2.4 Motion in the plane and periodic orbits

In this section we consider a system of differential equations in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \end{cases} \tag{2.7}$$

with C^k, $k \geq 1$, functions $f, g : \mathbb{R}^2 \to \mathbb{R}$. We discuss methods to study the phase space of (2.7) which work in two dimensions.

Polar coordinates

In $\mathbb{R}^2$, it is sometimes easier to study the phase space of a system when using polar coordinates. Let

$$\Omega := \left\{(\rho, \theta) \in \mathbb{R}^2 \,:\, \rho > 0\,,\, 0 \leq \theta \leq 2\pi\right\} / (\{\theta = 0\} = \{\theta = 2\pi\}),$$

that is Ω is a strip in the plane with the upper and the lower boundary identified, hence it is an open cylinder. The map $\psi : \Omega \to \mathbb{R}^2$, $(x, y) = \psi(\rho, \theta)$, with

$$\begin{cases} x(\rho, \theta) = \rho\, \cos\theta \\ y(\rho, \theta) = \rho\, \sin\theta \end{cases}$$

is a diffeomorphism from Ω to $\mathbb{R}^2$, with Jacobian $\det J\psi(\rho, \theta) = \rho$. We can then use ψ and its inverse to push a vector field $F(x, y) = (f(x, y), g(x, y))$ back to a vector field on Ω. An easy computation shows that the system (2.7) when written in polar coordinates reads

$$\begin{cases} \dot{\rho} = f(\rho\cos\theta, \rho\sin\theta)\, \cos\theta + g(\rho\cos\theta, \rho\sin\theta)\, \sin\theta \\ \dot{\theta} = \frac{g(\rho\cos\theta, \rho\sin\theta)\, \cos\theta - f(\rho\cos\theta, \rho\sin\theta)\, \sin\theta}{\rho} \end{cases} \tag{2.8}$$

for all $(\rho, \theta) \in \Omega$. In general one should not expect that the vector field in polar coordinates can be continuously extended to the boundary $\{\rho = 0\}$ of Ω. This is true only under particular conditions on the functions f, g.

An important application of the use of polar coordinates is the identification of circular periodic orbits. Using Proposition 2.18 one can show that

Proposition 2.20. *If there exists* $\rho_0 > 0$ *such that*

$$f(\rho_0\cos\theta, \rho_0\sin\theta)\, \cos\theta + g(\rho_0\cos\theta, \rho_0\sin\theta)\, \sin\theta = 0\,, \quad \forall\, \theta \in [0, 2\pi]$$

and $\dot\theta \neq 0$ *for all* $(\rho,\theta) \in \{\rho = \rho_0\}$*, then the set*

$$\Gamma = \{\rho = \rho_0\} = \left\{(x,y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = \rho_0^2\right\}$$

is a periodic orbit.

Isoclines

Here we introduce a method to find an analytic expression for the orbits of (2.7) in special situations.

Proposition 2.21. *Let* (x_0, y_0) *be a non-fixed point for* (2.7)*. Then there exists a neighbourhood* $U(x_0, y_0)$ *such that the set* $\mathcal{O}(x_0, y_0) \cap U$*, that is the orbit of* (x_0, y_0) *in* U*, is the graph of a function.*
In particular, if $f(x_0, y_0) \neq 0$ *there exist* $\varepsilon > 0$ *and a* C^k *function* $h : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ *such that*

$$\mathcal{O}(x_0, y_0) \cap (x_0 - \varepsilon, x_0 + \varepsilon) = \{(x, h(x)) \,:\, x \in (x_0 - \varepsilon, x_0 + \varepsilon)\}\,,$$

and $h(x)$ *satisfies the Cauchy system*

$$\begin{cases} \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \\ y(x_0) = y_0 \\ x \in (x_0 - \varepsilon, x_0 + \varepsilon) \end{cases} \tag{2.9}$$

Instead, if $g(x_0, y_0) \neq 0$ *the analogous statement holds by interchanging the roles of* x *and* y *and of* f *and* g*.*

Proof. If $f(x_0, y_0) \neq 0$ there exists $\varepsilon > 0$ such that $f(x,y) \neq 0$ for all $(x,y) \in B_{2\varepsilon}(x_0, y_0)$. Let $h(x)$ be a solution to system (2.9) and define the C^k function $I(x,y) = y - h(x)$ on $\{x \in (x_0 - \varepsilon, x_0 + \varepsilon)\} \cap B_{2\varepsilon}(x_0, y_0)$. Then with respect to system (2.7)

$$\begin{aligned} \dot I|_{\{I=0\}} &= \left(\dot y - h'(x)\,\dot x\right)|_{\{I=0\}} = \left(g(x,y) - h'(x)\, f(x,y)\right)|_{y=h(x)} = \\ &= g(x, h(x)) - h'(x)\, f(x, h(x)) \equiv 0. \end{aligned}$$

