Chapter 2

Continuous-time dynamical systems

2.1 Linear systems

The simplest case to study is that of an ordinary differential equation with linear vector field. Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix and consider the ordinary differential equation $\dot{\underline{x}} = A\underline{x}$. It is well known that the flow is given by $\phi_t(\underline{x}) = e^{At}\underline{x}$, and the behaviour of the orbits is determined by the eigenvalues of A. We state a result in the case that all the eigenvalues of A are simple, an analogous result holds counting the multiplicities of the eigenvalues and using the Jordan normal form of A.

Theorem 2.1. *Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix with k distinct real eigenvalues $\lambda_1, \dots, \lambda_k$, and $m = \frac{1}{2}(n-k)$ distinct couples of conjugate complex eigenvalues $a_j \pm i\, b_j$. Then there exists an invertible matrix $P \in M(n \times n, \mathbb{R})$ such that*

$$P^{-1}\, A\, P = \Lambda := diag\Big(\lambda_1, \dots, \lambda_k, B_1, \dots, B_m\Big)$$

where

$$B_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}\,, \quad \forall\, j = 1, \dots, m,$$

and the flow of the differential equation $\dot{\underline{x}} = A\underline{x}$ is given by

$$\phi_t(\underline{x}) = P\, e^{\Lambda\, t}\, P^{-1}\, \underline{x}$$

where

$$e^{\Lambda\, t} = diag\Big(e^{\lambda_1 t}, \dots, e^{\lambda_k t}, e^{tB_1}, \dots, e^{tB_m}\Big)$$

and

$$e^{tB_j} = e^{a_j t} \begin{pmatrix} \cos(b_j t) & -\sin(b_j t) \\ \sin(b_j t) & \cos(b_j t) \end{pmatrix}, \quad \forall j = 1, \dots, m.$$

Remark 2.2. Let us consider the case $n = 2, 3$, so that the matrix A can only have multiple real roots. If $n = 2$ the possible Jordan normal form of a matrix A with a double real eigenvalue λ are

$$\Lambda = \mathrm{diag}\Big(\lambda, \lambda\Big) \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

In the non-diagonal case, one writes $\Lambda = \lambda I + N$, where N is the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for which $N^2 = 0$. So that[1] $e^{\Lambda t} = e^{\lambda t} e^{Nt}$. It follows that

$$e^{\Lambda t} = \mathrm{diag}\Big(e^{\lambda t}, e^{\lambda t}\Big) \quad \text{or} \quad e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Analogously, in the $n = 3$ case, if A has eigenvalues with geometric multiplicities greater than or equal to 2, we are reduced to the previous case. If A has an eigenvalue λ with geometric multiplicity 1 its Jordan normal form is

$$\Lambda = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

and as before we write $\Lambda = \lambda I + N$, where N is a nilpotent matrix such that $N^3 = 0$. Then

$$e^{\Lambda t} = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{1}{2}t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

In the case of linear ordinary differential equations it is also particularly simple to find fixed points, periodic orbits, and invariant sets. First, using Definition 1.5 we find

[1] Here we use the fact that the matrices I and N commute.

Proposition 2.3. *The fixed points of the ordinary differential equation* $\dot{\underline{x}} = A\underline{x}$ *are the points in the kernel of* A.

In particular, the origin $\underline{x}_0 = \underline{0}$ is a fixed point for all A, and the other fixed points come in linear subspaces of $\mathbb{R}^n$. We'll see that the origin plays a special role in characterizing the dynamics of all the non-trivial orbits.

Concerning periodic orbits, it is straightforward from Theorem 2.1 that they can exist only if there is a couple of conjugate complex eigenvalues with null real part. If this is the case, all orbits leaving in the relative eigenspace are periodic, since they are of the form $e^{tB}\underline{x}$ with $a = 0$.

In general, the space $\mathbb{R}^n$ can be written as the direct sum of generalised eigenspaces of A, and according to the asymptotic behaviour of the orbits, it makes sense to consider the following decomposition.

