ∂W^R, we can use that T is area-preserving and has zero net flux to show that

$$\text{Area}\Big(W^R \cap \{y \geq y_0\}\Big) = \text{Area}\Big(T^{-1}(W^R \cap \{y \geq y_0\})\Big) =$$
$$= \text{Area}\Big(T^{-1}(W^R) \cap \{y \geq y_0\}\Big).$$

However, this is a contradiction with $T(W^R) \subsetneq W^R$. Hence $W^R = U$, and U has no lobes to the left.

The same argument shows that $W^L = U$. Hence, U has no lobes and ∂U is the graph of a function. $\square$

Theorem 5.16. *Let $T : S^1 \times (a, b) \to S^1 \times (a, b)$ be an area-preserving diffeomorphism with the positive twist condition with constant K and zero net flux. Then:*

(i) all RICs are graphs of continuous functions;

(ii) all RICs are graphs of Lipschitz functions;

(iii) given two RICs $\mathcal{C}_1$ and $\mathcal{C}_2$ which are graphs of the functions φ_1 and φ_2 respectively, if $\varphi_1 < \varphi_2$ then for the rotation numbers $\tau(\mathcal{C}_1) < \tau(\mathcal{C}_2)$;

(iv) the set of RICs contained in a bounded subset $S^1 \times [c, d]$ is compact (in the C^0 topology for the functions φ);

(v) the set of rotation numbers $\tau(\mathcal{C})$ of a compact set of RICs is compact;

(vi) *if there exist* $c, d \in \mathbb{R}$ *such that* $T^n(\{y < c\}) \subseteq \{y < d\}$ *for all* $n \in \mathbb{Z}$, *then there exists a RIC in the set* $\{c \le y \le d\}$;

(vii) *let* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *be two RICs with* $\tau(\mathcal{C}_1) < \tau(\mathcal{C}_2)$, *and such that there exist no RIC with rotation number* $\tau \in (\tau(\mathcal{C}_1), \tau(\mathcal{C}_2))$, *and let us denote by* U *the annulus bounded by* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *(a Birkhoff region of instability). Then for all* $\varepsilon > 0$ *there exists* $P \in U$ *such that* $d(P, \mathcal{C}_1) < \varepsilon$ *and such that there exists* $n(P) \in \mathbb{Z}$ *for which* $d(T^{n(P)}(P), \mathcal{C}_2) < \varepsilon$.

Proof. (i) It follows directly from Theorem 5.15.

(ii) Given a RIC $\mathcal{C}$ and a point $P \in \mathcal{C}$, if $v = (0,1)$ is a vector applied to P, it holds $dT(v) = (\partial T_1/\partial y(P), \partial T_2/\partial y(P))$, so that the slope of the vector $dT(v)$, which is applied to $T(P)$, is

$$\frac{\partial T_2}{\partial y}(P)\left(\frac{\partial T_1}{\partial y}(P)\right)^{-1} \le K^{-1}\max\left|\frac{\partial T_2}{\partial y}\right| =: S_+\,.$$

At the same time, let $Q = T^2(P)$, then if $w = (0,-1)$ is a vector applied to Q, it holds $dT^{-1}(w) = (\partial T_1/\partial y(T(P)), -\partial T_1/\partial x(T(P)))$, so that the slope of the vector $dT^{-1}(w)$, which is applied to $T(P)$, is

$$-\frac{\partial T_1}{\partial x}(T(P))\left(\frac{\partial T_1}{\partial y}(T(P))\right)^{-1} \ge -K^{-1}\max\left|\frac{\partial T_1}{\partial x}\right| =: S_-\,.$$

Since T preserves the orientation, it follows that $\mathcal{C}$ is contained in the cone of vectors with slopes between S_- and S_+, hence the function φ describing $\mathcal{C}$ is Lipschitz.

(iii) Given RIC $\mathcal{C}$, which is the graph of the function $\varphi : S^1 \to \mathbb{R}$, for each $x \in S^1$ one writes

$$T(x, \varphi(x)) = (g(x), \varphi(g(x))),$$

where $g : S^1 \to S^1$ is a diffeomorphism of the circle. We define $\tau(\mathcal{C}) := \tau(g)$. Let then $g_1, g_2 : S^1 \to S^1$ be the diffeomorphisms of the circle associated to the RICs $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. By definition, if $\varphi_1 < \varphi_2$

$$g_2(x) - g_1(x) = \int_0^1 \frac{d}{dt}\, T_1\Big(x, \varphi_1(x) + t(\varphi_2(x) - \varphi_1(x))\Big)\, dt =$$

$$= \int_0^1 \frac{\partial T_1}{\partial y}\Big(x, \varphi_1(x) + t(\varphi_2(x) - \varphi_1(x))\Big)\,(\varphi_2(x) - \varphi_1(x))\, dt \ge$$

$$\ge K\,(\varphi_2(x) - \varphi_1(x)) > 0\,.$$

By Proposition 4.14-(ii), it follows $\tau(\mathcal{C}_1) \leq \tau(\mathcal{C}_2)$. By Proposition 4.14-(iii), the inequality is strict if one of the rotation numbers is irrational. Let us consider the case $\tau(\mathcal{C}_1), \tau(\mathcal{C}_2) \in \mathbb{Q}$, and suppose that $\tau(\mathcal{C}_1) = \tau(\mathcal{C}_2) = p/q$, with $(p, q) = 1$.

We now consider the lift $L : \mathbb{R} \times (a, b) \to \mathbb{R} \times (a, b)$ of T. Let $P_1 = (x_1, y_1)$ be a fixed point for $R_{-p} \circ L^q$ on $\mathcal{C}_1$, and $P_2 = (x_2, y_2)$ a fixed point for $R_{-p} \circ L^q$ on $\mathcal{C}_2$ with $x_2 \geq x_1$. The segment P_1P_2 is tilted to the right by the action of L, hence the same is true for $R_{-p} \circ L^q(P_1P_2)$, and this implies that $x_2 > x_1$. Let now consider the points Q_1 and Q_2 defined as follows. The point Q_1 is on $\mathcal{C}_2$ and is of the form $(x_1, \varphi_2(x_1))$, whereas the point Q_2 is on $\mathcal{C}_1$ and is of the form $(x_2, \varphi_1(x_2))$. We can then consider the region U delimited by the segment P_1Q_1, the segment P_2Q_2, and the rotational invariant circles $\mathcal{C}_1$ and $\mathcal{C}_2$. The segment P_1Q_1 is tilted to the right by L, hence the same is true for $R_{-p} \circ L^q(P_1Q_1)$, and moreover the end points of $R_{-p} \circ L^q(P_1Q_1)$ are P_1 and a point $(\tilde{x}, \varphi_2(\tilde{x}))$ with $\tilde{x} \in (x_1, x_2)$, otherwise L would not be order preserving. Analogously, the curve $R_{-p} \circ L^q(P_2Q_2)$ has end points in P_2 and a point $(\bar{x}, \varphi_1(\bar{x}))$ with $\bar{x} \in (x_1, x_2)$. Since the other parts of the boundary are invariant, it follows that $R_{-p} \circ L^q(U) \subset U$, which is absurd since T is area-preserving.

