Prerequisiti: sistema dinamico discreto

$T: X \to X$ qualitativa

Meccanica Superiore: teoria ergodica, sistemi caotici

Fisica Matematica: dal regolare al comp. caotico.

Modello: mappa di Poincaré di un sist. dinamico continuo

- Il "caos" in ODE può esistere solo se spazio delle fasi ha dimensione ≥ 3. Mappa di Poincaré sarà su X di dim 2.

- Sistemi hamiltoniani. Esiste $H: \mathbb{R}^n_p \times \mathbb{R}^n_q \to \mathbb{R}$, n = gradi di libertà

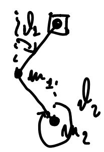
  $H(p,q)$ è l'hamiltoniana

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

  $H(p,q)$ è un integrale primo.

<u>ES</u> Se $n=2$ allora il flusso del sist. hamiltoniano è definito su una varietà di dim 3.

Pendolo doppio

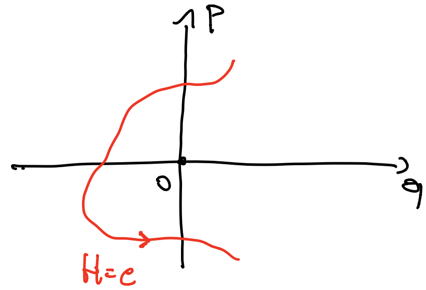


- Sistemi hamiltoniani a 1,5 gradi di libertà.

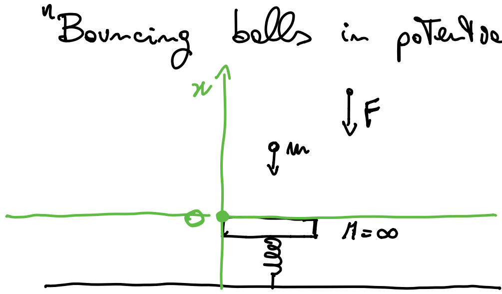
  $H(p,q,t)$, $(p,q) \in \mathbb{R}^2$

  Mappa di Poincaré $(p(0), q(0)) \xrightarrow{T} (p(T), q(T))$

oss Se $H(p,q): \mathbb{R}_p \times \mathbb{R}_q \to \mathbb{R}$



Esempi "Bouncing balls in potentials" (Accelerazione di Fermi)



$m = 1$

$(x, \dot{x})$ spazio delle fasi

urto elastico

$f(t)$ sia la posizione dell'ostacolo, $f(t)$ sia 1-periodica

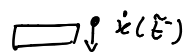
$F(x) = -g$

Semplificazione: approssimazione di ostacolo statico. Supponiamo che l'urto avvenga sempre all'altezza 0.

Sia $x(t)$ la posizione di m, e $y(t) = x(t) - f(t) \geq 0$

Se $\tilde{t}$ è l'istante di un urto, $\underbrace{\dot{y}(\tilde{t}^+)}_{\text{poco dopo}} = -\underbrace{\dot{y}(\tilde{t}^-)}_{\text{poco prima}}$

$$\dot{x}(\tilde{t}^+) = \dot{f}(\tilde{t}^+) + \dot{f}(\tilde{t}^-) - \dot{x}(\tilde{t}^-)$$



f supponiamo che sia almeno C^1, $\dot{f}(\tilde{t}^+) = \dot{f}(\tilde{t}^-)$

$$\dot{x}(\tilde{t}^-) = -\dot{x}(\bar{t}^+)$$ dove $\bar{t}$ è l'istante dell'urto precedente

Studio l'evoluzione di (t_n, v_n), dove $\{t_n\}$ istanti di urto e $\{v_n\}$ sono le velocità di m al tempo t_n^+.

$$T \quad \begin{cases} t_{n+1} = t_n + \tau(v_n) \overset{F=-g}{=} t_n + 2\dfrac{v_n}{g} \\ v_{n+1} = 2\dot{f}(t_{n+1}) + v_n \end{cases}$$

<u>OSS</u> $F(x) = -\nabla U(x)$, $U(x) = c\,x^{\alpha}$, $\alpha > 0$

$\alpha = 1 \Rightarrow F = -c$ ($\exists$ un insieme di mis. positiva di condiz. iniz. con orbite con velocità illimitate)

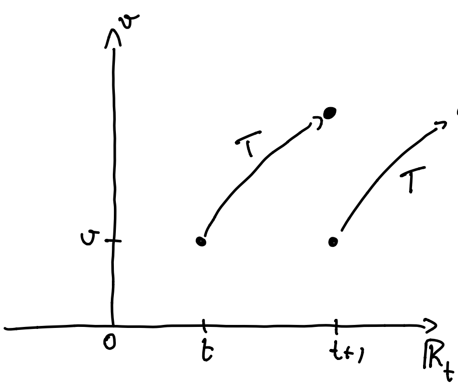
$\alpha = 2$

$\alpha \in (0,1)$

$$(t_{n+1}, v_{n+1}) = T(t_n, v_n) \qquad T: \mathbb{R}_t \times \mathbb{R}^+_v \to \mathbb{R}_t \times \mathbb{R}^+_v$$

$$T(t+1, v) = \left(t+1+2\frac{v}{g},\ v + 2\dot{f}\left(t+1+2\frac{v}{g}\right)\right) =$$

$$= T(t,v) + (1,0)$$

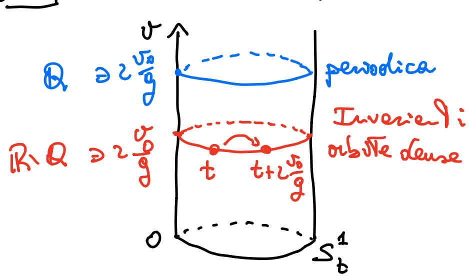


$$\pi: \mathbb{R}_t \to S^1_t = [0,1]/_{0\sim 1}$$

$$T: S^1_t \times \mathbb{R}^+_v \to S^1_t \times \mathbb{R}^+_v$$

$$T(t,v) = \left(\left\{t + 2\frac{v}{g}\right\},\ v + 2\dot{f}\left(t+2\frac{v}{g}\right)\right)$$

$\boxed{\text{CASO } \dot{f} \equiv 0}$ $\qquad T(t,v) = \left(t + 2\dfrac{v}{g},\ v\right)$



$\forall v_0 \in (0,+\infty)$

$$T\big|_{\{v=v_0\}} \sim Rot_{\alpha_0}: S^1 \circlearrowleft$$

$$\alpha_0 = 2\frac{v_0}{g}$$

Come cambia la dinamica se $\dot{f} \not\equiv 0$?

