Chapter 3

Discrete-time dynamical systems

In this chapter we consider discrete-time dynamical systems as defined in Definition 1.2. Hence we need to specify a set X and a map $T : X \to X$. The properties of X and T may vary and give rise to different areas of research. Here we assume that X is a locally compact connected metric space and T is a continuous map, and call (X, T) a *discrete-time continuous dynamical system*. In many situations one can simply consider X to be an interval of the real line, and in fact some results of this chapter hold only for one-dimensional spaces X or even for compact intervals of the real line.

We start with simple definitions.

Definition 3.1. Let (X, T) and $(\tilde{X}, \tilde{T})$ be two discrete-time continuous dynamical systems. We say that $(\tilde{X}, \tilde{T})$ is a *topological factor* of (X, T) if there exists a continuous map $h : X \to \tilde{X}$ that is surjective and satisfies

$$\tilde{T} \circ h = h \circ T\,. \tag{3.1}$$

If the map $h : X \to \tilde{X}$ is a homeomorphism and satisfies (3.1) then we say that (X, T) and $(\tilde{X}, \tilde{T})$ are *topologically conjugate* and h is a *topological conjugacy*.

Example 3.1. Let's consider the full shift $(\Omega_{\mathcal{A}}, \mathbb{N}_0, \sigma)$ on two symbols $\mathcal{A} = \{0, 1\}$ of Example 1.8, and the Bernoulli map T_2 on S^1 of Example 1.7. Let $J_0 = [0, 1/2)$ and $J_1 = [1/2, 1)$ be a partition of S^1, and let the map $h : \Omega_{\{0,1\}} \to S^1$ be defined by

$$\omega = (\omega_i)_{i \in \mathbb{N}_0} \mapsto h(\omega) = \bigcap_{i \in \mathbb{N}_0} T_2^{-i}(J_{\omega_i})\,.$$

The map h is continuous and surjective, and satisfies $T_2 \circ h = h \circ \sigma$. Then the Bernoulli map is a topological factor of the full shift on two symbols.

Example 3.2. Let's consider the Tent map T_s with $s = 2$ of Example 1.5, and the logistic map T_λ with $\lambda = 4$ of Example 1.6. Let the map $h : [0,1] \to [0,1]$ be defined by

$$[0,1] \ni x \mapsto h(x) = \sin^2\left(\frac{\pi}{2}x\right).$$

The map h is a homeomorphism, and satisfies $T_4 \circ h = h \circ T_2$. Hence the Tent map T_s with $s = 2$ is topologicall conjugate to the logistic map T_λ with $\lambda = 4$.

Remark 3.1. In some situations it is interesting to study the regularity of a conjugacy. For example, if T and $\tilde{T}$ are C^k maps, with $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$, a natural question is whether there exists a conjugacy h between the systems (X, T) and $(\tilde{X}, \tilde{T})$ which is of class C^k. If it exists we say that (X, T) and $(\tilde{X}, \tilde{T})$ are C^k *conjugate.*

3.1 Stability in one dimension

Let $T : X \to X$ be a continuous map of a one-dimensional space $X = [a,b], (a,b), [a,+\infty), (a,+\infty), (-\infty,b], (-\infty,b), \mathbb{R}, S^1$.

Definition 3.2. A fixed point $x_0 \in X$ of T is called *attractive* if there exists $\delta > 0$ such that, for all $x \in B_\delta(x_0)$, one has $T^n(x) \in B_\delta(x_0)$ for all $n \geq 0$, and $T^n(x) \to x_0$ as $n \to +\infty$.
A fixed point $x_0 \in X$ is called *repulsive* if there exists $\delta > 0$ such that, for all $x \in B_\delta(x_0)$, $x \neq x_0$, there exists $\bar{n} \in \mathbb{N}$ for which $T^{\bar{n}}(x) \notin B_\delta(x_0)$.