(iv) Immediate from continuity of the functions φ and of the map T.

(v) Immediate from Proposition 4.14-(i) and point (iv) of this theorem.

(vi) Let V be defined as

$$V := \bigcup_{n \in \mathbb{Z}} T^n(\{y < c\})$$

Then V is invariant and contained in $\{y < d\}$, but V is not necessarily homeomorphic to $S^1 \times \mathbb{R}$ because of the possible existence of elliptic islands in the strip $\{c \leq y \leq d\}$. However if W denotes the connected component of V^c which contains $\{y > d\}$, then the set $U := W^c$ satisfies the assumptions of Theorem 5.15, whence ∂U is a RIC.

(vii) It follows from point (vi) by contradiction. □

Before considering the variational approach to the RICs, we recall two results on the existence of RICs for twist maps which are small perturbations of *integrable* maps. These results are the discrete versions of the famous Kolmogorov-Arnold-Moser Theorems for Hamiltonian systems.

We begin with a result for analytic maps, whose proof is similar to that of Theorem 4.16 and can be found in [SM71]. To simplify notations, consider the map

$$T_0 : S^1 \times [0,1] \to S^1 \times [0,1]\,, \qquad T_0(x,y) = (x+y,y)\,.$$

This is the standard form of an integrable twist map of the cylinder. It's immediate to verify that it satisfies the conditions of Definition 5.1, and it is integrable since the horizontal circles $\{y = y_0\}$ are invariant for all $y_0 \in [0,1]$. We remark that a map of the form $(x,y) \mapsto (x+\alpha(y),y)$ with $\alpha(0) = 0$, $\alpha(1) = 1$, and $\alpha'(y) > 0$, can be reduced to T_0 by a change of variables. We now state the result on the existence of RICs for analytic perturbations of T_0.

Theorem 5.17 (Analytic KAM)**.** *Let $T : S^1 \times [0,1] \to S^1 \times [0,1]$ be an area-preserving diffeomorphism with the positive twist condition and zero net flux, given by*

$$T(x,y) = (x+y+f(x,y), y+g(x,y))\,.$$

Assume that f, g are real-analytic functions and 1-periodic in x, and that there exist a constant $r_0 \in (0,1)$ and a neighborhood $I \subset \mathbb{C}$ of the interval $[0,1]$, such that f, g can be extended to holomorphic functions in the domain

$$D_{r_0} := \left\{(x,y) \in \mathbb{C}^2 \,:\, |\Im(x)| < r_0\,,\, y \in I\right\}\,,$$

and denote by $\|\cdot\|_{\infty,r_0}$ the sup-norm of a function restricted to D_{r_0}.

Given $\omega \in D(c,\nu)$ for some $c,\nu > 0$ (see Definition A.1), for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|f\|_{\infty,r_0} + \|g\|_{\infty,r_0} < \delta$, then there exist functions u, v holomorphic in the strip

$$S_{r_0/2} := \left\{\xi \in \mathbb{C} \,:\, |\Im(\xi)| < \frac{r_0}{2}\right\}\,,$$

which are real-valued when restricted to $\Im(\xi) = 0$ and are 1-periodic in $\Re(\xi)$, such that $\|u\|_{\infty,r_0/2} + \|v-\omega\|_{\infty,r_0/2} < \varepsilon$ ($\|\cdot\|_{\infty,r_0/2}$ denotes the sup-norm of a function restricted to $S_{r_0/2}$), the curve

$$\mathcal{C} = \{(\xi + u(\xi), v(\xi)) \,:\, \Im(\xi) = 0\,,\, \Re(\xi) \in [0,1]\}$$

is a RIC, and $T|_{\mathcal{C}}$ satisfies $T(\xi + u(\xi), v(\xi)) = (\xi + \omega + u(\xi+\omega), v(\xi+\omega))$.

This formulation shows that, for any fixed Diophantine number ω, small enough analytic perturbations of the integrable map T_0 have a RIC with rotation number ω.

The next result deals with differentiable perturbations of T_0 and shows the existence of a positive area set of RICs.

Theorem 5.18 (Differentiable KAM). *Let $T : S^1 \times [0,1] \to S^1 \times [0,1]$ be an area-preserving C^k, $k > 3$, diffeomorphism with the positive twist condition and zero net flux, given by*

$$T(x,y) = (x + y + f(x,y), y + g(x,y)) \,.$$

Then, for all $\omega \in D(c,\nu)$ with $0 < \nu < (k-3)/2$, there exists $\varepsilon > 0$ such that if $\|f\|_{C^k} + \|g\|_{C^k} < c^2\varepsilon$, there exists a RIC $\mathcal{C}_\omega$ with the following properties:

(i) $\mathcal{C}_\omega$ is the graph of a Lipschitz function $\phi : S^1 \to \mathbb{R}$.

(ii) There exists $F_\omega : S^1 \to S^1$, an order-preserving homeomorphism of the circle with $\tau(F_\omega) = \omega$, such that

$$\mathcal{C}_\omega \ni (x,\phi(x)) \longmapsto T(x,\phi(x)) = \big(F_\omega(x), \phi(F_\omega(x))\big) \in \mathcal{C}_\omega,$$

and F_ω is conjugated to the rotation R_ω.

It is interesting to remark that the condition $k > 3$ in Theorem 5.18 is sharp, there are counterexamples for C^3 maps in [He83]. Moreover, the required smallness of the perturbation depends only on the constant c for numbers in $D(c,\nu)$. Hence, the existence of a RIC with rotation number ω holds for all $\omega \in D(c,\nu)$ and c sufficiently small. Since $m(D(c,\nu)) \to 1$ as $c \to 0^+$ for all $\nu > 0$, fixed $\varepsilon > 0$, we obtain the existence of a RIC $\mathcal{C}_\omega$ for a positive measure set of numbers ω. This implies the existence of a positive measure set of RICs in the cylinder.