Mappa standard

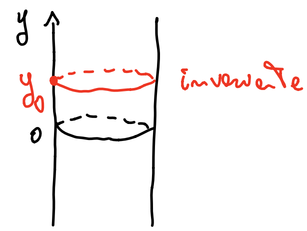
$$T_k : S^1_x \times \mathbb{R}_y \to S^1_x \times \mathbb{R}_y \,, \quad k \in \mathbb{R}$$

$$T_k(x,y) = \left(x + y - \frac{k}{2\pi} \sin(2\pi x) \,,\; y - \frac{k}{2\pi} \sin(2\pi x) \right)$$

$$JT_k(x,y) = \begin{pmatrix} 1 - k\cos(2\pi x) & 1 \\ -k\cos(2\pi x) & 1 \end{pmatrix} , \quad \det JT_k = 1 \quad \forall k \;\; \forall (x,y)$$

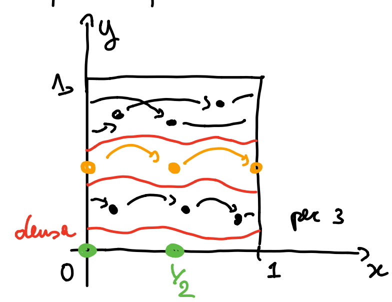
La misura di Lebesgue m è invariante, $m(T_k^{-1}(A)) = m(A)$ $\forall A$ misurabile

$\boxed{k=0}$



$$T_0 \big|_{\{y=y_0\}} \sim Rot_{y_0} : S^1 \circlearrowright$$

$\boxed{0<k \ll 1}$



$(0,0)$, $(\frac{1}{2},0)$ punti fissi $\forall k$

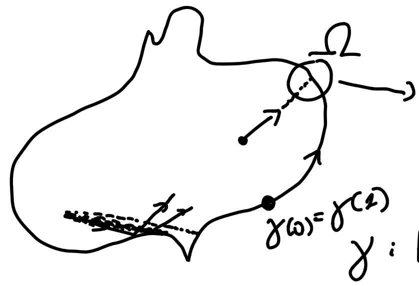
$\{(0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2})\}$ periodo 2 $\forall k$

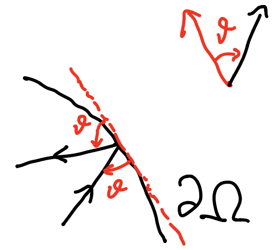
$h_{top}(T_k) > 0 \quad \forall k \neq 0$

Esistono orbite caotiche (esp. di Lyap.), non si sa se l'insieme di condizioni iniziali caotiche abbia misura di Lebesgue positiva.

Biliardi

$\Omega \subset \mathbb{R}^2$ limitato

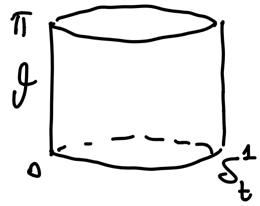




$\gamma : [0,1] \to \mathbb{R}^2$ param. di $\partial\Omega$

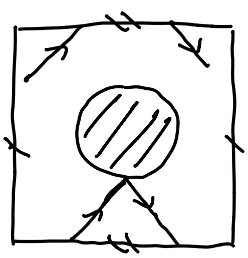
$\vartheta \in [0,\pi]$

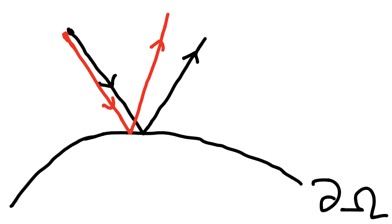
Mappa di Poincaré $\quad T : S^1_t \times (0,\pi) \to S^1_t \times (0,\pi)$



$\partial\Omega$ sia unione finita di curve di classe C^3

OSS Per ottenere dinamica caotica

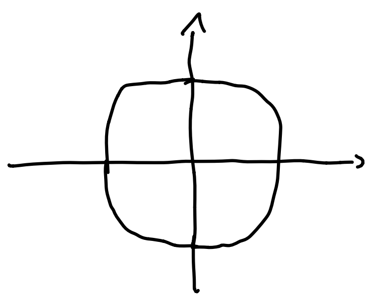




Biliardo di Sinai

- Ω sia strettamente convesso e $\partial\Omega \in C^3$

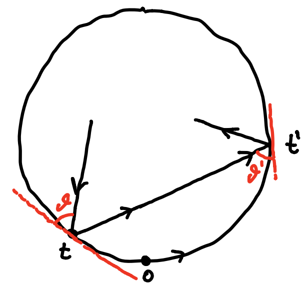
ES $\Omega = \{ x^4 + y^4 \leq 1 \}$



$T : S^1_t \times [0,\pi] \to S^1_t \times [0,\pi]$ Mappa del Biliardo di Birkhoff

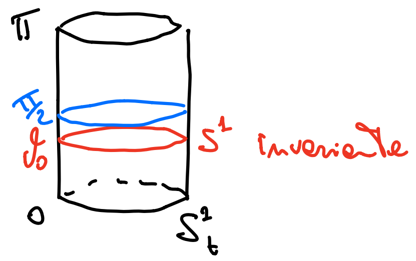
OSS $\quad T(t,0) = (t,0) \quad , \quad T(t,\pi) = (t,\pi)$

ES $\Omega = \{ x^2 + y^2 \leq R^2 \}$



$$(t', \vartheta') = T(t, \vartheta)$$

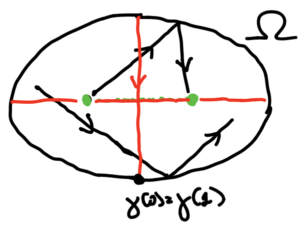
$$\begin{cases} \vartheta' = \vartheta \\ t' = t + f(\vartheta) \end{cases}$$



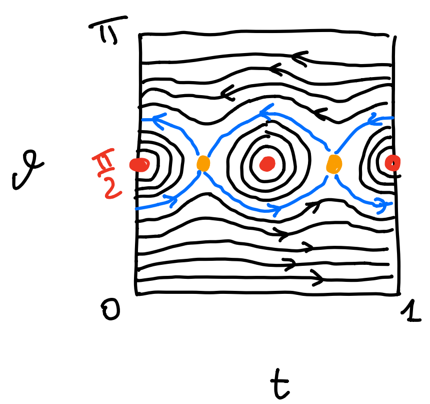
$$T\big|_{\{\vartheta = \vartheta_0\}} \sim Rot_{f(\vartheta_0)} : S^1 \circlearrowleft$$