Therefore $I_0 := \{y = h(x)\}$ is an invariant set in a neighbourhood $U(x_0, y_0)$ containing (x_0, y_0). Then $I_0 = \mathcal{O}(x_0, y_0) \cap U$, and the proposition is proved.
The analogous argument works if $g(x_0, y_0) \neq 0$. □

The solutions to system (2.9) are called *isoclines* for (2.7).

Example 2.7 (Predator-prey Lotka-Volterra models). We apply the method of finding isoclines to prove the existence of periodic orbits in a predator-prey Lotka-Volterra system. Let $x, y \in \mathbb{R}_0^+$ denote the population of two species in a predator-prey relationship. The population x predates on the population y, hence the system of differential equations for x and y is of the form

$$\begin{cases} \dot{x} = x\,(-A + b_1\,y) \\ \dot{y} = y\,(B - b_2\,x) \end{cases} \tag{2.10}$$

with $A, B, b_1, b_2 > 0$. The system has two fixed points, $P_0 = (0, 0)$ and $P_1 = (B/b_2\,,\, A/b_1)$. The point P_0 is hyperbolic and it is a saddle with stable and unstable manifolds given by the x and y axis respectively, whereas the point P_1 is not hyperbolic being a center.

Let us find the isoclines of (2.10). When $x_0 \neq 0$ and $y_0 \neq A/b_1$ we can write

$$\begin{cases} \frac{dy}{dx} = \frac{y\,(B - b_2\,x)}{x\,(-A + b_1\,y)} \\ y(x_0) = y_0 \end{cases}$$

which has a local solution given implicitly by the equality

$$\int_{y_0}^{y} \frac{-A + b_1\,y}{y}\,ds = \int_{x_0}^{x} \frac{B - b_2\,x}{x}\,dt \quad \Leftrightarrow \quad I(x, y) = I(x_0, y_0)$$

where

$$I(x, y) := A\,\log y + B\,\log x - b_1\,y - b_2\,x\,.$$

We have thus found that $I(x, y)$ is a first integral for (2.10), hence the orbits lie on its level sets. Then, it is immediate to find that $I(x, y)$ has a point of global minimum at P_1, therefore the levels sets $\{I(x, y) = c\}$ are closed curves for c bigger than $I(P_1)$ but sufficiently close to it. Hence, the orbits on these level sets are periodic[3].

The field and the symmetries of the system

Here we introduce two ideas to draw the phase portrait of a system. Both ideas work in all dimensions but are particularly simple to apply in the two dimensional case.

The first idea uses the property of the field to be tangent to the orbits of a system. Therefore, in principle, one can obtain the orbits of a system simply by drawing the field in all the points of the phase space. In practice,

[3] It can be proved that all level sets are closed curves, therefore all orbits different from the axes and the fixed points are periodic.

it is useful to draw the behaviour of the field on some curves. For example, it is a good idea to draw the lines on which the single components of the field vanish (the intersection of these lines give the fixed points) and to obtain the direction of the field in all the regions of the phase space between these lines.

A more theoretical idea to apply is to look for symmetries of the system. Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, the associated system $\dot{\underline{x}} = F(\underline{x})$, and a diffeomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$, we give the following definition.

Definition 2.10. Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ and a diffeomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$, we say that the system $\dot{\underline{x}} = F(\underline{x})$ is *symmetric* with respect to S if

$$d_{\underline{x}} S(F(\underline{x})) = \pm F(S(\underline{x}))\,, \quad \forall\, \underline{x} \in \mathbb{R}^n\,.$$

A simple case is the case of systems symmetric with respect to linear transformations. That is there exists an invertible matrix $S \in M(n \times n, \mathbb{R})$ such that $S\,F(\underline{x}) = \pm F(S\,\underline{x})$.

Proposition 2.22. *If the system* (2.7) *in* $\mathbb{R}^2$ *is symmetric with respect to a diffeomorphism* S*, given a trajectory* $(x(t), y(t))$ *of the system, the curve* $(\tilde{x}(t), \tilde{y}(t))$ *defined as*

$$(\tilde{x}(t), \tilde{y}(t)) = \begin{cases} S(x(t), y(t))\,, & \text{if } d_{\underline{x}} S(F(\underline{x})) = F(S(\underline{x})), \\ S(x(-t), y(-t))\,, & \text{if } d_{\underline{x}} S(F(\underline{x})) = -F(S(\underline{x})), \end{cases}$$

is a solution to (2.7).

