3.2 Existence of periodic orbits

In this section $[a, b]$ denotes a compact interval of the real line. Given a finite number of points $\{x_k\}_{k=0,\dots,n}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\,,$$

we consider the partition $\mathcal{J}$ of $[a, b]$ into the closed intervals $J_k = [x_{i-1}, x_i]$, $k = 1, \dots, n$.

Definition 3.6. Given a partition $\mathcal{J} = \{J_\ell\}$ of $[a, b]$ and two not necessarily distinct sets J_k and J_h of the partition, we say that J_k *T-covers* J_h *m-times*, with $m \in \mathbb{N} \cup \{\infty\}$, if there exist m open sub-intervals $K_1, \dots, K_m$ of J_k such that $K_i \cap K_j = \emptyset$ for $i \neq j$, and $T(\overline{K_i}) = J_h$ for all $i = 1, \dots, m$.

Definition 3.7. Given a partition $\mathcal{J} = \{J_\ell\}_{\ell=1,\dots,n}$ of $[a, b]$, the *T-graph of* $\mathcal{J}$ is a graph with nodes given by the indices $\{1, \dots, n\}$, and such that there are m-arcs from a set J_k to a set J_h if J_k T-covers J_h m-times. An *admissible path of length* $s \in \mathbb{N}$ on the T-graph of $\mathcal{J}$ is a sequence $J_{p(1)}J_{p(2)} \dots J_{p(s)}$ with $p(j) \in \{1, \dots, n\}$ and such that there is at least one arc from $J_{p(j)}$ to $J_{p(j+1)}$ for all $j = 1, \dots, s-1$. An admissible path of length $s \in \mathbb{N}$ is called *closed* if $p(s) = p(1)$.

Lemma 3.5. *If $J_{p(1)}J_{p(2)} \dots J_{p(s)}J_{p(s+1)}$ is an admissible closed path on the T-graph of a partition $\mathcal{J}$ with $s \in \mathbb{N}_0$, then there exists a point $x \in J_{p(1)}$ which is periodic for T with period s and such that $T^j(x) \in J_{p(j+1)}$ for all $j = 0, \dots, s$.*

Proof. Let us fix $K_{s+1} = \mathring{J}_{p(s+1)}$. Since the path $J_{p(1)}J_{p(2)} \dots J_{p(s)}J_{p(s+1)}$ is admissible, there exists a family $K_j \subset J_{p(j)}$, $j = 1, \dots, s$, of open intervals such that $T(K_j) = K_{j+1}$. Hence there exists an interval $K_1 \subset J_{p(1)}$ such that $T^s(K_1) = K_{s+1} \supseteq K_1$. The fixed-point theorem implies that there exists $x \in \overline{K_1}$ such that $T^s(x) = x$, moreover by construction $T^j(x) \in \overline{K_{j+1}} \subseteq J_{p(j+1)}$ for all $j = 0, \dots, s$. □

Remark 3.6. It is important to notice that Lemma 3.5 does not prove the existence of a periodic point with minimal period s. That the period s is minimal may be obtained by looking at the path used in the proof of the lemma.

Proposition 3.7. *Let $T : [a, b] \to [a, b]$ be a continuous map for which there exists a periodic orbit of odd period $m > 1$. Then T admits periodic orbits of minimal period n for all $n > m$, for all even $n < m$, and for $n = 1$.*

Proof. Let's assume that m is the smallest odd number greater than 1 for which T has a periodic orbit of period m[1]. In particular, m is the minimal period of the orbit. Let us denote by $p_1, p_2, \dots, p_m$ the points of the periodic orbit ordered in $[a, b]$, so that $T(p_1) > p_1$ and $T(p_m) < p_m$. It follows that there exists $\bar{h}$ such that $T(p_{\bar{h}}) > p_{\bar{h}}$ and $T(p_k) < p_k$ for all $k = \bar{h}+1, \dots, m$. Finally let $\mathcal{J}$ be the partition given by the points a, b and the points of the periodic orbit $p_1, p_2, \dots, p_m$, and let $J_0 := [a, p_1]$, $J_m := [p_m, b]$, and $J_k := [p_k, p_{k+1}]$ for $k \in \mathcal{N} := \{1, \dots, m-1\}$. By construction and the fact that $m > 2$ we have that one of the inequalities $T(p_{\bar{h}+1}) \leq p_{\bar{h}}$ and $T(p_{\bar{h}}) \geq p_{\bar{h}+1}$ is strict, hence $J_{\bar{h}}$ T-covers itself at least once. By Lemma 3.5, this gives the result for $n = 1$.