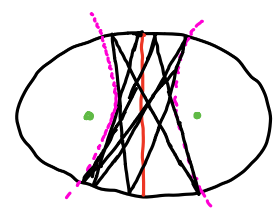
$f(\frac{\pi}{2}) = \frac{1}{2}$ $\quad (t, \frac{\pi}{2})$ periodici di periodo 2

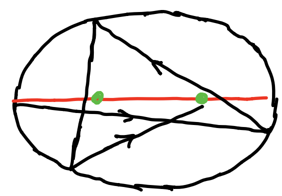
ES Ω ellisse



Orbite periodiche







Congettura di Birkhoff: se un intorno del bordo di $S^1_t \times [0, \pi]$ è interamente foliato da cerchi invarianti allora Ω è un'ellisse.

$\longleftrightarrow$

Teoria dell'espansione in frazioni continue

Sia $\alpha \in \mathbb{R}$, poniamo $a_0 = \lfloor \alpha \rfloor \in \mathbb{Z}$ e

$\alpha_0 = \alpha - a_0 \in [0, 1)$

Poniamo $a_1 = \left\lfloor \frac{1}{\alpha_0} \right\rfloor$ e $\alpha_1 = \frac{1}{\alpha_0} - a_1 \in [0, 1)$

" $a_2 = \left\lfloor \frac{1}{\alpha_1} \right\rfloor$ e $\alpha_2 = \frac{1}{\alpha_1} - a_2 \in [0, 1)$

Genera $\underbrace{a_1, \dots, a_n, \dots}_{\in \mathbb{N}}$ e $\underbrace{\alpha_1, \alpha_2, \dots, \alpha_n,}_{\in [0,1)}$ t.c.

$$\alpha_{n+1} = \frac{1}{\alpha_n} - a_{n+1}, \quad a_{n+1} = \left\lfloor \frac{1}{\alpha_n} \right\rfloor.$$

Prop L'algoritmo termina (ossia $\exists$ k t.c. $\alpha_k = 0$) se e solo se $\alpha \in \mathbb{Q}$.

$$\alpha = a_0 + \alpha_0 = a_0 + \frac{1}{a_1 + \alpha_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \alpha_2}}$$

$$\alpha \in \mathbb{Q} \iff \alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_k}}}}$$

Notazione $\alpha = [a_0; a_1, a_2, \dots]$ (se $a_0 = 0$, $\alpha = [a_1, a_2, \dots]$)

$$\alpha \in \mathbb{Q} \iff \alpha = [a_0; a_1, a_2, \dots, a_k]$$

OSS Se $\alpha \in \mathbb{Q}$, posso scrivere

$$\alpha = [a_0; a_1, a_2, \dots, a_k] = [a_0; a_1, a_2, \dots, a_k - 1, 1]$$

$$a_k > 1$$

$$\alpha = a_0 + \cfrac{1}{a_1 + \cdots + \cfrac{1}{a_k}} \qquad + \frac{1}{a_k} = + \cfrac{1}{a_k - 1 + \frac{1}{1}}$$

ES

$$1 : \alpha = \alpha : 1-\alpha \iff \alpha^2 = 1-\alpha \iff \alpha = \frac{-1+\sqrt{5}}{2} \in (0,1)$$

$$\alpha > 0$$

$$\alpha^2 = 1-\alpha \iff \alpha = \underbrace{\frac{1}{\alpha} - 1}_{\alpha_1} \iff \alpha = \alpha_k \quad \forall k \geq 1$$

$$a_1 = \left\lfloor \frac{1}{\alpha} \right\rfloor = 1 \iff \alpha \in \left(\tfrac{1}{2}, 1\right)$$

$$\alpha_1 = \frac{1}{\alpha} - a_1 = \frac{1}{\alpha} - 1$$

$$\alpha = [1, 1, \dots] = [\overline{1}]$$

Prop Sia $\alpha = [a_1, a_2, \dots] \in (0,1) \setminus \mathbb{Q}$, e definiamo

$$p_{-1} = 1, \quad p_0 = 0, \quad p_n = a_n p_{n-1} + p_{n-2}$$

$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \qquad \forall n \geq 1.$$

Allora:

(i) $\{p_n\}$ e $\{q_n\}$ sono divergenti;

(ii) $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$ e $(p_n, q_n) = 1$

(iii) $\dfrac{p_n}{q_n} = [a_1, a_2, \dots, a_n] \in \mathbb{Q}$

(iv) $\dfrac{1}{q_n(q_n + q_{n+1})} < \left| \alpha - \dfrac{p_n}{q_n} \right| < \dfrac{1}{q_n q_{n+1}}$

Proposition Let $\alpha \in [0,1] \setminus \mathbb{Q}$, $\alpha = [a_1, a_2, \dots]$ with $a_i \in \mathbb{N}$

and let
$$p_{-1} = 1, \quad p_0 = 0, \quad p_n = a_n p_{n-1} + p_{n-2}$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2} \qquad \forall n \geq 1.$$

Then: (i) $\{p_n\}$ and $\{q_n\}$ are increasing and diverging sequences;

(ii) $q_n p_{n-1} - p_n q_{n-1} = (-1)^n \quad \forall n \geq 0$ and $(p_n, q_n) = 1$

(iii) $\dfrac{p_n}{q_n} = [a_1, a_2, \dots, a_n] \in \mathbb{Q}$

(iv) $\dfrac{1}{q_n(q_n+q_{n+1})} < \left| \alpha - \dfrac{p_n}{q_n} \right| < \dfrac{1}{q_n q_{n+1}}$

<u>proof of (iv)</u> (ii) $\Leftrightarrow$
$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}}$$
$+$
$$\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^{n+1}}{q_{n+1} q_n}$$