Example 5.6. It is interesting to recall from [La73] the following application of Theorem 5.18 to the Birkhoff billiards. Let $T : S^1 \times [-1,1] \to S^1 \times [-1,1]$ be the billiard map inside a domain Ω from Example 5.1. Then, Lazutkin has introduced in [La73] coordinates (X,Y), with Y vanishing close to the boundaries $\{y = \pm 1\}$, such that if Ω is strictly convex and $\partial\Omega$ is of class C^6 with strictly positive curvature, then the map T becomes

$$\tilde{T}(X,Y) = (X + Y + O(Y^3), Y + O(Y^4)) \,.$$

Hence, for $Y \approx 0$, the map $\tilde{T}$ is a small perturbation of the integrable map T_0. In particular, there exists a positive measure set of RICs in $S^1 \times [-1,1]$.

If Ω is a circle, these RICs give a foliation of the phase space $S^1 \times [-1,1]$, and if Ω is an ellipse the foliation is restricted to a neighborhood of the boundaries $\{y = \pm 1\}$. The well-known *Birkhoff conjecture* states that this is the case if and only if Ω is an ellipse.

Finally, we now assume that T is an exact symplectic twist map and ask what we obtain about the existence of RICs by the variational approach introduced in Section 5.2. Unfortunately not much more than the previous results. But it follows that RICs are a special case of a more general kind of invariant sets, the *Mather sets*. Here, we sketch the basic definitions and results. The reader may consult [Ba88, Go01, Si04] for more details.

Given an exact symplectic twist map T with generating function h, recall Definition 5.4 for the action W of a finite sequence. It has been used in Section 5.2 to characterize periodic orbits as critical points of W. To study the RICs, we need to extend the approach to non-periodic orbits. We introduce the useful notion and collect some of the results of the theory in a single statement.

In the variational approach, we look at the projection of the orbits on the x-axis. Let $\{(x_k, y_k\}_{k\in\mathbb{Z}}$ be an orbit for L, the lift of T to the strip $\mathbb{R} \times (a, b)$. Then, we look at the sequence $\{x_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, from which it is possible to reconstruct the orbit using the generating function h.

Definition 5.5. A sequence $\{x_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ is called *minimal* if each finite segment $\{x_m, \ldots, x_n\}$ is a minimizing point of the action W, that is

$$W(x_m, x_{m+1}, \ldots, x_n) \leq W(\xi_m, \ldots, \xi_n)$$

for every $\{\xi_m, \ldots, \xi_n\}$ with $\xi_m = x_m$ and $\xi_n = x_n$.

Definition 5.6. For a sequence $\{x_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, we call *rotation number* the quantity

$$\omega := \lim_{n\to+\infty} \frac{x_n - x_0}{n},$$

if it exists.

Theorem 5.19 (Mather)**.** *Let T be an exact symplectic twist map of a cylinder, then:*

(i) A minimal sequence $\{x_k\}_{k\in\mathbb{Z}}$ is monotone, that is if for some $i, j \in Z$ there exists $p \in \mathbb{Z}$ such that $x_i < x_j + p$, then $x_{i+1} < x_{j+1} + p$.

(ii) For a monotone sequence $\{x_k\}_{k\in\mathbb{Z}}$, the rotation number exists and it is continuous with respect to pointwise convergence of sequences.

(iii) The sequences $\{x_k\}_{k\in\mathbb{Z}}$ associated to the periodic orbits of type (p, q) which have minimizing segments are minimal, and their rotation number is $p/q \in \mathbb{Q}$.

(iv) *For $\omega \in \mathbb{R} \setminus \mathbb{Q}$ in the interval of twist (ω_-, ω_+) of T, let $\{p_n/q_n\} \subset \mathbb{Q}$ be a sequence converging to ω. Then, the minimal periodic orbits of type (p_n, q_n) converge to a minimal orbit with rotation number ω.*

(v) *A minimal orbit of T lives on the graph of a Lipschitz function.*

Thus, for all $\omega \in (\omega_-, \omega_+)$, the set $\mathcal{M}_\omega$ of minimal sequences $\{x_k\}_{k\in\mathbb{Z}}$ with rotation number ω is non-empty. If:

- *$\omega \in \mathbb{Q}$, the set M_ω is given by periodic orbits and orbits heteroclinic or homoclinic to periodic orbits;*
- *$\omega \in \mathbb{R} \setminus \mathbb{Q}$, the set M_ω is given by orbits whose closure in $S^1 \times (a, b)$ is a RIC or a Cantor set.*

In particular, Theorem 5.19 shows that if there exists $\omega \in (\omega_-, \omega_+)$ for which there is no RIC with rotation number ω, nonetheless we can find an invariant closed Cantor set of orbits with rotation number ω. Contrarily to the case of the RICs, these Cantor sets do not bound the orbits with initial conditions below them, hence transport through them is admitted.

5.4 Converse KAM

We conclude our analysis of the twist maps on cylinders, by considering methods of *converse KAM*. These are methods to prove the non-existence of a RIC for a given rotation number or for all of them. The importance of these methods lies in the following result, whose proof can be found in [An90, An92].

Theorem 5.20 (Angenent)**.** *Let T be an exact symplectic twist map of a cylinder with interval of twist (ω_-, ω_+). If the topological entropy $h_{top}(T)$ of T vanishes (see Definition 3.9), then T must have a RIC with rotation number ω, for any $\omega \in (\omega_-, \omega_+)$.*

In particular, showing the non-existence of a single RIC implies the existence of a Birkhoff region of instability, see Theorem 5.16-(vii). Restricting T to this region we find that the map is chaotic, in the sense that its topological entropy is positive.

In this section, T is an exact symplectic twist map of the cylinder $S^1 \times (a, b)$ or a specific map introduced before.

Wandering orbits

This first set of methods uses the idea of producing orbits which move along the cylinder.

For example, if one can show the existence of an orbit in the cylinder $S^1 \times (a, b)$ which goes arbitrarily close to the boundaries $\{y = a\}$ and $\{y = b\}$, then there is no RIC. This follows directly from Theorem 5.16.