We now proceed by proving intermediate statements.

Step 1. There exists an admissible path on the T-graph of the partition $\mathcal{J}$ from $J_{\bar{h}}$ to any set J_k of the partition with $k \in \mathcal{N}$.

Let us define by recurrence the following subsets of the nodes $\mathcal{N}$ of the T-graph. We put $N_1 := \{\bar{h}\}$,

$$N_2 := \{r \in \mathcal{N} \,:\, J_{\bar{h}} \; T\text{-covers } J_r\} \;,$$

and for $i \geq 3$

$$N_i := \{r \in \mathcal{N} \,:\, \exists\, s \in N_{i-1} \text{ such that } J_s \; T\text{-covers } J_r\} \;.$$

Since $m > 2$, each J_s with $s \in \mathcal{N}$, T-covers at least one set J_r with $r \neq s$. Moreover the fact that $J_{\bar{h}}$ T-covers itself implies that $\bar{h} \in N_i$ for all $i \geq 1$, hence $\{N_i\}$ is a non-decreasing sequence of sets. We conclude that there exists ℓ such that $N_\ell = N_{\ell+1} = \mathcal{N}$, since $N_\ell \neq \mathcal{N}$ implies that m is not the minimal period of the periodic orbit. This finishes the proof of this step.

Step 2. There exists $k \in \mathcal{N}$ such that J_k T-covers $J_{\bar{h}}$.

We argue by contradiction. If the thesis of this step is false, all points p_j of the periodic orbit with $j \leq \bar{h}$ have distinct images in the set $\{p_{\bar{h}+1}, \dots, p_m\}$, and analogously all points p_j of the periodic orbit with $j \geq \bar{h}+1$ have distinct images in the set $\{p_1, \dots, p_{\bar{h}}\}$. Since m is odd we get the contradiction.

Step 3. The T-graph of the partition $\mathcal{J}$ contains a loop starting from $J_{\bar{h}}$ through all the sets J_k with $k \in \mathcal{N}$, and contains one single arc from a set J_k with $k \in \mathcal{N}$ to $J_{\bar{h}}$.

We first show that the shortest admissible path from $J_{\bar{h}}$ to itself is of length m. Let $J_{\bar{h}} J_{p(2)} \dots J_{p(s)} J_{\bar{h}}$ be such path with length $s+1 < m$, there are two

[1] If not, we prove the result for such smallest odd number greater than 1 and obtain the proposition.

cases. If s is odd, by Lemma 3.5 there exists $x \in J_{\bar{h}}$ such that $T^s(x) = x$, but $s < m-1$ and we have a contradiction by the choice of m. If s is even, we can consider the admissible path $J_{\bar{h}} J_{p(2)} \dots J_{p(s)} J_{\bar{h}} J_{\bar{h}}$ which is of length $s+2$ and gives, by Lemma 3.5, the existence of a periodic point of period $s+1 < m$. Again we have a contradiction by the choice of m.

Let $J_{\bar{h}} J_{p(2)} \dots J_{p(m-1)} J_{\bar{h}}$ be the shortest admissible path from $J_{\bar{h}}$ to itself. All J_k appear at most once in this path, indeed if one J_k appears twice, we can construct a shorter admissible path from $J_{\bar{h}}$ to itself. It follows that this path is actually a loop starting from $J_{\bar{h}}$ through all the sets J_k with $k \in \mathcal{N}$. The same argument shows that the T-graph of the partition $\mathcal{J}$ contains one single arc from a set J_k with $k \in \mathcal{N}$ to $J_{\bar{h}}$.

Let us now relabel the sets of the partition $\mathcal{J}$ by letting $I_1 := J_{\bar{h}}$ and $I_2, \dots, I_{m-1}$ be chosen so that there exists an arc from I_k to I_{k+1} for all $k \in \mathcal{N}$.