Proof. It is enough to compute $(\dot{\tilde{x}}(t), \dot{\tilde{y}}(t))$. □

Example 2.8. We show how the proposition works in two easy cases. Let us consider the system (2.7) with the assumption that $f(-x,-y) = -f(x,y)$ and $g(-x,-y) = -g(x,y)$. The field $F(x,y) = (f(x,y), g(x,y))$ satisfies $F(-x,-y) = -F(x,y)$, hence it is symmetric with respect to the linear transformation $S(x,y) = (-x,-y)$ and

$$d_{(x,y)} S(F(x,y)) = -F(x,y) = F(S(x,y)).$$

Then, given a trajectory $(x(t), y(t))$ of the system, we show that another trajectory is given by $(\tilde{x}(t), \tilde{y}(t)) = (-x(t), -y(t))$. Indeed, we have

$$\dot{\tilde{x}}(t) = -\dot{x}(t) = -f(x(t), y(t)) = f(-x(t), -y(t)) = f(\tilde{x}(t), \tilde{y}(t))\,,$$

$$\dot{\tilde{y}}(t) = -\dot{y}(t) = -g(x(t), y(t)) = g(-x(t), -y(t)) = g(\tilde{x}(t), \tilde{y}(t))\,.$$

The other case considered in the proposition is obtained for Hamiltonian systems in $\mathbb{R}^2$ with Hamiltonian function of the form (2.2). In this case the field is $F(x,y) = (y, -W'(x))$ and the system is symmetric with respect to the linear transformation $S(x,y) = (x,-y)$ since

$$d_{(x,y)}S(F(x,y)) = (y, W'(x)) = -F(x,-y) = -F(S(x,y))\,.$$

Then, given a trajectory $(x(t), y(t))$ of the system, we show that another trajectory is given by $(\tilde{x}(t), \tilde{y}(t)) = (x(-t), -y(-t))$. Indeed,

$$\begin{aligned}\dot{\tilde{x}}(t) &= -\dot{x}(-t) = -y(-t) = \tilde{y}(t)\,,\\ \dot{\tilde{y}}(t) &= \dot{y}(-t) = -W'(x(-t)) = -W'(\tilde{x}(t))\,.\end{aligned}$$

Periodic orbits: non-existence

We describe two methods to prove non-existence of periodic orbits in a region of the phase space. The first is of pure topological nature and the second uses the analytical nature of the differential equation (2.7).

Definition 2.11. Let $\Gamma \subset \mathbb{R}^2$ be a simple closed curve. Given a vector field $F(x,y) = (f(x,y), g(x,y))$ without fixed points on Γ, the *Poincaré index of* Γ, denoted by $I_F(\Gamma)$, is the number of turns that F makes counterclockwise as a point goes round Γ. It can be computed as

$$I_F(\Gamma) := \frac{1}{2\pi}\int_\Gamma d\left(\arctan\frac{g}{f}\right) = \frac{1}{2\pi}\int_\Gamma \frac{f\,dg - g\,df}{f^2+g^2}$$

Proposition 2.23. *Given a vector field F on $\mathbb{R}^2$, the Poincaré index of a curve has the following properties:*

(i) let $t \mapsto \Gamma_t$ be a continuous family of simple closed curves, then $I_F(\Gamma_t)$ is constant as long as no Γ_t contains a fixed point of F;

(ii) let Γ be a simple closed curve not containing fixed points of F which can be written as $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 and Γ_2 are two simple closed curves not containing fixed points of F. Then $I_F(\Gamma) = I_F(\Gamma_1) + I_F(\Gamma_2)$;

(iii) if Γ is a periodic orbit then $I_F(\Gamma) = +1$.

Definition 2.12. Let (x_0, y_0) be an isolated fixed point of a vector field F. The *Poincaré index of* (x_0, y_0), $I_F(x_0, y_0)$, is the Poincaré index of any simple closed curve Γ encircling (x_0, y_0) and no other fixed point of F.

Proposition 2.24. *Let (x_0, y_0) be a fixed point of a C^1 vector field F on $\mathbb{R}^2$ with $\det(JF(x_0, y_0)) \neq 0$. Then:*

(i) if (x_0, y_0) is a node, a star, an improper node, a focus or a centre, then $I_F(x_0, y_0) = +1$;

(ii) if (x_0, y_0) is a saddle, then $I_F(x_0, y_0) = -1$.