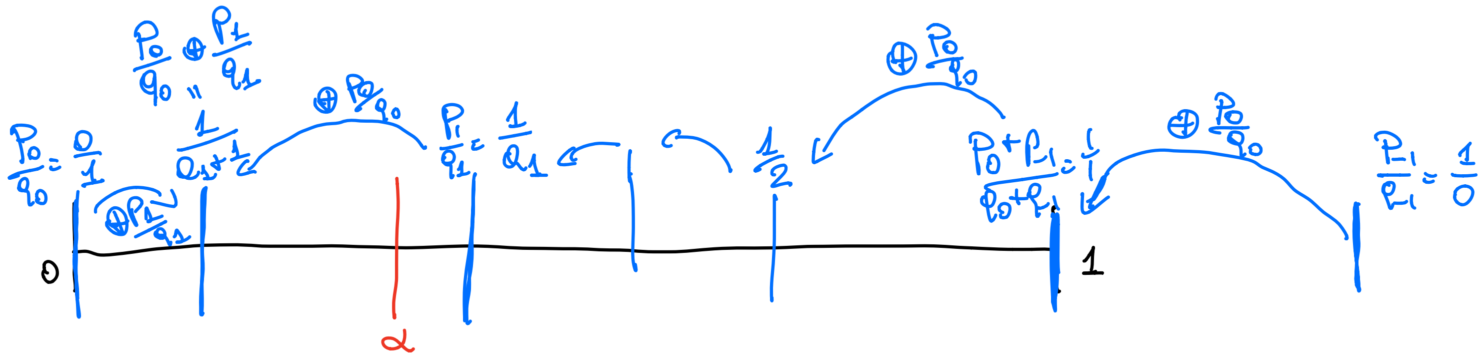
$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n+1}}{q_{n+1}} = \frac{(-1)^n}{q_n} \underbrace{\left(\frac{q_{n+1} - q_{n-1}}{q_{n-1} q_{n+1}} \right)}_{>0}$$

$\left\{ \dfrac{p_{2n}}{q_{2n}} \right\}$ is an increasing sequence, hence converging

$\left\{ \dfrac{p_{2n+1}}{q_{2n+1}} \right\}$ is a decreasing sequence, hence converging

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}} \xrightarrow[n \to +\infty]{} 0$$

Hence $\exists \lim_{n \to +\infty} \frac{p_n}{q_n}$.



$p_1 = Q_1 p_0 + p_{-1} = 1$

$q_1 = Q_1 q_0 + q_{-1} = Q_1$

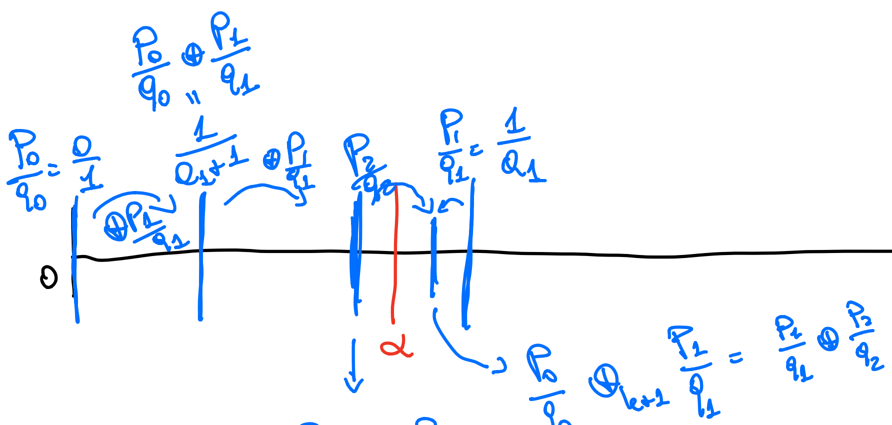
$$\frac{p_1}{q_1} = \frac{Q_1 p_0 + p_{-1}}{Q_1 q_0 + q_{-1}} = \frac{p_{-1}}{q_{-1}} \oplus_{Q_1} \frac{p_0}{q_0}$$

FAREY SUM

<u>Notation</u> $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$, $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$

$0 < \frac{a}{b} < \frac{c}{d}$, $\frac{a}{b} \oplus \frac{c}{d} \in \left(\frac{a}{b}, \frac{c}{d}\right)$

$$\alpha = \cfrac{1}{Q_1 + \cfrac{1}{Q_2 + \cfrac{1}{Q_3 + \dots}}} < \frac{1}{Q_1}$$

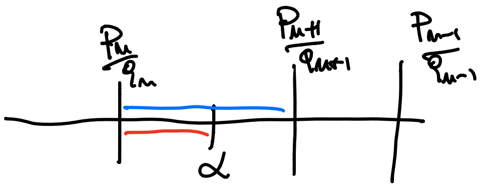


$k = Q_2$

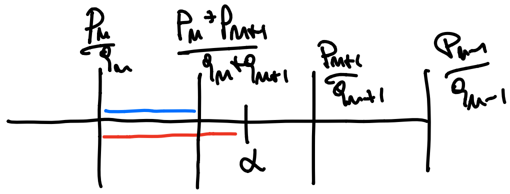
$$\underbrace{\frac{p_0}{q_0} \oplus_k \frac{p_1}{q_1}}_{\frac{Q_2 p_1 + p_0}{Q_2 q_1 + q_0} = \frac{p_2}{q_2}}$$

$$\frac{p_{2n}}{q_{2n}} < \alpha < \frac{p_{2n+1}}{q_{2n+1}} \quad \forall n \in \mathbb{N}_0$$

Def We call <u>intermediate approximants of α</u> the fractions $\left\{ \dfrac{h\, p_{n-1} + p_{n-2}}{h\, q_{n-1} + q_{n-2}} \right\}$ with $1 \leq h \leq a_n$, $n \in \mathbb{N}$.



$$\left| \alpha - \frac{p_n}{q_n} \right| < \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n+1}}$$



$$\left| \alpha - \frac{p_n}{q_n} \right| > \left| \frac{p_n + p_{n+1}}{q_n + q_{n+1}} - \frac{p_n}{q_n} \right| = \frac{1}{q_n (q_n + q_{n+1})}$$

Ex $\alpha = \dfrac{\sqrt{5}-1}{2} = [1, 1, \dots] = [\overline{1}]$

p_n, q_n are Fibonacci numbers

$p_{-1} = 1,\ p_0 = 0,\ p_1 = 1,\ p_2 = 1,\ p_3 = 2,\ p_4 = 3, \dots$

$q_{-1} = 0,\ q_0 = 1,\ q_1 = 1,\ q_2 = 2,\ q_3 = 3,\ q_4 = 5, \dots$

Def Sia $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, we say that $\frac{p}{q} \in \mathbb{Q}$, $q > 0$, is a