In the same way, if one can show that two hyperbolic fixed points have stable and unstable manifolds intersecting, thus creating a heteroclinic point, then there is no RIC passing between the two fixed points. Hence, we have a Birkhoff region of instability.

Residues criterion

This method was introduced by Greene in [Gr79] and is based on the characterization of periodic orbits into hyperbolic and elliptic orbits as described in Section 5.2. In particular, we have seen in Proposition 5.9 that this characterization is obtained by looking at the trace of JT. Greene has introduced a reformulation of the trace.

Let $P \in S^1 \times (a, b)$ be a periodic point of type (p, q), its *residue* is the quantity

$$\mathcal{R}(P) := \frac{1}{4}\Big(2 - \mathrm{trace}(JT^q(P))\Big). \tag{5.8}$$

Then:

- P is hyperbolic if and only if $\mathcal{R}(P) \in (-\infty, 0) \cup (1, +\infty)$.
- P is parabolic if and only if $\mathcal{R}(P) \in \{0, 1\}$.
- P is elliptic if and only if $\mathcal{R}(P) \in (0, 1)$.

Greene has formulated the following conjecture:

Conjecture [Gr79]. Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and $\{p_n/q_n\} \subset \mathbb{Q}$, with $(p_n, q_n) = 1$, converging to ω. Let $\{P_n\}$ be a sequence of periodic points of type (p_n, q_n). Then,

$$\mu(\omega) := \lim_{n\to\infty} \frac{1}{q_n} \log \mathcal{R}(P_n)$$

exists, and:

- If $\mu(\omega) \leq 0$, there exists a RIC $\mathcal{C}_\omega$ with rotation number ω.
- If $\mu(\omega) > 0$, there exists no RIC $\mathcal{C}_\omega$ with rotation number ω.

The conjecture has not been proved completely. There are only partial results.

Theorem 5.21 (MacKay)**.** *Using the notations in the conjecture, let I_ω denote the union of all invariant sets with rotation number $\omega \in \mathbb{R} \setminus \mathbb{Q}$. Then,*

$$\limsup_{n\to\infty} \frac{1}{q_n} \log |\mathcal{R}(P_n)|$$

is bounded from above by the supremum of the Lyapunov exponents of the orbits in I_ω. Therefore, if I_ω is a RIC with rotation number ω, then $\mu(\omega) \leq 0$, if the limit exixts.

Theorem 5.22 (Arnaud-Berger)**.** *In the conjecture, choose periodic points $\{P_n\}$ which are minimizing points of the action. Then, if*

$$\limsup_{n\to\infty} |\mathcal{R}(P_n)|^{1/q_n} > 1,$$

there is no RIC $\mathcal{C}_\omega$ with rotation number ω.

Lipschitz condition

This method uses the Lipschitz property for RICs proved in Theorem 5.16-(ii). The idea is the following. Assume that there exists an orbit along which the tangent vector $v = (0, 1)$ gets tilted to the right such that, after a finite number of iterations, it points to the left. Then, this orbit does not lie on a RIC.

We now apply this method to the Standard map of Example 1.11.

Proposition 5.23. *The Standard map*

$$T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}\,, \quad T(x, y) = \left(\left\{x + y - \frac{k}{2\pi} \sin(2\pi x)\right\}, y - \frac{k}{2\pi} \sin(2\pi x)\right)$$

has no RICs for $|k| \geq 2$.

Proof. Without loss of generality consider k positive. Then, given $v \in (0, 1)$, the vector applied to a point $P \in S^1 \times \mathbb{R}$, we have

$$w = JT(P)v = \begin{pmatrix} \dfrac{\partial T_1}{\partial x}(P) & \dfrac{\partial T_1}{\partial y}(P) \\ \dfrac{\partial T_2}{\partial x}(P) & \dfrac{\partial T_2}{\partial y}(P) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial T_1}{\partial y}(P) \\ \dfrac{\partial T_2}{\partial y}(P) \end{pmatrix}.$$

Analogously,

$$u = JT(T(P))w = JT^2(P)v = \begin{pmatrix} \frac{\partial T_1}{\partial x}(T(P)) & \frac{\partial T_1}{\partial y}(T(P)) \\ \frac{\partial T_2}{\partial x}(T(P)) & \frac{\partial T_2}{\partial y}(T(P)) \end{pmatrix} \begin{pmatrix} \frac{\partial T_1}{\partial y}(P) \\ \frac{\partial T_2}{\partial y}(P) \end{pmatrix}.$$

The condition we want to check is whether

$$\frac{\partial T_1}{\partial x}(T(P))\frac{\partial T_1}{\partial y}(P) + \frac{\partial T_2}{\partial x}(T(P))\frac{\partial T_2}{\partial y}(P) > 0.$$

This is equivalent to u being a vector pointing to the right.

Let now $P = (x_0, y_0)$, $T(P) = (x_1, y_1)$. By using the definition of the Standard map, the condition becomes

$$(1 - k\cos(2\pi x_1)) \cdot (1) + (1) \cdot (1) = 2 - k\cos(2\pi x_1) > 0\,.$$

However, if $k \geq 2$, there exists $\bar{x}_1$ such that $2 - k\cos(2\pi\bar{x}_1) \leq 0$. Hence, there is no RIC passing through the vertical line $\{x = \bar{x}_1\}$. Therefore, there is no RIC. □

In the proposition above, we have considered only two iterates of the map. One can try to get sharper conditions on k by considering more iterates. This has been done in [MP85] obtaining that the Standard map has no RICs for $|k| > 63/64$.

Generating function

Finally, we consider a method which uses the generating function. First, we show two results, for the Standard map and for the Birkhoff billiards.

Proposition 5.24 (Mather)**.** *The Standard map*

$$T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}\,, \quad T(x,y) = \left(\left\{x + y - \frac{k}{2\pi}\sin(2\pi x)\right\}, \, y - \frac{k}{2\pi}\sin(2\pi x)\right)$$

has no RICs for $|k| > 4/3$.