Putting together Propositions 2.23 and 2.24, we obtain information on regions of a phase space where a periodic orbit may exist or not. For example, it may not exist a periodic orbit encircling only a saddle. Each periodic orbit has to encircle sets of isolated fixed points for which the sum of their Poincaré indices is +1.

Example 2.9. Let us consider the system

$$\begin{cases} \dot{x} = x \\ \dot{y} = y^2 \end{cases}$$

then $I_F(0,0) = 0$.

Example 2.10. Let us consider the system

$$\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = 2xy \end{cases}$$

then $I_F(0,0) = 2$.

Proposition 2.25 (Curl method)**.** *Let $U \subset \mathbb{R}^2$ be a simply connected open set and assume that the vector field $F(x,y) = (f(x,y), g(x,y))$ satisfies*

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial g}{\partial x}(x,y)\,, \quad \forall\,(x,y) \in U$$

Then in U there exist no periodic orbits for the vector field F.

Proof. Let $\Gamma \subset U$ be a periodic orbit of period T parametrised by the solution $\gamma(t)$ of the Cauchy problem

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \\ (x(0), y(0)) = \gamma(0) \end{cases}$$

Then $\gamma(T) = \gamma(0)$ and $\gamma'(t) = F(\gamma(t))$ for all $t \in \mathbb{R}$.

By assumption and Poincaré's lemma, the vector field F is conservative in U, that is there exists a C^1 function $h : U \to \mathbb{R}$ such that $F = \nabla h$. Then

$$0 = h(\gamma(T)) - h(\gamma(0)) = \int_0^T \frac{d}{dt}(h \circ \gamma)(t)\,dt = \int_0^T \big\langle \nabla h(\gamma(t)), \gamma'(t) \big\rangle\,dt =$$

$$= \int_0^T \langle F(\gamma(t)), F(\gamma(t)) \rangle\,dt = \int_0^T \|F(\gamma(t))\|^2\,dt$$

which is a contradiction because $\|F(\gamma(t))\| \neq 0$ for all t. □

Remark 2.26. The curl method can be easily extended to a differential equation in $\mathbb{R}^n$ with vector field F. By repeating the last part of the proof of Proposition 2.25 one can show that

Proposition 2.27 (Gradient systems). *If there exists $h : \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla h$, then there are no periodic orbits for the differential equations $\dot{\underline{x}} = F(\underline{x})$.*

Proposition 2.28 (Bendixson-Dulac method)**.** *Let $U \subset \mathbb{R}^2$ be a simply connected open set and assume that there exists a C^1 function $\rho : U \to \mathbb{R}$ such that for the vector field $F(x, y) = (f(x, y), g(x, y))$ it holds*

$$\frac{\partial(\rho \cdot f)}{\partial x}(x, y) + \frac{\partial(\rho \cdot g)}{\partial y}(x, y) > 0 \;(\text{or} < 0), \quad \forall\,(x, y) \in U$$

Then in U there exist no periodic orbits for the vector field F.

Proof. Let $\Gamma \subset U$ be a periodic orbit of period T and let A be the region enclosed by Γ. Then applying Gauss-Green Theorem

$$0 < \iint_A \left(\frac{\partial(\rho \cdot f)}{\partial x}(x, y) + \frac{\partial(\rho \cdot g)}{\partial y}(x, y) \right) dxdy = \int_\Gamma (-\rho\, g\, dx + \rho\, f\, dy) =$$

$$= \int_0^T \rho(x(t), y(t))\, \big(-g(x(t), y(t))\, \dot{x}(t) + f(x(t), y(t))\, \dot{y}(t)\big) = 0$$

where we have used that $\Gamma = (x(t), y(t))$ for $t \in [0, T]$ and $(x(t), y(t))$ is a solution of the differential equation associated to the vector field F. □

Example 2.11 (Species in competition). In Example 2.7 we have shown that predator-prey Lotka-Volterra models admit periodic orbits. Now we show that there are no periodic orbits in a Lotka-Volterra model for species in competition. Let $x, y \in \mathbb{R}_0^+$ denote the population of two species in competition for the same resources on a finite environment. The system of differential equations for x and y is of the form

$$\begin{cases} \dot{x} = x\,(A - a_1\,x - b_1\,y) = f(x,y) \\ \dot{y} = y\,(B - b_2\,x - a_2\,y) = g(x,y) \end{cases} \tag{2.11}$$

with $A, B, a_1, a_2, b_1, b_2 > 0$.