- <u>best approximation of α of the II kind</u> if
  $\forall \frac{a}{b} \in \mathbb{Q}$ with $0 < b \leq q$ we have $|q\alpha - p| < |b\alpha - a|$
- <u>best approximation of α of the I kind</u> if
  $\forall \frac{a}{b} \in \mathbb{Q}$ with $0 < b \leq q$ we have $\left|\alpha - \frac{p}{q}\right| < \left|\alpha - \frac{a}{b}\right|$

Rem II kind $\Rightarrow$ I kind
$\not\Leftarrow$

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q} |q\alpha - p| < \frac{1}{q} |b\alpha - a| \le \frac{1}{b} |b\alpha - a| = \left|\alpha - \frac{a}{b}\right|$$
$$\forall \frac{a}{b} \text{ with } 0 < b \le q$$

Prop Let $\alpha \in [0,1] \setminus \mathbb{Q}$ with $\alpha = [a_1, a_2, \ldots]$ then:

(i) $\frac{p}{q}$ is a best approx of α of the II kind if and only if $\frac{p}{q}$ is an approximant of α ($\exists\, n$ s.t. $\frac{p}{q} = [a_1, a_2, \ldots, a_n]$)

(ii) $\frac{p}{q}$ is a best approx of α of the I kind if and only if $\frac{p}{q}$ is an intermediate approximant of α

$$\left(\exists\, n, h \text{ with } 1 \le h \le a_n \text{ s.t. } \frac{p}{q} = \frac{h\, p_{n-1} + p_{n-2}}{h\, q_{n-1} + q_{n-2}}\right)$$

Def Given a function $\psi : \mathbb{N} \to \mathbb{R}$ decreasing (not necessarily strictly decreasing) we define

$$W(\psi) := \{ \alpha \in \mathbb{R} \,/\, \exists \text{ infinitely many } p, q \text{ with } (p,q) = 1 \text{ s.t. } |q\alpha - p| < \psi(q) \}$$
$p \in \mathbb{Z}$, $q \in \mathbb{N}$

Prop $W\left(\frac{1}{q}\right) = \mathbb{R} \setminus \mathbb{Q}$ $\quad \left(|q_n \alpha - p_n| < \frac{1}{q_{n+1}} < \frac{1}{q_n} \right)$

$W\left(\frac{c}{q}\right) = \mathbb{R} \setminus \mathbb{Q}$ if $c \ge \frac{1}{\sqrt{5}}$.

EX $\alpha = \frac{\sqrt{5} - 1}{2}$ $\qquad q_n = a_n q_{n-1} + q_{n-2} = q_{n-1} + q_{n-2}$ $\forall n \ge 1$

For all α, $\quad q_n \ge q_{n-1} + q_{n-2}$, $a_n \ge 1$.

Def The <u>Diophantine exponent of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$</u> is

$$\omega(\alpha) := \sup \left\{ \gamma \geq 1 \;/\; \alpha \in W\left(\tfrac{1}{q^\gamma}\right) \right\}$$

(The measure of irrationality of a number $\mu(\alpha) = 1 + \omega(\alpha)$)

Ex $\omega(\pi) = ?$ $\quad \omega(\pi) < 6.103\ldots$

$\omega(e) = 1 \quad e = [2; 1, 1, 2, 1, 1, 4, 1, 1, 6, \ldots, \ldots, 1, 1, 2n, \ldots]$

Prop (i) $\omega(\alpha) \geq 1 \quad \forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$

(ii) $m\{\alpha : \omega(\alpha) > 1\} = 0$

(iii) α is an algebraic number then $\omega(\alpha) = 1$;

$\omega(\alpha) = +\infty$ then α is a transcendental number.

(iv)

$$\omega(\alpha) = 1 + \limsup_{n \to +\infty} \frac{\log q_{n+1}}{\log q_n}$$

Def Let $c > 0$, $\nu > 0$ we let

$$D(c, \nu) = \left\{ \alpha \in \mathbb{R} \setminus \mathbb{Q} \;/\; \forall\, p, q \text{ with } p \in \mathbb{Z},\ q \in \mathbb{N} \quad \left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{\nu+2}} \right\}$$

<u>Diophantine numbers of type (c, ν)</u>

Ex $\alpha \notin W\left(\frac{1}{q^\sigma}\right)$, $\sigma > 1$. $\iff$ $|q\alpha - p| < \frac{1}{q^\sigma}$ has only finitely many solutions p, q

$\iff \exists\, c > 0$ s.t. $|q\alpha - p| > \frac{c}{q^\sigma}$ for all p, q

$\iff \exists\, c > 0$ s.t. $\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{\sigma+1}}$ for all p, q $\iff \alpha \in D(c, \sigma - 1)$ for some c

Def Diophantine numbers $D := \bigcup_{c,\nu} D(c,\nu) = \{ \alpha \mid \omega(\alpha) < +\infty \}$

Liouville numbers $L := (\mathbb{R} \setminus \mathbb{Q}) \setminus D = \{ \omega(\alpha) = +\infty \}$

$\longleftrightarrow$

$T : [0,1] \circlearrowleft \qquad T(x) = \begin{cases} 0 & , \ x = 0 \\ \frac{1}{x} - \lfloor \frac{1}{x} \rfloor & , \ x \neq 0 \end{cases}$ Gauss map

$A_n = \left(\frac{1}{n+1}, \frac{1}{n} \right) , \ n \geq 1$

$x \in \mathbb{R} \setminus \mathbb{Q}, \qquad T^{k-1}(x) \in A_n \iff a_k = n$

$x = [a_1, a_2, \dots] \qquad \forall k \geq 1$

x is pre-periodic for T ($\exists \bar{n}$ s.t. $T^{\bar{n}}(x)$ is periodic)

if and only if $x = [a_1, \dots, a_{\bar{n}}, \overline{b_1, \dots, b_k}]$

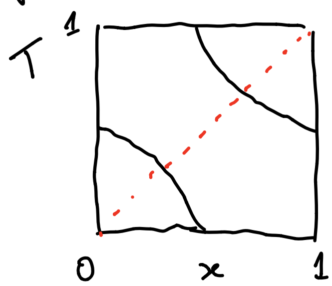
if and only if x is a quadratic irrational

(Lagrange Thm)