Definition 2.1. Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix and let E_λ denote the generalised eigenspace of an eigenvalue λ. We call:
Stable eigenspace of $\underline{0}$ the linear space $E^s(\underline{0})$ defined as

$$E^s(\underline{0}) := \operatorname{Span}\{v \in E_\lambda \,:\, \Re(\lambda) < 0\}\ ;$$

Central eigenspace of $\underline{0}$ the linear space $E^c(\underline{0})$ defined as

$$E^c(\underline{0}) := \operatorname{Span}\{v \in E_\lambda \,:\, \Re(\lambda) = 0\}\ ;$$

Unstable eigenspace of $\underline{0}$ the linear space $E^u(\underline{0})$ defined as

$$E^u(\underline{0}) := \operatorname{Span}\{v \in E_\lambda \,:\, \Re(\lambda) > 0\}\ .$$

Theorem 2.4. *Let* $A \in M(n \times n, \mathbb{R})$ *be a real* $n \times n$ *matrix and consider the ordinary differential equation* $\dot{\underline{x}} = A\underline{x}$. *Then:*

(i) $n = \dim E^s(\underline{0}) + \dim E^c(\underline{0}) + \dim E^u(\underline{0})$;

(ii) the eigenspaces $E^s(\underline{0}), E^c(\underline{0}), E^u(\underline{0})$ *are invariant;*

(iii) the following dynamical characterisation holds:

$$E^s(\underline{0}) = \{\underline{x} \in \mathbb{R}^n \,:\, \phi_t(\underline{x}) \to \underline{0} \text{ as } t \to +\infty\}\,;$$

$$E^u(\underline{0}) = \{\underline{x} \in \mathbb{R}^n \,:\, \phi_t(\underline{x}) \to \underline{0} \text{ as } t \to -\infty\}\,.$$

Proof. It is a simple application of Theorem 2.1. □

Remark 2.5. It is interesting to notice that we haven't given a dynamical interpretation for the central eigenspace of $\underline{0}$. The reason is that if $\dim E^c(\underline{0}) \neq 0$ we can find different behaviours for the orbits. Let us consider the simple case $n = \dim E^c(\underline{0}) = 2$ with $\lambda = 0$ being a double eigenvalue. Then there are two possibilities for the matrix A (up to use of the Jordan normal form):

$$A = \operatorname{diag}\Big(0, 0\Big) \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the first case the flow is the identity, that is $\phi_t(x, y) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$, whereas in the second case the flow is given by $\phi_t(x, y) = (x + ty, y)$ for all $(x, y) \in \mathbb{R}^2$. Using Definition 2.2, in the first case $(0, 0)$ is Lyapunov stable and in the second case it is unstable.

Theorem 2.4 gives the characterisation of the dynamics with respect to the fixed point $\underline{0}$. In particular if $\ker(A) = \{\underline{0}\}$ and $\dim E^c(\underline{0}) = 0$, all orbits converge to $\underline{0}$, either for $t \to +\infty$ or for $t \to -\infty$. If instead the kernel of A consists of a non-trivial linear subspace W with $\dim W = \dim E^c(\underline{0})$, it is easy to see that the dynamics of non-fixed points is determined by that of the points in the space $W^\perp$.

Linear systems in the plane

In the case of linear systems in $\mathbb{R}^2$ it is possible to characterise the dynamical properties of the system without explicitly computing the eigenvalues of the matrix A. We also introduce a terminology for fixed points with different local dynamics.

The nature of the origin $\underline{0} = (0, 0)$ as a fixed point of a system $\dot{\underline{x}} = A\underline{x}$, with $\underline{x} = (x, y) \in \mathbb{R}^2$ is determined by the relation between the determinant and the trace of A. Indeed the characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 - \operatorname{tr}(A)\,\lambda + \det(A)\,,$$

so that the eigenvalues are

$$\lambda_\pm = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}^2(A) - 4\det(A)}}{2}\,,$$

and we distinguish different cases according to the sign of the determinant of A and of the discriminant $\Delta := \operatorname{tr}^2(A) - 4\det(A)$.