The axes are invariant sets, hence the simply connected set

$$U = \left\{(x,y) \in \mathbb{R}^2 \, : \, x > 0, \, y > 0\right\}$$

is also invariant. Consider the C^1 function $\rho(x,y) = 1/(xy)$ on U. We have

$$\frac{\partial(\rho \cdot f)}{\partial x}(x,y) + \frac{\partial(\rho \cdot g)}{\partial y}(x,y) = -\frac{a_1}{y} - \frac{a_2}{x} < 0\,, \quad \forall\,(x,y) \in U.$$

Hence, by Proposition 2.28, there are no periodic orbits in U.

Remark 2.29. The Bendixson-Dulac method uses the divergence of a vector field. For differential equations in $\mathbb{R}^n$, $n \geq 3$, it gives different information.

Proposition 2.30. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field such that there exists a constant $k > 0$ for which $div(F)(\underline{x}) \leq -k$ for all $\underline{x} \in \mathbb{R}^n$. Then the flow associated to F contracts the volumes.*

Proof. For $A \subset \mathbb{R}^n$ let $\phi_t(A)$ be the evolution of the set at time t, and let m be the n-dimensional Lebesgue measure. By applying the same ideas in the proof of Liouville Theorem 2.15, we obtain

$$\mathrm{vol}(\phi_t(A)) = \int_A \exp\left(\int_0^t \mathrm{div}(F)(\phi_s(\underline{x}))\,ds\right)\,dm.$$

If $\mathrm{div}(F)(\underline{x}) \leq -k$ for all $\underline{x} \in \mathbb{R}^n$ then

$$\mathrm{vol}(\phi_t(A)) \leq e^{-kt}\,\mathrm{vol}(A)\,, \quad \forall\, t \geq 0,$$

and the proof is finished. □

Periodic orbits: existence in general

Theorem 2.31 (Poincaré - Bendixson). *Let F be a C^1 vector field in $\mathbb{R}^2$, and assume that there exists a non-empty region $D \subset \mathbb{R}^2$ which is compact and does not contain fixed points of F. If for some $\underline{x}_0$ there exists t_0 such that $\phi_t(\underline{x}_0) \in D$ for all $t \geq t_0$, then there exists a periodic orbit $\Gamma \subset D$ and $\Gamma = \omega(\underline{x}_0)$.*

For the proof we need some preliminaries. Given the differential equation $\dot{\underline{x}} = F(\underline{x})$ in $\mathbb{R}^2$ with $F \in C^1$ and any non-fixed point $\underline{y}$ of F, we call *transversal line at* $\underline{y}$ the line $\ell(\underline{y})$ which is the image of the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $\gamma(u) = \underline{y} + u\underline{v}$, where $\underline{v}$ is a vector applied at $\underline{y}$ which satisfies $\langle \underline{v}, F(\underline{y}) \rangle = 0$.

Definition 2.13. Given a non-fixed point $\underline{y}$ of F and a constant $k \in [0, 1)$, we call *k-wide local section at $\underline{y}$* the set $S_k(\underline{y})$ obtained by taking the connected component containing $\underline{y}$ of the set of points $\underline{z} \in \ell(\underline{y})$ for which $|\sin(\widehat{\underline{v}F(\underline{z})})| > k$.

The k-wide local section at $\underline{y}$ is non-empty since $\underline{y} \in S_k(\underline{y})$, and there exists $\varepsilon > 0$ such that $\gamma(-\varepsilon, \varepsilon) \subseteq S_k(\underline{y})$.

Proposition 2.32 (Local rectifiability of a vector field)**.** *Given a C^1 vector field F in $\mathbb{R}^2$, a non-fixed point $\underline{y}$ of F, and a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, there exists a diffeomorphisms $\psi : U(\underline{0}) \to V(\underline{y})$ which maps horizontal lines into the orbits of $\dot{\underline{x}} = F(\underline{x})$ passing through $S_k(\underline{y})$. That is $\psi(s, u) = \phi_s(\gamma(u))$ for all $(s, u) \in U(\underline{0})$.*

Applying Proposition 2.32, let $\sigma > 0$ and $N_\sigma := \{(s, u) \in U(\underline{0}) : |s| < \sigma\}$. Then we call *$\sigma$-rectangle of flux in $\underline{y}$* the set $\mathcal{N}_\sigma := \psi(N_\sigma)$. Then for each $\underline{z} \in \mathcal{N}_\sigma$ there exists a unique $s \in (-\sigma, \sigma)$ such that $\phi_s(\underline{z}) \in S_k(\underline{y})$.