CIRCLE HOMEOMORPHISMS

$T : S^1 \to S^1$ orientation preserving homeos

Rem If T were orientation reversing



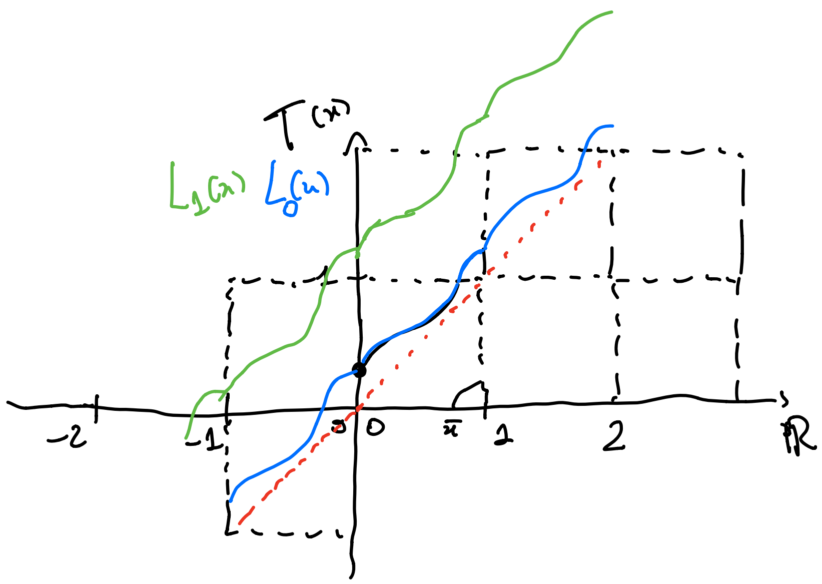
$\Rightarrow$ T^2 is orientation preserving

$S^1 = [0,1]/_{0\sim 1}$, $\pi : \mathbb{R} \to S^1$ i.e. $\pi(x) = \{x\}$

A *lift* $L : \mathbb{R} \to \mathbb{R}$ of T is a homeo s.t.

$$\pi \circ L = T \circ \pi$$

$$\begin{tikzcd} S^1 \arrow[r, "T"] & S^1 \\ \mathbb{R} \arrow[u, "\pi"] \arrow[r, "L"'] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{R} \arrow[u, "\pi"'] \end{tikzcd}$$



$$L(x) = \begin{cases} T(x) , & 0 \le x \le \bar{x} \\ T(x)+1 , & \bar{x} < x < 1 \\ L(\{x\}) + \lfloor x \rfloor , & \forall x \in \mathbb{R} \setminus [0,1] \end{cases}$$

Let $c = \lfloor L(0) \rfloor$ then all lifts are obtained by

$$L_c(x) = L_0(x) + c$$

Prop (i) For all c, $L_c^n(x+k) = L_c^n(x) + k \quad \forall n, k \in \mathbb{Z} \ \forall x \in \mathbb{R}$

(ii) $L_{c_1}^n(x) - L_{c_2}^n(x) = n(c_1 - c_2) \quad \forall n \in \mathbb{Z}, \ \forall x \in \mathbb{R}$

(iii) If $x, y \in \mathbb{R}$ with $|x-y|<1$ then $|L_c^n(x) - L_c^n(y)| < 1$ $\forall n \in \mathbb{Z}$.

proof (i) $\underline{n=1}$ $L_c(x+k) = L_c(x) + k$

$\underline{n=2}$

$$L_c^2(x+k) = L_c(L_c(x+k)) = L_c(L_c(x)+k) \overset{(i)}{=} L_c(L_c(x)) + k$$
$$= L_c^2(x) + k$$

By induction on n.

(ii) $\underline{n=1}$ $L_{c_1}(x) - L_{c_2}(x) = c_1 - c_2$ $(*)$

$\underline{n=2}$ $L_{c_1}^2(x) \overset{(*)}{=} L_{c_1}(L_{c_2}(x) + c_1 - c_2) \overset{(i)}{=} L_{c_1}(L_{c_2}(x)) + c_1 - c_2 \overset{(*) \text{ for } y = L_{c_2}(x)}{=}$

$$= L_{c_2}(L_{c_2}(x)) + c_1 - c_2 + c_1 - c_2 = L_{c_2}^2(x) + 2(c_1 - c_2)$$

By induction on n

(iii) Take $x, y \in \mathbb{R}$ with $|x-y|<1$ and I can assume that $x, y \in [0,1]$. Then $L(0) \leq L(x) < L(y) \leq L(1)$ and $L(1) = L(0) + 1$. $\Rightarrow |L(y) - L(x)| < 1$.

$$|L^n(y) - L^n(x)| = |L(L^{n-1}(y)) - L(L^{n-1}(x))| \underset{\Uparrow}{<} 1$$
$$|L^{n-1}(y) - L^{n-1}(x)| < 1.$$

$\square$

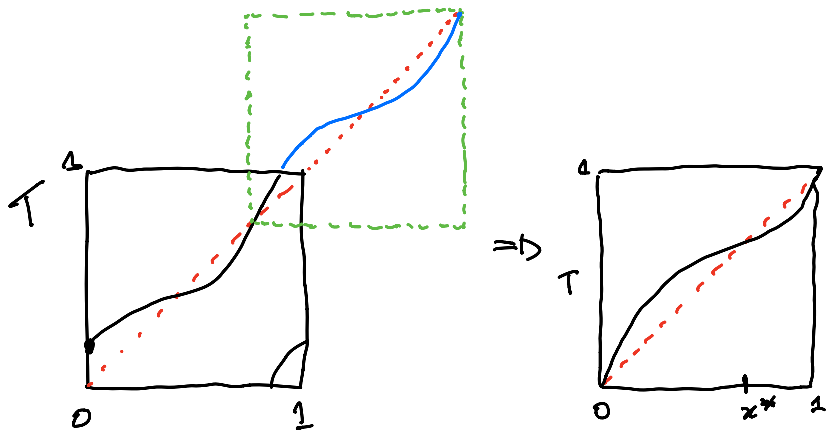
Def (Rotation number)

Let $T: S^1 \to S^1$ be an OPCH and $L: \mathbb{R} \to \mathbb{R}$ be a lift of T.