To study the dynamics in a neighbourhod of a fixed point x_0, first it is useful to try the linearization approach. Let T be differentiable at x_0. Then, there exists $\varepsilon > 0$ such that for all $x \in B_\varepsilon(x_0)$

$$T(x) = T(x_0) + T'(x_0)(x - x_0) + o(|x - x_0|) = x_0 + T'(x_0)(x - x_0) + o(|x - x_0|).$$

Hence,

$$|T(x) - x_0| = |T'(x_0)|\,|x - x_0| + o(|x - x_0|). \tag{3.2}$$

We deduce that, at the first order, it is the derivative $T'(x_0)$ which may determine whether the orbit of a point $x \in B_\varepsilon(x_0)$ gets closer or further from the fixed point x_0. This justifies the following definition.

Definition 3.3. Let T be differentiable at a fixed point x_0. The fixed point $x_0 \in X$ is called *hyperbolic* if $|T'(x_0)| \neq 1$.

Theorem 3.2. *Let x_0 be a hyperbolic fixed point for a map T which is differentiable at x_0. If $|T'(x_0)| < 1$ then the point is attractive, if $|T'(x_0)| > 1$ then the point is repulsive.*

Proof. Let $|T'(x_0)| < 1$ and fix $c \in (|T'(x_0)|, 1)$. If we choose $\delta > 0$ such that $|T'(x)| \leq c$ for all $x \in B_\delta(x_0)$, then we have that for all $n \geq 1$

$$|T^n(x) - x_0| \leq c^n\, |x - x_0|\,, \quad \forall\, x \in B_\delta(x_0)\,. \tag{3.3}$$

From (3.3) and $c \in (0, 1)$, it follows that $T^n(x) \in B_\delta(x_0)$ for all $n \geq 0$ and $T^n(x) \to x_0$ as $n \to +\infty$.

We now prove (3.3) by induction. For $n = 1$, for all $x \in B_\delta(x_0)$ there exists ξ_1 between x and x_0 such that

$$|T(x) - x_0| = |T(x) - T(x_0)| = |T'(\xi_1)|\, |x - x_0| \leq c\, |x - x_0|\,,$$

where $|T'(\xi_1)| \leq c$ since $\xi_1 \in B_\delta(x_0)$. Then, let's assume that (3.3) holds for a given n, and show that it holds for $n + 1$. There exists ξ_n between $T^n(x)$ and x_0 such that

$$|T^{n+1}(x) - x_0| = |T(T^n(x)) - T(x_0)| = |T(\xi_n)|\, |T^n(x) - x_0| \leq$$
$$\leq c \cdot c^n\, |x - x_0| = c^{n+1}\, |x - x_0|\,,$$

since $\xi_n \in B_\delta(x_0)$.

Let now $|T'(x_0)| > 1$, and first consider the case $T'(x_0) > 1$. Then we fix $c \in (1, T'(x_0))$ and choose $\delta > 0$ such that $T'(x) \geq c$ for all $x \in B_\delta(x_0)$. We now argue by contradiction and assume that there exists $x \in B_\delta(x_0)$, $x \neq x_0$, such that $T^n(x) \in B_\delta(x_0)$ for all $n \geq 1$. Then, we can repeat the argument above to show that

$$|T^n(x) - x_0| \geq c^n\, |x - x_0|\,, \quad \forall\, n \geq 1\,,$$

from which we find that $|T^n(x) - x_0| \to +\infty$ as $n \to +\infty$ since $c > 1$. This gives the contradiction with the assumption $T^n(x) \in B_\delta(x_0)$ for all $n \geq 1$.

A similar argument works in the case $|T'(x_0)| > 1$ and $T'(x_0) < -1$. □

When the fixed point is not hyperbolic, the approach in (3.2) suggests that the higher derivatives of T at x_0 may give some information.