Proposition 2.33. *Given a C^1 vector field F in $\mathbb{R}^2$, a non-fixed point $\underline{y}$ of F, and a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, let $\underline{z}$ be a point such that $\underline{y} = \phi_{t_0}(\underline{z})$ for some t_0. Then there exist $\varepsilon > 0$ and a continuous function $\tau : B_\varepsilon(\underline{z}) \to \mathbb{R}$ such that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{z})$.*

Proof. Let us define the function $p : \mathbb{R}^2 \to \mathbb{R}$ by $p(\underline{x}) = \langle \underline{x}, F(\underline{y}) \rangle$. We notice that $p(\underline{x}) = p(\underline{y})$ if and only if $\underline{x} \in \ell(\underline{y})$, in fact if $\underline{x} = \underline{y} + \underline{w}$ then

$$p(\underline{x}) = p(\underline{y}) + p(\underline{w}) = p(\underline{y}) \quad \Leftrightarrow \quad \langle \underline{w}, F(\underline{y}) \rangle = 0$$

Let then consider the regular function $G : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by $G(\underline{x}, t) = p(\phi_t(\underline{x}))$. Then by definition $G(\underline{z}, t_0) = p(\phi_{t_0}(\underline{z})) = p(\underline{y})$ and

$$\frac{\partial G}{\partial t}(\underline{z}, t_0) = p(\dot{\phi}_t(\underline{x}))\Big|_{\underline{x}=\underline{z}, t=t_0} = p(F(\phi_{t_0}(\underline{z}))) = p(F(\underline{y})) = \|F(\underline{y})\|^2 \neq 0$$

Hence we can apply the Implicit Function Theorem to G at $(\underline{z}, t_0)$ and prove the existence of $\varepsilon > 0$ and $\delta > 0$, and of a continuous function $\tau : B_\varepsilon(\underline{z}) \to (t_0 - \delta, t_0 + \delta)$ such that

$$p(\underline{y}) = G(\underline{x}, \tau(\underline{x})) = p(\phi_{\tau(\underline{x})}(\underline{x})) \qquad \forall\, \underline{x} \in B_\varepsilon(\underline{z})$$

It follows that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{z})$. □

We are now ready to prove Poincaré-Bendixson Theorem.

Proof of Theorem 2.31. Choose $\underline{x}_0 \in \mathbb{R}^2$ such that there exists t_0 for which $\phi_t(\underline{x}_0) \in D$ for all $t \geq t_0$. By Proposition 1.1, the set $\omega(\underline{x}_0) \subset D$ is non-empty, compact, and invariant. For any $\underline{x} \in \omega(\underline{x}_0)$ we show that $\mathcal{O}(\underline{x})$ is a periodic orbit Γ, and that $\Gamma = \omega(\underline{x}_0)$.

Fix $\underline{x} \in \omega(\underline{x}_0)$, and let $\underline{y} \in \omega(\underline{x}) \subset \omega(\underline{x}_0)$, which is not a fixed point by assumption. Consider a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, and a σ-rectangle of flux $\mathcal{N}_\sigma$ in $\underline{y}$.

Lemma 2.34. *The forward orbit of $\underline{x}$ intersects $S_k(\underline{y})$ exactly once.*

Proof. Since $\underline{y} \in \omega(\underline{x})$, there exists a point of $\mathcal{O}^+(\underline{x})$ in $\mathcal{N}_\sigma$, hence $\mathcal{O}^+(\underline{x}) \cap S_k(\underline{y}) \neq \emptyset$. Let's assume by contradiction that there exist $\underline{x}_1, \underline{x}_2 \in \mathcal{O}^+(\underline{x}) \cap S_k(\underline{y})$ with $\underline{x}_1 \neq \underline{x}_2$. Since $\underline{x} \in \omega(\underline{x}_0)$, also $\underline{x}_1, \underline{x}_2 \in \omega(\underline{x}_0)$ by invariance of the omega limit. Hence if $\mathcal{N}_\sigma(\underline{x}_1)$ and $\mathcal{N}_\sigma(\underline{x}_2)$ are disjoint σ-rectangles of flux, the forward orbit of $\underline{x}_0$ has countable points both in $\mathcal{N}_\sigma(\underline{x}_1)$ and in $\mathcal{N}_\sigma(\underline{x}_2)$. By the properties of the rectangles of flux, this implies that $\mathcal{O}^+(\underline{x}_0)$ intersects $S_k(\underline{y})$ countable many times, alternatively close to $\underline{x}_1$ and to $\underline{x}_2$. We now show that this is not possible.