Proof. First, using Proposition 5.23, we assume $k \in (0, 2)$. Recall that the Standard map has generating function

$$h(x_0, x_1) = \frac{1}{2}(x_1 - x_0)^2 + \frac{k}{4\pi^2}\cos(2\pi x_0).$$

Assume that a RIC $\mathcal{C}_\omega$ exists and let (x_0, y_0), (x_1, y_1), and (x_2, y_2) be consecutive points of an orbit on $\mathcal{C}_\omega$. Then, $y_1 = \partial_2 h(x_0, x_1) = -\partial_1 h(x_1, x_2)$, which implies

$$x_1 - x_0 = (x_2 - x_1) + \frac{k}{2\pi}\sin(2\pi x_1),$$

or equivalently,

$$2x_1 - x_0 - x_2 - \frac{k}{2\pi}\sin(2\pi x_1) = 0.$$

Since $T|_{\mathcal{C}_\omega}$ is a Lipschitz order-preserving circle homeomorphism F, we have $x_2 = F(x_1)$ and $x_0 = F^{-1}(x_1)$. Hence,

$$2x_1 - F^{-1}(x_1) - F(x_1) - \frac{k}{2\pi}\sin(2\pi x_1) = 0.$$

Differentiating with respect to x_1,

$$2 - k\cos(2\pi x_1) = (F^{-1})'(x_1) + F'(x_1) = F'(x_1) + \frac{1}{F'(F^{-1}(x_1))} > 0, \quad (5.9)$$

where we have used that F is almost everywhere differentiable.

Let

$$C := \max\left\{\sup_{S^1} F',\ \sup_{S^1}\frac{1}{F'}\right\},$$

and write $L := \sup_{S^1} F'$ and $\ell := \inf_{S^1} F'$, so that

$$C = \max\left\{L, \frac{1}{\ell}\right\}.$$

From (5.9) we get

$$2 + k = \sup_{S^1}(2 - k\cos(2\pi x_1)) \geq \max\left\{\sup F' + \inf\frac{1}{F'},\ \inf F' + \sup\frac{1}{F'}\right\},$$

that is,

$$2 + k \geq \max\left\{L + \frac{1}{L},\ \ell + \frac{1}{\ell}\right\} \geq C + \frac{1}{C}.$$

Similarly,

$$2 - k = \inf_{S^1}(2 - k\cos(2\pi x_1)) \geq \inf F' + \inf\frac{1}{F'} = \ell + \frac{1}{L} \geq \frac{2}{C}.$$

Since $k \in (0, 2)$, from the second inequality, we get

$$C \geq \frac{2}{2-k} > 1.$$

Hence, in the first, we can write

$$2+k \geq g(C) \geq g\left(\frac{2}{2-k}\right) = \frac{2}{2-k} + \frac{2-k}{2},$$

using that the function

$$g(t) = t + \frac{1}{t}$$

is increasing on $[1, \infty)$. Therefore,

$$4-k^2 \geq 2 + \frac{(2-k)^2}{2} = 2+2+\frac{k^2}{2} - 2k,$$

which gives

$$\frac{3}{2}k^2 - 2k \leq 0 \quad \Leftrightarrow \quad k\left(\frac{3}{2}k - 2\right) \leq 0.$$

Thus, assuming that $k \in (0,2)$ and that a RIC exists, we obtain $k \leq 4/3$. □

Theorem 5.25 (Mather [Ma82])**.** *Let Ω be a C^3-smooth strictly convex set in $\mathbb{R}^2$, such that the curvature of the boundary vanishes at one point. Then the Billiard map $T : S^1 \times [-1,1] \to S^1 \times [-1,1]$. has no Rics.*

Proof. Let $\gamma : [0,1] \to \mathbb{R}^2$ be the arc length parameterisation of $\partial\Omega$ with components $\gamma(s) = (x(s), y(s))$, and $d(\cdot,\cdot)$ denote the Euclidean distance in $\mathbb{R}^2$. If we consider the function

$$G(t,u,w) := d(\gamma(t),\gamma(u)) + d(\gamma(u),\gamma(w)),$$

then a straightforward computation gives

$$\frac{\partial G}{\partial u}(t,u,w) = \frac{\dot{x}(u)(x(u)-x(t)) + \dot{y}(u)(y(u)-y(t))}{d(\gamma(t),\gamma(u))} + \\ + \frac{\dot{x}(u)(x(u)-x(w)) + \dot{y}(u)(y(u)-y(w))}{d(\gamma(u),\gamma(w))}\,.$$

Let now s_{-1}, s_0, s_1 be the length-arc coordinates of three consecutive points of a billiard trajectory. As shown in Example 5.5, the Euclidean distance is minus the generating function of the billiard map T, hence by the properties of the generating functions we have

$$\frac{\partial G}{\partial u}(s_{-1}, s_0, s_1) = 0\,.$$

Moreover, if the curvature of $\partial\Omega$ vanishes at $\gamma(s_0)$, so that $\ddot{\gamma}(s_0) = 0$, we again obtain by a straightforward computation that

$$\frac{\partial^2 G}{\partial u^2}(s_{-1}, s_0, s_1) = \frac{1 - \left\langle \dot{\gamma}(s_0), P_{\gamma(s_{-1})\gamma(s_0)} \right\rangle^2}{d(\gamma(s_{-1}), \gamma(s_0))} + \frac{1 - \left\langle \dot{\gamma}(s_0), P_{\gamma(s_0)\gamma(s_1)} \right\rangle^2}{d(\gamma(s_0), \gamma(s_1))} > 0$$

since $\|\dot{\gamma}(s_0)\| = 1$, where $P_{\gamma(u)\gamma(w)}$ denotes the unitary vector from $\gamma(u)$ to $\gamma(w)$.

It follows that we can apply the Implicit Function Theorem to the function $\partial G/\partial u$ in (s_{-1}, s_0, s_1) to get the existence of a function

$$\eta : U(s_{-1}, s_1) \to V(s_0)$$

defined in a neighbourhood U of (s_{-1}, s_1) with values in a neighbourhood V of s_0, such that $\eta(s_{-1}, s_1) = s_0$ and

$$\frac{\partial G}{\partial u}(t, \eta(t, w), w) = 0, \qquad \forall\, (t, w) \in U\,.$$

Hence $t, \eta(t, w), w$ are the length-arc coordinates of three consecutive points of a billiard trajectory.