Definition 3.4. A fixed point $x_0 \in X$ is called *semi-attractive from the left* if there exists $\delta > 0$ such that it is attractive for points on $(x_0 - \delta, x_0)$ and repulsive for points on $(x_0, x_0 + \delta)$. A fixed point $x_0 \in X$ is called *semi-attractive from the right* if there exists $\delta > 0$ such that it is attractive for points on $(x_0, x_0 + \delta)$ and repulsive for points on $(x_0 - \delta, x_0)$.

Proposition 3.3. *Let x_0 be a fixed point for a map T which is differentiable at x_0 with $|T'(x_0)| = 1$. The following possibilities hold:*

(i) Let $T'(x_0) = 1$ and assume that $T \in C^2(B_\varepsilon(x_0))$ for some $\varepsilon > 0$, and $T''(x_0) \neq 0$. Then,

- *If $T''(x_0) > 0$, then x_0 is semi-attractive from the left;*
- *If $T''(x_0) < 0$, then x_0 is semi-attractive from the right;*

(ii) Let $T'(x_0) = 1$ and assume that $T \in C^3(B_\varepsilon(x_0))$ for some $\varepsilon > 0$, that $T''(x_0) = 0$, and $T'''(x_0) \neq 0$. Then,

- *If $T'''(x_0) > 0$, then x_0 is repulsive;*
- *If $T'''(x_0) < 0$, then x_0 is attractive;*

(iii) Let $T'(x_0) = -1$ and assume that $T \in C^3(B_\varepsilon(x_0))$ for some $\varepsilon > 0$. Then we look at $ST(x_0)$, the Schwarzian derivative of T at x_0, where

$$ST(x) := \frac{T'''(x)}{T'(x)} - \frac{3}{2}\left(\frac{T''(x)}{T'(x)}\right)^2 . \tag{3.4}$$

Then,

- *If $ST(x_0) > 0$, then x_0 is repulsive;*
- *If $ST(x_0) < 0$, then x_0 is attractive.*

Proof. The proofs of (i) and (ii) are immediate from the graphical approach. Let us prove (iii). Since $T'(x_0) = -1$, in a neighborhood of x_0 the map T is order-reversing. We look at $G := T^2$ for which $G(x_0) = x_0$, and use that x_0 has the same stability for G and T. We have

$$\begin{aligned} G'(x) &= T'(T(x))\,T'(x) \quad \Rightarrow \quad G'(x_0) = (T'(x_0))^2 = 1\,, \\ G''(x) &= T''(T(x))\,(T'(x))^2 + T'(T(x))\,T''(x) \\ &\qquad \Rightarrow \quad G''(x_0) = T''(x_0)\Big((T'(x_0))^2 - T'(x_0)\Big) = 0\,. \end{aligned}$$

Moreover $G \in C^3(B_\varepsilon(x_0))$, hence we can compute $G'''(x_0)$. It holds

$$\begin{aligned} G'''(x) &= T'''(T(x))\,(T'(x))^3 + 3\,T''(T(x))\,T'(x)\,T''(x) + T'(T(x))\,T'''(x) \\ &\Rightarrow \quad G'''(x_0) = T'''(x_0)\Big((T'(x_0))^3 + T'(x_0)\Big) + 3\,(T''(x_0))^2\,T'(x_0) \\ &\Rightarrow \quad G'''(x_0) = 2\,ST(x_0)\,. \end{aligned}$$

The result follows from (ii). □

We conclude this section by studying the stability for periodic orbits.

Definition 3.5. Let x_0 be a periodic point for T with minimal period p. The orbit $\mathcal{O}(x_0)$ is called *attractive* (respectively *repulsive*) if x_0 is an attractive (respectively repulsive) fixed point for T^p.

Remark 3.4. Let x_0 be a periodic point for T with minimal period p. If $T \in C^1$, it is a straightforward corollary of the chain rule that the derivative of T^p is the same on all the points of the orbit of x_0, i.e. $(T^p)'(T^i(x_0)) = (T^p)'(x_0)$ for all $i = 0, \dots, p-1$, since

$$(T^p)'(x_0) = \prod_{j=0}^{n-1} T'(T^j(x_0))\,.$$