Let us denote by $\{\underline{z}_1, \underline{z}_2, \dots\}$ the points in $\mathcal{O}^+(\underline{x}_0) \cap S_k(\underline{y})$ cronologically ordered, that is $\underline{z}_1 = \phi_{t_1}(\underline{x}_0)$, $\underline{z}_2 = \phi_{t_2}(\underline{x}_0)$, and so on, with $t_1 < t_2 < \dots$. Given three points $\underline{z}_{n-1}, \underline{z}_n, \underline{z}_{n+1}$ and an ordering on $S_k(\underline{y})$ it must hold $\underline{z}_{n-1} < \underline{z}_n < \underline{z}_{n+1}$ or $\underline{z}_{n+1} < \underline{z}_n < \underline{z}_{n-1}$. Indeed let Σ denotes the Jordan curve given by the segment $\overline{\underline{z}_{n-1}\underline{z}_n}$ and the orbit $\cup_{t_{n-1} \leq t \leq t_n} \phi_t(\underline{x}_0)$, and let R be the region bounded by Σ. Then $\phi_t(\underline{x}_0) \in R$ for all $t > t_n$, because it cannot intersect any part of ∂R. It cannot intersect the orbit $\cup_{t_{n-1} \leq t \leq t_n} \phi_t(\underline{x}_0)$ by the uniqueness of solutions of a differential equation, and it cannot intersect the segment $\overline{\underline{z}_{n-1}\underline{z}_n}$ which is in $S_k(\underline{y})$, because the vector field points in the same direction in all the points of a local section. It follows that $\underline{z}_{n+1} \in R$ and it lies on the other side of $\underline{z}_{n-1}$ with respect to $\underline{z}_n$. It follows that the countable intersections of $\mathcal{O}^+(\underline{x}_0)$ with $S_k(\underline{y})$ must be ordered, so cannot be alternatively close to $\underline{x}_1$ and to $\underline{x}_2$. This shows that the forward orbit of $\underline{x}$ intersects $S_k(\underline{y})$ exactly once. $\square$

We have thus proved that $\mathcal{O}^+(\underline{x}) \cap S_k(\underline{y}) = \{\phi_{\bar{t}}(\underline{x})\}$. Since $\underline{y} \in \omega(\underline{x})$ there is a sequence $\{t_m\}$ such that $\phi_{t_m}(\underline{x}) \to \underline{y}$, hence for m big enough $\phi_{t_m}(\underline{x}) \in \mathcal{N}_\sigma$. It follows that for m big enough, there exist $\tau_m \in \mathbb{R}$ such that $\phi_{t_m+\tau_m}(\underline{x}) \in S_k(\underline{y})$ for all m, hence $\phi_{t_m+\tau_m}(\underline{x}) = \phi_{\bar{t}}(\underline{x})$ for all m. It follows that there exists $T > 0$ such that $\phi_T(\underline{x}) = \underline{x}$. We have thus proved that $\mathcal{O}(\underline{x})$ is a periodic orbit Γ.

It remains to show that $\Gamma = \omega(\underline{x}_0)$. By invariance of the omega limit $\Gamma \subset \omega(\underline{x}_0)$. Let now $\underline{y} \in \Gamma$ and consider a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, and a σ-rectangle of flux $\mathcal{N}_\sigma$ in $\underline{y}$. As discussed above, there exists a sequence $\{t_m\}$ such that $\phi_{t_m}(\underline{x}_0) \to \underline{y}$ and $\phi_{t_m}(\underline{x}_0) \in S_k(\underline{y})$, with $\phi_t(\underline{x}_0) \notin S_k(\underline{y})$ for $t \in (t_m, t_{m+1})$ for all m. Since $\phi_T(\underline{y}) = \underline{y}$, we can apply Proposition 2.33 and find $\varepsilon > 0$, $\delta > 0$, and a continuous function $\tau : B_\varepsilon(\underline{y}) \to (T-\delta, T+\delta)$ such that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{y})$, and $\tau(\underline{y}) = T$. Hence, choosing $\tilde{\varepsilon} < \varepsilon$ if necessary, we have $\phi_T(\underline{x}) \in \mathcal{N}_\sigma$ for all $\underline{x} \in B_{\tilde{\varepsilon}}(\underline{y})$. Since for m big enough $\phi_{t_m}(\underline{x}_0) \in B_{\tilde{\varepsilon}}(\underline{y})$, it follows that $\phi_T(\phi_{t_m}(\underline{x}_0)) = \phi_{T+t_m}(\underline{x}_0) \in \mathcal{N}_\sigma$, and there exists $s_m \in (-\sigma, \sigma)$ such that $\phi_{T+t_m+s_m}(\underline{x}_0) \in S_k(\underline{y})$. Since $\phi_t(\underline{x}_0) \notin S_k(\underline{y})$ for $t \in (t_m, t_{m+1})$, it must hold $t_{m+1} = T + t_m + s_m$, hence $t_{m+1} - t_m \leq T + \sigma$ for all m big enough.