Then we call ROTATION NUMBER of T the fractional part $\tau(T)$ of

$$\tau(L) := \lim_{n\to+\infty} \frac{L^n(x) - x}{n} \quad \text{for } x \in \mathbb{R}.$$

Example



$$\tau(T) = 0$$

Rem $\tau(L_{c_1}) - \tau(L_{c_2}) = c_1 - c_2$

$$\Uparrow$$

$$L^n_{c_1}(x) - L^n_{c_2}(x) = n(c_1 - c_2)$$

Prop $\tau(L)$ is well defined (the limit exists and does not depend on x)

Lemma Let $\{a_n\}$ be a sequence in $\mathbb{R}$ for which there exist a $c \in \mathbb{R}^+$ s.t. $a_{n+m} \le a_n + a_m + c \quad \forall n, m \in \mathbb{N}$.

Then there exists the $\lim_{n\to+\infty} \frac{a_n}{n}$.

proof of lemma Let $l := \liminf_{n\to+\infty} \frac{a_n}{n}$ then for all fixed $k \in \mathbb{N}$

$$l \le \limsup_{n\to+\infty} \frac{a_n}{n} \underset{\uparrow}{=} \limsup_{m\to+\infty} \frac{a_{mk+r}}{mk+r} \underset{\uparrow}{\le} \limsup_{m\to+\infty} \frac{m a_k + a_r + (m+1)c}{mk+r}$$

$n = mk + r$ $\qquad a_{mk+r} \le a_{mk} + a_r + c \le$

$0 \le r \le k-1$ $\qquad \le a_{(m-1)k} + a_k + c + a_r + c \le \dots \le$

$\le m a_k + a_r + (m+1)c$

$= \frac{a_k}{k} + 0 + \frac{c}{k}$

Let $\{k_j\}$ a diverging seq. s.t. $\frac{a_{k_j}}{k_j} \xrightarrow[j\to+\infty]{} \ell$ and for a fixed $\varepsilon > 0$

let $k = k_{\bar{j}}$ s.t. $\left|\frac{a_k}{k} - \ell\right| < \varepsilon$ and $\frac{c}{k} < \varepsilon$.

For all $\varepsilon > 0$ it holds $\limsup \frac{a_n}{n} \le \frac{a_k}{k} + \frac{c}{k} < \ell + 2\varepsilon \Rightarrow \ell = \lim_{n\to+\infty} \frac{a_n}{n}$

$\square$

<u>proof of Prop</u> $\quad \lim_{n\to+\infty} \frac{L^n(x) - x}{n} = \tau(L)$

$a_n = L^n(x) - x$, $\qquad 0 \le L^n(x) - x - \lfloor a_n \rfloor < 1$

$a_{n+m} = L^{n+m}(x) - x = L^m(L^n(x)) - L^n(x) + \overbrace{L^n(x) - x}^{a_n} =$

$= \left[L^m(L^n(x)) - L^m(x + \lfloor a_n \rfloor)\right] + \left[L^m(x + \lfloor a_n \rfloor) - (x + \lfloor a_n \rfloor)\right] +$

$+ \left[(x + \lfloor a_n \rfloor) - L^n(x)\right] + \left[\underbrace{L^n(x) - x}_{a_n}\right]$

$L^m(x + \lfloor a_n \rfloor) = L^m(x) + \lfloor a_n \rfloor$

$L^m(L^n(x)) - L^m(x + \lfloor a_n \rfloor) < 1$ bec. $L^n(x) - (x + \lfloor a_n \rfloor) < 1$

$x + \lfloor a_n \rfloor - L^n(x) \le 0$

$a_{n+m} \le 1 + \left(L^m(x) + \lfloor a_n \rfloor - x - \lfloor a_n \rfloor\right) + 0 + \left(L^n(x) - x\right)$

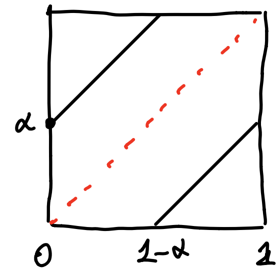
$= 1 + a_m + a_n \quad \forall\, n, m.$

By the lemma, $\lim_{n\to+\infty} \frac{L^n(x) - x}{n}$ exists for all x.

Moreover $\left|\frac{L^n(x) - x}{n} - \frac{L^n(y) - y}{n}\right| \le \frac{|L^n(x) - L^n(y)| + |x - y|}{n} \le \frac{2}{n}$ for all $x, y \in \mathbb{R}$ s.t. $|x-y|<1$. Then the limit does not depend on $x \in \mathbb{R}$ and the convergence $\frac{L^n(x) - x}{n} \xrightarrow[n\to+\infty]{} \tau(L)$ is uniform on $[0,1]$. $\square$

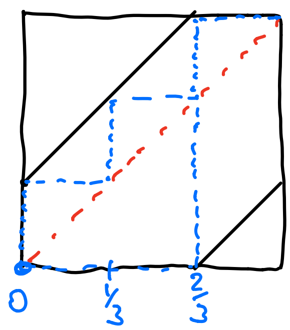
<u>Example</u> $\quad R_\alpha(x) = \{x+\alpha\}$

$\tau(R_\alpha) = \alpha$



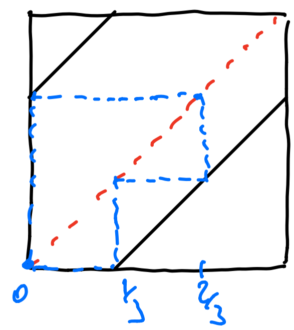
If $\alpha = \frac{p}{q} \in \mathbb{Q}$, all orbits are periodic of period q.

$\alpha = \frac{1}{3}$



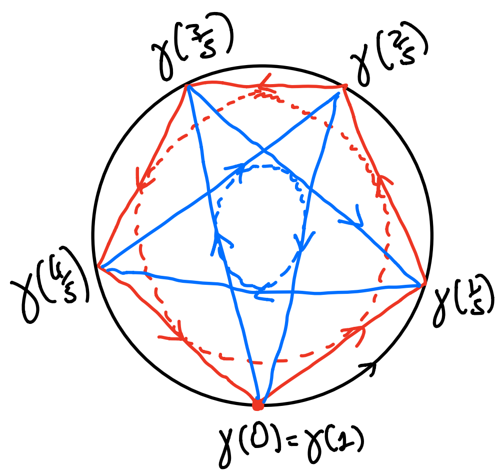
$\mathcal{O}(0) = \{0, \frac{1}{3}, \frac{2}{3}\}$

