2.6 Exercises

2.1. Draw the phase portrait of the linear system $\dot{\underline{x}} = A\,\underline{x}$ in $\mathbb{R}^2$ and find the stable, unstable, and central eigenspace of $\underline{0}$, with A given by:

(a) $A = \begin{pmatrix} 5 & 4 \\ 2 & 7 \end{pmatrix}$ (b) $A = \begin{pmatrix} -8 & 0 \\ 1 & -6 \end{pmatrix}$ (c) $A = \begin{pmatrix} -8 & 6 \\ -9 & 13 \end{pmatrix}$

(d) $A = \begin{pmatrix} -8 & 4 \\ -1 & -4 \end{pmatrix}$ (e) $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$ (f) $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

(g) $A = \begin{pmatrix} -7 & -5 \\ 1 & -5 \end{pmatrix}$ (h) $A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$ (i) $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$

2.2. For the following systems, find the critical points and study their linear stability.

$$(a) \begin{cases} \dot{x} = -2x(x-1)(2x-1) \\ \dot{y} = -2y \end{cases} \qquad (b) \begin{cases} \dot{x} = x(4-2x-y) \\ \dot{y} = y(3-x-y) \end{cases}$$

$$(c) \begin{cases} \dot{x} = -y + x^3 \\ \dot{y} = x + y^3 \end{cases} \qquad (d) \begin{cases} \dot{x} = e^{(x+y)} + y \\ \dot{y} = y - xy \end{cases}$$

$$(e) \begin{cases} \dot{x} = 2xy \\ \dot{y} = y^2 - x^2 \end{cases} \qquad (f) \begin{cases} \dot{x} = x(60-4x-3y) \\ \dot{y} = y(42-3x-2y) \end{cases}$$

2.3. Find a Lyapunov function to study the stability of the fixed point $(0,0)$ for the following systems:

$$(a) \begin{cases} \dot{x} = y - 3x^3 \\ \dot{y} = -x - 7y^3 \end{cases} \qquad (b) \begin{cases} \dot{x} = -xy^4 \\ \dot{y} = yx^4 \end{cases}$$

$$(c) \begin{cases} \dot{x} = x - xy^4 \\ \dot{y} = y - y^3x^2 \end{cases} \qquad (d) \begin{cases} \dot{x} = x^2 - xy - x \\ \dot{y} = y^2 + 2xy - 7y \end{cases}$$

2.4. Determine the stability of the fixed point $(0,0)$ varying $\mu \in \mathbb{R}$ for the system

$$\begin{cases} \dot{x} = (\mu x + 2y)(z+1) \\ \dot{y} = (-x + \mu y)(z+1) \\ \dot{z} = -z^3 \end{cases}$$

2.5. Find the fixed points and study their stability varying $\mu \in \mathbb{R}$, $\mu \neq 4$, for the system

$$\begin{cases} \dot{x} = \mu\, x^3 - x^5 \\ \dot{y} = (2\mu y + z)(x-2) \\ \dot{z} = (-2y + \mu z)(x-2) \end{cases}$$

2.6. Draw the phase portrait for a mechanical Hamiltonian system with $H(x,y)$ of the form (2.4) with $m=1$ and potential energy W given by:

(a) $W(x) = \frac{1}{3}x^2 + \frac{1}{9}x^3 - \frac{1}{4}x^4$;

(b) $W(x) = x \log(1+x^2)$;

(c) $W(x) = \begin{cases} e^{-x^2}\,, & x \le 0 \\ \cos(\sqrt{2}\,x)\,, & x \ge 0 \end{cases}$;

(d) $W(x) = -\frac{\sin x}{x}$.

2.7. Consider the system

$$\begin{cases} \dot{x} = \frac{1}{2}y \\ \dot{y} = -(1+\mu)x + \mu x^2 + x^3 \end{cases}$$

varying $\mu \in \mathbb{R}$. Show that it is a mechanical Hamiltonian system writing down the Hamiltonian function. Let denote by $(x_\mu(t,0), y_\mu(t,y_0))$ the solution to the system with initial condition $(x(0),y(0)) = (0,y_0)$, then find

$$y^*(\mu) := \inf\{y_0 > 0 \,:\, \lim_{t\to+\infty} x_\mu(t) = +\infty\}\,.$$

2.8. Draw the phase portrait for the following systems:

$$(a) \begin{cases} \dot{x} = y - x^2 \\ \dot{y} = x - 2 \end{cases} \qquad (b) \begin{cases} \dot{x} = \sin x\,(-0.1\cos x - \cos y) \\ \dot{y} = \sin y\,(\cos x - 0.1\cos y) \end{cases} \text{ on } [0,\pi]^2$$

$$(c)\ \begin{cases} \dot{x} = x^2 - 1 \\ \dot{y} = -xy + x^2 - 1 \end{cases} \qquad (d)\ \begin{cases} \dot{x} = y\ \cos x \\ \dot{y} = \sin x \end{cases}$$

$$(e)\ \begin{cases} \dot{x} = y \\ \dot{y} = x^3 - x \end{cases} \qquad (f)\ \begin{cases} \dot{x} = y \\ \dot{y} = x^3 - x + \frac{1}{2}y \end{cases}$$

2.9. For the following systems, study the existence of a periodic orbit entirely contained in $\{x^2 + y^2 \geq 2\}$:

$$(a)\ \begin{cases} \dot{x} = x^3 - x + y^2 \\ \dot{y} = -2y \end{cases} \qquad (b)\ \begin{cases} \dot{x} = \frac{x^3}{1+x^4+y^4} \\ \dot{y} = \frac{y^3}{1+x^4+y^4} \end{cases}$$

2.10. Study the existence of a periodic orbit for the system

$$\begin{cases} \dot{x} = x\ \sqrt{x^2 + y^2} - 3x\,(x^2 + y^2) + \frac{1}{10}y^5 \\ \dot{y} = y\ \sqrt{x^2 + y^2} - 3y\,(x^2 + y^2) - \frac{1}{10}x^5 \end{cases}$$