We now consider a fixed $\eta > 0$. By continuity of the flux ϕ_t, there exists $\delta > 0$ such that if $d(\underline{z}_1, \underline{z}_2) < \delta$ then $d(\phi_t(\underline{z}_1), \phi_t(\underline{z}_2)) < \eta$ for all $t \in (-T-\sigma, T+\sigma)$. Hence, for m big enough such that $d(\phi_{t_m}(\underline{x}_0), \underline{y}) < \delta$, we have

$$d\Big(\phi_t(\phi_{t_m}(\underline{x}_0)), \phi_t(\underline{y})\Big) < \eta \qquad \forall\, t \in (-T-\sigma, T+\sigma)$$

Since $\underline{y} \in \Gamma$, so that $\mathcal{O}(\underline{y}) = \Gamma$, and $t_{m+1} - t_m \leq T + \sigma$ for all m big enough, we have that

$$d\Big(\phi_t(\underline{x}_0), \Gamma\Big) < \eta \qquad \forall\, t \in (t_m, t_{m+1})$$

for m big enough. We can conclude that $d(\phi_t(\underline{x}_0), \Gamma) \to 0$ as $t \to +\infty$. Hence $\omega(\underline{x}_0) \subset \Gamma$. This shows that $\Gamma = \omega(\underline{x}_0)$, and concludes the proof of the theorem. □

Example 2.12. Let us consider the following system in polar coordinates

$$\begin{cases} \dot{\rho} = \rho(1-\rho^2) + \varepsilon\, f(\rho, \theta) \\ \dot{\theta} = 1 + \varepsilon\, g(\rho, \theta) \end{cases}$$

with $f, g \in C^1(\mathbb{R}^2)$. For $\varepsilon = 0$ the system admits the orbitally asymptotically stable periodic orbit $\Gamma = \{\rho = 1\}$. We now show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a periodic orbit Γ_ε. Let

$$L = \max_{\rho \leq 5} \,(|f| + |g|)$$

and $\varepsilon_0 = \frac{1}{4L}$. We now prove that if $\varepsilon < \varepsilon_0$ the set $D = \left\{\frac{1}{2} \leq \rho \leq 2\right\}$ satisfies the assumptions of Poincaré-Bendixson Theorem 2.31.

First of all for all $(\rho, \theta) \in D$

$$1 + \varepsilon\, g(\rho, \theta) \geq 1 - \varepsilon L > 1 - \varepsilon_0 L = \frac{3}{4}$$

so that D contains no fixed points of the system. Moreover

$$\dot{\rho}|_{\rho=2} = -6 + \varepsilon\, f(2, \theta) < -6 + \varepsilon\, L < -6 + \varepsilon_0\, L = -6 + \frac{1}{4} < 0$$

and

$$\dot{\rho}|_{\rho=\frac{1}{2}} = \frac{3}{8} + \varepsilon\, f\left(\frac{1}{2}, \theta\right) > \frac{3}{8} - \varepsilon\, L > \frac{3}{8} - \varepsilon_0\, L = \frac{1}{8} > 0$$

so that on ∂D the vector field is always directed towards the inside of D. This implies that for all $\underline{x} \in \partial D$ and for all $t > 0$ it holds $\phi_t(\underline{x}) \in D$, and completes the proof.

Finally, we state a result which extends Theorem 2.31 to the case of regions with fixed points.

Theorem 2.35. *Let F be a C^1 vector field in $\mathbb{R}^2$, and let $D \subset \mathbb{R}^2$ be a non-empty bounded positively invariant region containing at most a finite number of fixed points for F. Then, for all $\underline{x} \in D$, the set $\omega(\underline{x})$ is non-empty and one of the following possibilities holds:*

- *$\omega(\underline{x})$ is a fixed point;*
- *$\omega(\underline{x})$ is a periodic orbit;*
- *$\omega(\underline{x})$ consists of a finite number of fixed points and heteroclinic orbits connecting them.*