Let now assume by contradiction that s_{-1}, s_0, s_1 are on a RIC $\mathcal{C}$, then by Theorem 5.16-(ii) $\mathcal{C}$ is the graph of a Lipschitz function $\phi : S^1 \to \mathbb{R}$, and consequently the billiard map restricted on $\mathcal{C}$ is of the form $(s, \phi(s)) \mapsto (F(s), \phi(F(s)))$ where $F : S^1 \to S^1$ is an order-preserving Lipschitz homeomorphism. It follows that $s_0 = F(s_{-1})$ and $s_0 = F^{-1}(s_1)$. Moreover by the Implicit Function Theorem we can write

$$\frac{\partial \eta}{\partial t}(s_{-1}, s_1) = -\frac{\frac{\partial^2 G}{\partial t\, \partial u}(s_{-1}, s_0, s_1)}{\frac{\partial^2 G}{\partial u^2}(s_{-1}, s_0, s_1)} \tag{5.10}$$

$$\frac{\partial \eta}{\partial w}(s_{-1}, s_1) = -\frac{\frac{\partial^2 G}{\partial w\, \partial u}(s_{-1}, s_0, s_1)}{\frac{\partial^2 G}{\partial u^2}(s_{-1}, s_0, s_1)} \tag{5.11}$$

As a final ingredient, we use that as observed in Example 5.1,

$$\frac{\partial^2 G}{\partial t\, \partial u}(s_{-1}, s_0, s_1) = \frac{\partial}{\partial u}\, \frac{\partial d(\gamma(t), \gamma(u))}{\partial t}\Big|_{t = s_{-1}, u = s_0} =$$

$$= -\frac{\partial}{\partial s_0}\, \cos\theta_{-1} = \sin\theta_{-1}\, \frac{\partial \theta_{-1}}{\partial s_0},$$

where θ_i denotes the angle of incidence of the billiard trajectory in the point $\gamma(s_i)$, and we are using that θ_{-1} can be written as a function of (s_{-1}, s_0).

Moreover, $\sin\theta_{-1} > 0$ and the twist property of the billiard ball map implies that

$$\frac{\partial s_0}{\partial \theta_{-1}} = \left(\frac{\partial \theta_{-1}}{\partial s_0}\right)^{-1} > 0\,.$$

Hence in (5.10) we find $\partial\eta/\partial t(s_{-1}, s_1) < 0$. Analogously

$$\frac{\partial^2 G}{\partial w\,\partial u}(s_{-1}, s_0, s_1) = \frac{\partial}{\partial w}\,\frac{\partial d(\gamma(u),\gamma(w))}{\partial u}\Big|_{u=s_0, w=s_1} =$$

$$= -\frac{\partial}{\partial s_1}\,\cos\theta_0 = \sin\theta_0\,\frac{\partial \theta_0}{\partial s_1},$$

whence in (5.11) we find $\partial\eta/\partial w(s_{-1}, s_1) < 0$. Using the homeomorphism F we can write

$$\frac{\partial \eta}{\partial t}(s_{-1}, s_1) = F'(s_{-1}) < 0\,, \qquad \frac{\partial \eta}{\partial w}(s_{-1}, s_1) = (F^{-1})'(s_1) < 0,$$

which gives the contradiction because F is order-preserving. □

Remark 5.26. The two previous results are a particular formulation of a more general principle. Let h be the generating function of an exact symplectic twist map, then if (x_0, y_0), (x_1, y_1), and (x_2, y_2), are consecutive points of an orbit on a RIC $\mathcal{C}$, then

$$\partial_{22}h(x_0, x_1) + \partial_{11}h(x_1, x_2) > 0\,.$$

Hence, converse KAM results may be obtained by violating the previous condition at some point x_1, writing x_0 and x_2 as functions of x_1 via an order-preserving Lipschitz circle homeomorphism.

Part II

Ergodic Theory (to appear)

Part III

Thermodynamic Formalism for maps (to appear)

Part IV

Flows on hyperbolic surfaces (to appear)

Part V

Hyperbolic flows on surfaces with non-constant negative curvature (to appear)

Appendix A

Continued fractions

A.1 The algorithm

For $\alpha \in \mathbb{R}$ let's consider the following algorithm. First we define $a_0 = \lfloor \alpha \rfloor \in \mathbb{Z}$ and let $\alpha_0 := \alpha - a_0 \in [0,1)$. Then we let

$$\begin{aligned} a_1 &:= \left\lfloor \frac{1}{\alpha_0} \right\rfloor \in \mathbb{N}\,, \quad \alpha_1 := \frac{1}{\alpha_0} - a_1 \in [0,1) \\ a_{n+1} &:= \left\lfloor \frac{1}{\alpha_n} \right\rfloor \in \mathbb{N}\,, \quad \alpha_{n+1} := \frac{1}{\alpha_n} - a_{n+1} \in [0,1) \quad \forall\, n \geq 1 \end{aligned} \tag{A.1}$$

The algorithm stops if $\alpha_{\bar{n}} = 0$ for some $\bar{n}$. Notice that we may write

$$\alpha = a_0 + \frac{1}{a_1 + \alpha_1} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \alpha_2}} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n + \alpha_n}}}} \tag{A.2}$$

for all $n \geq 1$ until the algorithm stops.

Proposition A.1. *The algorithm* (A.1) *stops if and only if* $\alpha \in \mathbb{Q}$.

Proof. If the algorithm stops and $\alpha_{\bar{n}} = 0$, from (A.2) it follows that

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{\bar{n}}}}}} \in \mathbb{Q}$$

Conversely, if $\alpha \in \mathbb{Q}$, let $\alpha = a_0 + \frac{p}{q}$ with $0 \leq p < q$, and apply the algorithm. If $p = 0$, $\alpha_0 = 0$ and we are done. Otherwise, working out the first steps we get

$$\alpha_1 = \frac{q - a_1 p}{p}\,, \quad \alpha_2 = \frac{p - a_2(q - a_1 p)}{q - a_1 p}\,, \quad \ldots$$

If $\alpha_1 = 0$ we are done, otherwise $0 < q - a_1 p < p$. Again if $\alpha_2 = 0$ we are done, otherwise for the numerator we can write $0 < p - a_2(q - a_1 p) < p$. It follows that the algorithm produces rational numbers α_n with decreasing positive numerators and denominators. Hence the algorithm necessarily stops. □

Given a real number α, its *continued fraction expansion* is then denoted by

$$\alpha = [a_0;\, a_1, a_2, a_3, \ldots] \tag{A.3}$$

and the coefficients $\{a_k\}$ are obtained as in (A.1) and are called *partial quotients*.

Without loss of generality we now consider the case $\alpha \in [0, 1)$, so that $a_0 = 0$ and $\alpha_0 = \alpha$. In this case we drop a_0 from notation (A.3).