Case 1. $\det(A) > 0$ *and* $\Delta > 0$. The matrix A has two real distinct eigenvalues satisfying $\lambda_+ > \lambda_- > 0$ if $\operatorname{tr}(A) > 0$, and $\lambda_- < \lambda_+ < 0$ if $\operatorname{tr}(A) < 0$.

In both cases the orbits are generalised parabola through $\underline{0}$, at which they are tangent to the line generated by the eigenvector relative to eigenvalue of smallest modulus. If $\operatorname{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable node*. We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\operatorname{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable node*. We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

Note that $\underline{0}$ being a node is an open property since sufficiently small perturbations of A don't change the nature of the origin.

Case 2. $\det(A) > 0$ *and* $\Delta < 0$. The matrix A has a couple of complex conjugate eigenvalues $\lambda_\pm$ with $\Re(\lambda_\pm) = \frac{1}{2}\operatorname{tr}(A)$.

If $\operatorname{tr}(A) > 0$ all orbits are spirals out of $\underline{0}$ and they are either clockwise or anti-clockwise according for example to the sign of $\dot{x}$ when $y = 0$. In this case the origin is called an *unstable focus* and $E^u(\underline{0}) = \mathbb{R}^2$. If $\operatorname{tr}(A) < 0$ all orbits are spirals into $\underline{0}$ and as before they are either clockwise or anti-clockwise. In this case the origin is called a *stable focus* and $E^s(\underline{0}) = \mathbb{R}^2$. If $\operatorname{tr}(A) = 0$ all orbits are concentric circles about $\underline{0}$ and again they are either clockwise or anti-clockwise. In this case the origin is called a *center* and $E^c(\underline{0}) = \mathbb{R}^2$.

Notice that $\underline{0}$ being a focus is an open property. Instead $\underline{0}$ being a center is a closed property and arbitrarily small perturbations of A may turn the origin into an unstable or stable focus.

Case 3. $\det(A) > 0$ *and* $\Delta = 0$. The matrix A has one double real eigenvalue $\lambda = \frac{1}{2}\operatorname{tr}(A) \neq 0$.

If A is diagonalisable then the orbits lie on straight lines through $\underline{0}$. If $\operatorname{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable star*. We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\operatorname{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable star*. We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

If A is not diagonalisable then we use its Jordan normal form to understand the behaviour of the orbits. The differential equation in normal form reads

$$\begin{cases} \dot{x} = \lambda\, x + y \\ \dot{y} = \lambda y \end{cases}$$

so that there exists an invariant line, which is generated by the eigenvector of A, and the behaviour of the orbits can be found by looking at the sign of the two components of the vector field. If $\text{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable improper node*. We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\text{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable improper node*. We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

Both $\underline{0}$ being a star and being an improper node are closed properties. An arbitrarily small perturbation can turn the origin into a focus or a node, not changing the stability but the nature of the fixed point.

Case 4. $\det(A) < 0$. The matrix A has a couple of distinct real eigenvalues $\lambda_- < 0 < \lambda_+$.

In this case the orbits are generalised hyperbolae, and the origin is called a *saddle*. It holds $\dim E^u(\underline{0}) = \dim E^s(\underline{0}) = 1$, and none of the orbits outside the eigenspaces approaches the origin as $t \to \pm\infty$. Being a saddle is an open property.

Case 5. $\det(A) = 0$. The matrix A has two real eigenvalues, $\lambda_- = 0$ and $\lambda_+ = \text{tr}(A)$.

If $\text{tr}(A) \neq 0$, then A is diagonalisable and there is a line of fixed points. All the other orbits lie in straight lines which are parallel to the eigenspace of λ_+. If $\text{tr}(A) = 0$ we are reduced to the case of Remark 2.5 up to a change of coordinates, hence either all points are fixed or there is a line of fixed points and all other orbits lie in straight lines which are parallel to the eigenspace of λ_-.

Clearly, the properties of the origin considered in this case are closed and can be changed by arbitrarily small perturbations.