3.3 Topological chaos

Definition 3.8. Let $T : X \to X$ be a continuous map on a metric space X. We say that T is *chaotic in the sense of Devaney* if there exists a compact forward invariant set $\Lambda \subset X$ such that:

(i) the set of periodic orbits is dense in Λ;

(ii) T is topologically transitive on Λ, that is for all open sets $U, V \subset X$ with non-empty intersection with Λ, there exists $n \in \mathbb{N}$ such that $T^n(U \cap \Lambda) \cap (V \cap \Lambda) \neq \emptyset$;

(iii) T has sensitive dependence on initial conditions on Λ, that is there exists $c > 0$ such that for all $x \in \Lambda$ and all $\varepsilon > 0$ one can find $y \in B_\varepsilon(x) \cap \Lambda$ for which there exists $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) > c$.

Example 3.3. Show that the Symbolic dynamics of Example 1.8 is chaotic in the sense of Devaney.

Remark 3.9. Conditions (i) and (ii) in Definition 3.8 imply (iii) (see [Ru17, Thm 7.4]).

Definition 3.9. Let $T : X \to X$ be a continuous map on a compact metric space X. For $n \in \mathbb{N}$ and $\varepsilon > 0$, a set $S \subset X$ is called (n, ε)*-separated* if for all $x, y \in S$ there exists $k = 0, \ldots, n$ such that $d(T^k(x), T^k(y)) > \varepsilon$. Then the quantity

$$h_{top}(T) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \Big(\max \{ \#S \,:\, S \text{ is } (n, \varepsilon)\text{-separated} \} \Big)$$

is well-defined and is called *topological entropy of* T.

Proposition 3.10. *Let* (X, T) *and* $(\tilde{X}, \tilde{T})$ *be two discrete-time continuous dynamical systems on compact metric spaces, and assume that* $(\tilde{X}, \tilde{T})$ *is a topological factor of* (X, T)*. Then* $h_{top}(T) \leq h_{top}(\tilde{T})$*. In particular, topological entropy is invariant under topological conjugacy.*

Example 3.4. Using Definition 3.9 and Proposition 3.10, show that: The Symbolic dynamics has positive topological entropy; The Tent map of Example 1.5 with $s = 2$, the Bernoulli map of Example 1.7, and the Logistic map of Example 1.6 with $\lambda = 4$ have topological entropy $\log 2$; The rotations of the circle of Example 1.4 have null topological entropy.

We now move to the case of maps of the interval. First, we give a simple criterion to compute the topological entropy in a special case.

Proposition 3.11. *Let $T : [a, b] \to [a, b]$ be a piecewise continuous monotone map with respect to a partition $\mathcal{J} = \{J_1, \ldots, J_N\}$ of the compact interval $[a, b]$ into closed subintervals. Assume that $T(J_i) = [a, b]$ for all $i = 1, \ldots, N$. Then*

$$h_{top}(T) = \lim_{k\to\infty} \frac{1}{k} \log\left(\#Fix(T^k)\right) = \log N\,.$$

We now introduce another notion of chaotic behaviour.

Definition 3.10. Let $T : X \to X$ be a continuous map on a compact interval $X = [a, b]$. We say that T has a *horseshoe* if there exists a closed sub-interval $J \subseteq X$ which T-covers itself 2-times.

Proposition 3.12. *Let $T : X \to X$ be a continuous map on a compact interval $X = [a, b]$. Then:*

(i) if T has a horseshoe then has periodic orbits with minimal period n for all $n \geq 1$;

(ii) if T has a periodic point with minimal odd period $m > 1$, then T^2 has a horseshoe.

Proof. (i) Let $J \subseteq [a, b]$ be the closed interval which covers itself 2-times, and let K_1 and K_2 be the open sub-intervals of J such that $K_1 \cap K_2 = \emptyset$ and $T(\overline{K_1}) = T(\overline{K_2}) = J$. We consider the T-graph of K_1, K_2, which is a full graph on the indices $\{1, 2\}$.

Let $K_1 = (\alpha, \beta)$ and $K_2 = (\beta + \varepsilon, \gamma)$, there are two cases. If $\varepsilon > 0$ or $\varepsilon = 0$ and β is not a fixed point, we apply Lemma 3.5 to the admissible path $K_1K_2K_2K_1$ to find a periodic point of period 3 which is not fixed, so it has minimal period 3 and we can apply Sharkovsky Theorem 3.8. If $\varepsilon = 0$ and β is a fixed point, then it follows that there exists $\delta \in (\beta, \gamma)$ such that $T([\delta, \gamma]) = J$, so we can repeat the argument with $K_1 = (\alpha, \beta)$ and $K_3 = (\delta, \gamma)$.

(ii) Let m be the smallest odd number for which T has a periodic orbit of minimal period m, and let $\{p_1, \ldots, p_{m-1}\}$ be the points of the periodic orbit in dynamical order, that is $T(p_i) = p_{i+1}$ for all $i = 1, \ldots, m-2$, and $T(p_{m-1}) = p_1$. By Step 5 in the proof of Proposition 3.7, the point of the periodic orbit are ordered in $[a, b]$ as

$$a \leq p_{m-1} < p_{m-3} < \cdots < p_5 < p_3 < p_1 < p_2 < p_4 < \cdots < p_{m-4} < p_{m-2} \leq b$$

or specularly. In the first case, we find $T(p_1, p_2) = (p_3, p_2)$ so that there exists $\delta \in (p_1, p_2)$ such that $T(\delta) = p_1$, and hence $T^2(\delta) = p_2$. We now show that $J = [p_{m-1}, p_2]$ T^2-covers itself 2-times. Let $K_1 = (p_{m-1}, p_{m-3})$, then $T^2(p_{m-1}) = p_2$ and $T^2(p_{m-3}) = p_{m-1}$, hence $T^2(\overline{K_1}) = J$. If we also let $K_2 = (p_{m-3}, \delta)$, then as shown before again $T^2(\overline{K_2}) = J$. Since $K_1 \cap K_2 = \emptyset$, we are done. □

Definition 3.11. Let $T : X \to X$ be a continuous map on a compact interval $X = [a, b]$. We say that T is *chaotic in the horseshoe sense* if there exists $n \in \mathbb{N}$ such that T^n has a horseshoe.

Theorem 3.13 ([Ru17], Thm 4.58 and Thm 7.3)**.** *Let $T : X \to X$ be a continuous map on a compact interval $X = [a, b]$. Then the following are equivalent:*

(i) T is chaotic in the sense of Devaney;

(ii) $h_{top}(T) > 0$;

(iii) T is chaotic in the horseshoe sense;

(iv) T has a periodic point with minimal period not a power of 2.

Example 3.5. The Tent map T_s of Example 1.5 is chaotic for all $s > 1$. If $s \geq \sqrt{2}$ one shows that T_s^2 has a horseshoe by using the interval $J_s = [\frac{1}{s+1}, \frac{s}{s+1}]$, since $\frac{1}{2} \in J_s$ and $T^2(\frac{1}{2}) \leq \frac{1}{s+1}$, whereas $T^2(\frac{1}{s+1}) = T^2(\frac{s}{s+1}) = \frac{s}{s+1}$. If $s \in (1, \sqrt{2})$, the result follows by observing that there exist intervals J_1 and J_2 on which T_s^2 is equal to T_{s^2} after rescaling.

Remark 3.14. For a $C^{1+\alpha}$ diffeomorphism of a manifold, positive topological entropy is equivalent to existence of a Smale horseshoe [Ka80].