Proposition A.2. *Let $\alpha \in [0, 1)$ and $\alpha = [a_1, a_2, \ldots]$. Define by recurrence the following sequences of natural numbers*

$$\begin{aligned} &p_{-1} = 1,\ p_0 = 0,\ p_n = a_n p_{n-1} + p_{n-2}\,, \quad \forall\, n \geq 1 \\ &q_{-1} = 0,\ q_0 = 1,\ q_n = a_n q_{n-1} + q_{n-2}\,, \quad \forall\, n \geq 1 \end{aligned} \tag{A.4}$$

Then

(i) p_n and q_n are divergent sequences;

(ii) $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$ for all $n \geq 1$ and $(p_n, q_n) = 1$;

(iii) $\frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n]$ for $n \geq 1$ (and $n \leq \bar{n}$ if the continued fraction expansion of α is finite with $\alpha_{\bar{n}} = 0$).

Proof. (i) It is immediate from the definition.
(ii) The statement is true $n = 1$ since $q_1 = a_1$ and $p_1 = 1$. Assume that it is true up to $n - 1$, then

$$q_n p_{n-1} - q_{n-1} p_n = (a_n q_{n-1} + q_{n-2}) p_{n-1} - q_{n-1}(a_n p_{n-1} + p_{n-2}) =$$

$$= -(q_{n-1} p_{n-2} - q_{n-2} p_{n-1}) = -(-1)^{n-1} = (-1)^n$$

Moreover, if $c := (p_n, q_n) > 1$ we have an absurd from $c|(-1)^n$.
(iii) Let us now look at the terms of the form $\frac{p_n}{q_n}$. By definition we have

$$\frac{p_1}{q_1} = \frac{1}{a_1}\,, \quad \frac{p_2}{q_2} = \frac{a_2}{a_2\,a_1 + 1} = \frac{1}{a_1 + \frac{1}{a_2}}\,, \quad \dots$$

By induction, we assume that from representation (A.4) we obtain $\frac{p_k}{q_k} = [a_1, a_2, \dots, a_k]$ for any sequence $\{a_1, a_2, \dots, a_k\}$ of natural numbers with $k \leq n$ terms.

We now wnat to show that given $\alpha = [a_1, a_2, \dots]$ and p_{n+1}, q_{n+1} defined as in (A.4) we have $\frac{p_{n+1}}{q_{n+1}} = [a_1, a_2, \dots, a_n, a_{n+1}]$. Let us consider the number $\alpha_1 = \frac{1}{\alpha} - a_1$. Then by the algorithm (A.1), $\alpha_1 = [a_2, a_3, \dots]$, and we introduce the notation $[a_2, a_3, \dots, a_k, a_{k+1}] = \frac{\tilde{p}_k}{\tilde{q}_k}$ for all $k \leq n$. Moreover

$$[a_1, a_2, \dots, a_k, a_{k+1}] = \frac{1}{a_1 + [a_2, a_3, \dots, a_k, a_{k+1}]} = \frac{1}{a_1 + \frac{\tilde{p}_k}{\tilde{q}_k}} = \frac{\tilde{q}_k}{a_1\tilde{q}_k + \tilde{p}_k}$$

for all $k \leq n$. By the inductive hypothesis

$$p_{k+1} = \tilde{q}_k\,, \quad q_{k+1} = a_1\tilde{q}_k + \tilde{p}_k \qquad \forall k \leq n-1$$

since $(\tilde{p}_k, \tilde{q}_k) = 1$, and

$$\tilde{p}_k = a_{k+1}\tilde{p}_{k-1} + \tilde{p}_{k-2}\,, \quad \tilde{q}_k = a_{k+1}\tilde{q}_{k-1} + \tilde{q}_{k-2} \qquad \forall k \leq n$$

Hence we can now write

$$[a_1, a_2, \dots, a_n, a_{n+1}] = \frac{\tilde{q}_n}{a_1\tilde{q}_n + \tilde{p}_n}$$

with

$$\tilde{q}_n = a_{n+1}\tilde{q}_{n-1} + \tilde{q}_{n-2} = a_{n+1}p_n + p_{n-1}$$

and

$$\begin{aligned} a_1\tilde{q}_n + \tilde{p}_n &= a_1(a_{n+1}\tilde{q}_{n-1} + \tilde{q}_{n-2}) + a_{n+1}\tilde{p}_{n-1} + \tilde{p}_{n-2} = \\ &= a_{n+1}(a_1\tilde{q}_{n-1} + \tilde{p}_{n-1}) + a_1\tilde{q}_{n-2} + \tilde{p}_{n-2} = \\ &= a_{n+1}q_n + q_{n-1} \end{aligned}$$

It follows that

$$[a_1, a_2, \dots, a_n, a_{n+1}] = \frac{\tilde{q}_n}{a_1\tilde{q}_n + \tilde{p}_n} = \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}$$

and the proof is finished. $\square$

The fractions $p_n/q_n = [a_1, a_2, \dots, a_n]$ defined in terms of the coefficients of the continued fraction expansion of a real number $\alpha \in [0,1)$ are called *convergents* of α. The following result explains why.

Proposition A.3. *Let $\alpha \in [0,1)$ be an irrational number with $\alpha = [a_1, a_2, \dots]$, and let p_n/q_n the fractions defined in Proposition A.2. Then for all $n \geq 1$*

$$\frac{1}{q_n(q_n + q_{n+1})} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n\, q_{n+1}}$$

Proof. We first notice that from (A.2) and $\alpha_n \in [0,1)$ for all $n \geq 1$, it follows that $\frac{p_n}{q_n} > \alpha$ if n is odd, and $\frac{p_n}{q_n} < \alpha$ if n is even. Moreover from Proposition A.2-(ii) one obtains

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n\, q_{n-1}} \quad \text{and} \quad \frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} = \frac{(-1)^{n-1}\, a_n}{q_n\, q_{n-2}}$$

In particular one obtains that the subsequence $\frac{p_{2k}}{q_{2k}}$ is increasing, and the subsequence $\frac{p_{2k-1}}{q_{2k-1}}$ is decreasing. Hence the two subsequences admit limits, and the limits coincide since q_n is a divergent sequence. Hence $\frac{p_n}{q_n}$ is a convergent sequence. Since α is a separating element between the two subsequences, it follows that $\frac{p_n}{q_n} \to \alpha$ as $n \to \infty$.