Step 4. The map T admits periodic orbits of minimal period n for all $n > m$.
This follows from step 3 by applying Lemma 3.5 to the closed ammissible path $I_1 I_2 \dots I_{m-1} I_1 \dots I_1$ of length $n+1$.

Step 5. For each odd $k \in \mathcal{N}$ there exists an arc from I_{m-1} to I_k.
The statement is clearly true for $m = 3$. If $m > 3$ we show that the sets I_k are ordered in $[a, b]$ in a precise way. From step 3 we know that I_1 T-covers itself and I_2, and no other set. So $T(p_{\bar{h}}) = p_{\bar{h}+2}$ and $T(p_{\bar{h}+1}) = p_{\bar{h}}$, or $T(p_{\bar{h}}) = p_{\bar{h}+1}$ and $T(p_{\bar{h}+1}) = p_{\bar{h}-1}$. In the first case $I_2 = [p_{\bar{h}+1}, p_{\bar{h}+2}]$, and since I_2 T-covers only I_3 we have $I_3 = [p_{\bar{h}-1}, p_{\bar{h}}]$. We can continue repeating the argument to conclude that $I_{m-1} = [p_{m-1}, p_m]$, and $T(p_{m-1}) = p_1$, $T(p_1) = p_m$ and $T(p_m) = p_{\bar{h}}$. Since I_k with k odd are of the form $[p_h, p_{h+1}]$ with $h < \bar{h}$, the thesis of the step follows.

Step 6. The map T admits periodic orbits of minimal period n for all even $n < m$.
This follows from step 5 by applying Lemma 3.5 to the closed admissible path of length $n+1$ from I_{m-1} to itself of the form $I_{m-1} I_j I_{j+1} \dots I_{m-1}$ where $j = m - n$ is odd. □

Theorem 3.8 (Sharkovsky)**.** *Let $T : [a, b] \to [a, b]$ be a continuous map and consider the following ordering on $\mathbb{N}$*

$$
\begin{aligned}
&1 \prec 2 \prec 4 \prec 8 \prec \dots \prec 2^n \prec 2^{n+1} \prec \dots 2^{n+1}5 \prec 2^{n+1}3 \prec \dots \\
&\dots \prec 2^n 5 \prec 2^n 3 \prec \dots \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \dots 7 \prec 5 \prec 3
\end{aligned}
\tag{3.5}
$$

If T admits a periodic orbit of minimal period m then it admits a periodic *orbit of minimal period n for all $n \prec m$ in the ordering* (3.5).

Proof. If m is odd, the thesis follows from Proposition 3.7.
If $m = 2 \cdot \tilde{m}$ with $\tilde{m}$ odd and T admits no periodic orbits with odd period, then we can repeat the same argument of the proof of Proposition 3.7 up to step 2. This shows that $\bar{h} = \tilde{m}$ and, in the T-graph of the partition including the sets J_k with $k \in \mathcal{N}$, there exists an admissible path from the set $[p_{\tilde{m}}, p_{\tilde{m}+1}]$ to all the sets J_k with $k \in \mathcal{N}$. This implies that T admits a fixed point. However there is no arc to $[p_{\tilde{m}}, p_{\tilde{m}+1}]$ from a different set, since otherwise by Lemma 3.5 we could find a periodic orbit of T with odd period. It follows that $T(p_j) \geq p_{\tilde{m}+1}$ for all $j \leq \tilde{m}$ and $T(p_j) \leq p_{\tilde{m}}$ for all $j \geq \tilde{m}+1$, so the points $p_1, \dots, p_{\tilde{m}}$ give a periodic orbit of period $\tilde{m}$ for T^2. We can then repeat the argument for T^2 and find periodic orbits of T^2 with period $\tilde{n}$ for all $\tilde{n} \prec \tilde{m}$ in the ordering (3.5). The thesis for T follows.
If $m = 2^r \cdot \tilde{m}$ with $r > 1$, $\tilde{m}$ odd and T admits no periodic orbits with odd period, then we do one step as in the previous case, and we are reduced to the case $m = 2^{r-1} \cdot \tilde{m}$. So we can repeat the argument and obtain the thesis. We remark that when $\tilde{m} = 1$, we only obtain periodic orbits with period powers of 2. □