To obtain the estimates in the thesis, first we argue that

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right|$$

since $\frac{p_{n+1}}{q_{n+1}}$ and $\frac{p_n}{q_n}$ are on different sides with respect to α. Hence

$$\left|\alpha - \frac{p_n}{q_n}\right| < \left|\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n}\right| = \left|\frac{(-1)^n}{q_{n+1}\, q_n}\right| = \frac{1}{q_{n+1}\, q_n}$$

Let us now fix n odd, and consider $\frac{p_{n-1}}{q_{n-1}}$, $\frac{p_n}{q_n}$, and the following rationals

$$\frac{p_n + p_{n-1}}{q_n + q_{n-1}}, \quad \frac{2p_n + p_{n-1}}{2q_n + q_{n-1}}, \quad \frac{3p_n + p_{n-1}}{3q_n + q_{n-1}}, \dots, \quad \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}$$

For all $k = 1, \dots, a_{n+1}$ we have

$$\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}} - \frac{(k-1)p_n + p_{n-1}}{(k-1)q_n + q_{n-1}} = \frac{-(q_n p_{n-1} - q_{n-1}p_n)}{(kq_n + q_{n-1})((k-1)q_n + q_{n-1})} > 0$$

since $q_n p_{n-1} - q_{n-1}p_n = (-1)^n < 0$, and

$$\frac{kp_n + p_{n-1}}{kq_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{q_n p_{n-1} - q_{n-1}p_n}{q_n(kq_n + q_{n-1})} < 0$$

hence the rationals $\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}}$ with $k=1,\dots,a_{n+1}$ are an increasing sequence in the interval $(\frac{p_{n-1}}{q_{n-1}},\frac{p_n}{q_n})$. Since

$$\frac{p_n}{q_n}-\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}}=\frac{1}{q_n(kq_n+q_{n-1})}\geq\frac{1}{q_n\,q_{n+1}}>\frac{p_n}{q_n}-\alpha$$

where we used that n is odd, it follows that $\frac{kp_n+p_{n-1}}{kq_n+q_{n-1}}$ with $k=1,\dots,a_{n+1}$ are actually contained in the interval $(\frac{p_{n-1}}{q_{n-1}},\alpha)$. For $k=a_{n+1}$ we find that $\frac{p_{n+1}}{q_{n+1}}$ is smaller than α, as expected.

Repeating the same argument, we find that for all $n\geq 1$ the rationals $\frac{p_n}{q_n}$ and $\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n}$ with $k=1,\dots,a_{n+2}$ are on the same side with respect to α, and the rationals $\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n}$ are closer to α than $\frac{p_n}{q_n}$. If n is odd $\{\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n}\}_k$ define a decreasing sequence, which is increasing if n is even.

We can now use the previous results to write

$$\left|\alpha-\frac{p_n}{q_n}\right|>\left|\frac{p_{n+1}+p_n}{q_{n+1}+q_n}-\frac{p_n}{q_n}\right|=\frac{1}{q_n\,(q_{n+1}+q_n)}$$

and the proof is finished. □

Definition A.1. Given two constants $c,\nu>0$, we say that α is *Diophantine with parameters c and ν*, and we write $\alpha\in D(c,\nu)$, if

$$\left|\alpha-\frac{p}{q}\right|>\frac{c}{q^{2+\nu}}\quad\forall\,p\in\mathbb{Z},\,q\in\mathbb{N}\,.$$

We say that α is *Liouville* if it's not Diophantine with any $c,\nu>0$.

Definition A.2. We say that α is *Brjuno*, and we write $\alpha\in\mathcal{B}$, if the denominators $\{q_n\}$ of its convergents satisfy

$$\sum_{n=1}^{\infty}\frac{\log q_{n+1}}{q_n}<+\infty\,.$$

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List of symbols

- $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- $\mathbb{R}^+ := (0, +\infty)$ and $\mathbb{R}_0^+ := [0, +\infty)$.
- $\lfloor x \rfloor$ the integer part of a real number.
- $\{x\} = x - \lfloor x \rfloor$ the fractional part of a real number.
- $\underline{x}$ points of a Euclidean space $\mathbb{R}^n$ with $n > 1$.
- $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ the Euclidean distance.
- $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Euclidean scalar product.
- $B_\varepsilon(\underline{x}) := \left\{ \underline{y} \in \mathbb{R}^n \, : \, d(\underline{x}, \underline{y}) < \varepsilon \right\}$.
- X is a locally compact connected metric space.
- $M(n \times m, F)$ is the set of $n \times m$ matrices with coefficients in the set F.
- $A^c := X \setminus A$ for a set $A \subset X$.
- $\mathring{A}$ is the interior of $A \subset X$.
- $\bar{A}$ is the closure of $A \subset X$.
- ∂A is the boundary of $A \subset X$.
- $f_+ := \max \{f, 0\}$ is the positive part of a real-valued function f.
- $f_- := \max \{-f, 0\}$ is the negative part of a real-valued function f.
- $C^0(X)$ is the set of continuous functions on X.
- $C_b^0(X)$ is the set of continuous and bounded functions on X.

- $C_c^0(X)$ is the set of continuous functions on X with compact support.
- $C^{0,\alpha}(X)$ with $\alpha \in (0,1]$ is the set of α-Hölder (Lipschitz if $\alpha = 1$) functions on X.
- $C^k(X)$ with $k \in \mathbb{N} \cup \{\infty\}$ is the set of functions of class C^k on X.
- $C^{k,\alpha}(X)$ with $k \in \mathbb{N} \cup \{\infty\}$ and $\alpha \in (0,1]$ is the set of functions of class C^k on X for which the k-th derivative is in $C^{0,\alpha}(X)$.
- $C^\omega(X)$ is the set of real analytic functions on X.
- $BV(a,b)$ is the set of functions with bounded variation on the interval (a,b).
- $\mathcal{M}(X)$ is the set of positive Radon measures on X.
- $\mathcal{M}_1(X)$ is the set of positive Radon probability measures on X.
- $\mu(f) := \int_X f\,d\mu$ for $\mu \in \mathcal{M}$ and $f \in L^1(X,\mu)$.
- $\mu_g(f) := \int_X f\,g\,d\mu$ for $\mu \in \mathcal{M}$, g measurable and $f \in L^1(X, g\mu)$.
- $\mathcal{F}^*$ is the dual of a space $\mathcal{F}$.
- $L^* : \mathcal{G}^* \to \mathcal{F}^*$ is the dual of a linear operator $L : \mathcal{F} \to \mathcal{G}$.