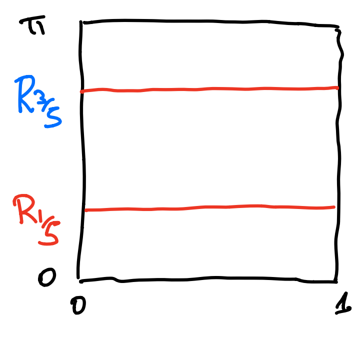
$\alpha = \frac{2}{3}$



$\mathcal{O}(0) = \{0, \frac{2}{3}, \frac{1}{3}\}$

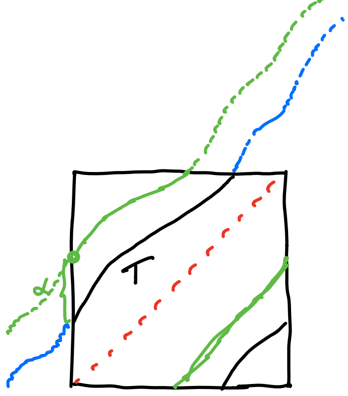
<u>Example</u>





<u>Example</u> $\quad T : S^1 \to S^1 \qquad\qquad R_\alpha \circ T : S^1 \to S^1$

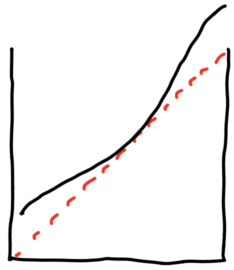
$\qquad x \mapsto T(x) \qquad\qquad\qquad x \mapsto T(x) + \alpha$



$\tau(R_\alpha \circ T) \neq \alpha + \tau(T)$

$\tau(T) = \frac{p}{q}$ and $T \neq R_{\frac{p}{q}}$ then $\exists$ an interval (a,b) s.t. $\forall \alpha \in (a,b)$

$\tau(R_\alpha \circ T) = 0$



Def Let (X_1, T_1) and (X_2, T_2) be continuous dynamical systems ($T_1 : X_1 \to X_1$, $T_2 : X_2 \to X_2$ continuous maps). We say that (X_2, T_2) is a <u>topological factor</u> of (X_1, T_1) if $\exists\ h : X_1 \to X_2$ surjective and continuous s.t.

$$T_2 \circ h = h \circ T_1$$

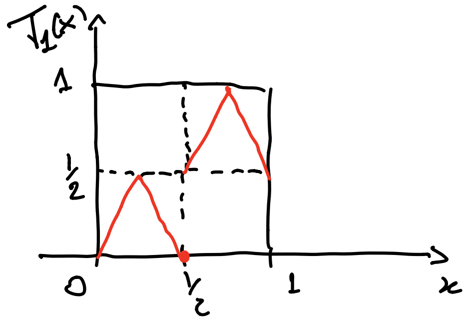
$$\begin{tikzcd} X_1 \arrow[r, "T_1"] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowright"] & X_1 \arrow[d, "h"] \\ X_2 \arrow[r, "T_2"'] & X_2 \end{tikzcd}$$

EX $X_1 = [0,1]$, $T_1 : [0,1] \to [0,1]$

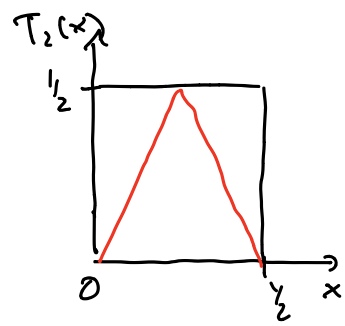
$X_2 = \{p\}$, $T_2(p) = p$

$h : [0,1] \to \{p\}$

EX $X_1 = [0,1]$ $T_1 : [0,1] \to [0,1]$



$X_2 = [0, \tfrac{1}{2}]$ $T_2 : [0, \tfrac{1}{2}] \to [0, \tfrac{1}{2}]$



$h : [0,1] \to [0, \tfrac{1}{2}]$

$$h(x) = \begin{cases} 0 , & 0 \le x \le \frac{1}{2} \\ x - \frac{1}{2} , & \frac{1}{2} \le x \le 1 \end{cases}$$

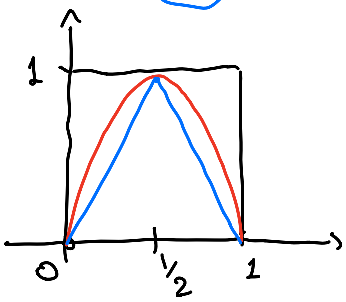
Def If the map $h : X_1 \to X_2$ is a homeomorphism then we say that (X_1, T_1) and (X_2, T_2) are <u>topologically</u>

conjugate and h is called a topological conjugacy.

Ex (X_1, T_1) $T_1 : [0,1] \to [0,1]$ $T_1(x) = 4x(1-x)$

(X_2, T_2) $T_2 : [0,1] \to [0,1]$ $T_2(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} \le x \le 1 \end{cases}$

tent map



$h^{-1}(x) = \sin^2\left(\frac{\pi}{2} x\right)$

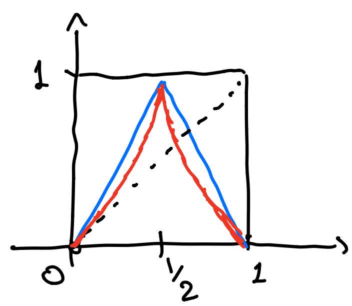
$T_1 \circ h^{-1}(x) = h^{-1} \circ T_2(x)$

Rem If $T_1, T_2 \in C^k$, $k \ge 1$, are topologically conjugate, can we have a conjugacy $h \in C^k$? NO

Ex $T_2 : [0,1] \to [0,1]$ is the tent map as before.

$T_1(x) = \begin{cases} \frac{x}{1-x}, & 0 \le x \le \frac{1}{2} \\ \frac{1-x}{x}, & \frac{1}{2} \le x \le 1 \end{cases}$

FAREY MAP



are topologically conjugate.

$T : S^1 \to S^1$ orientation preserving circle homeomorphism (OPCH)

$L : \mathbb{R} \to \mathbb{R}$ principal lift $(L(0) = T(0))$

$\tau(T) := \pi\left(\lim_{n \to +\infty} \frac{L^n(x) - x}{n}\right)$ rotation number of T

Prop If T_1 and T_2 are top conjugate OPCHs then $\tau(T_1) = \tau(T_2)$.

proof Let L_1 and L_2 be the lifts of T_1 and T_2, let $h : S^1 \to S^1$ be a homeo s.t. $h \circ T_1 = T_2 \circ h$, then H is a lift of h then $H \circ L_1 = L_2 \circ H$.

We prove that $L_1 = H^{-1} \circ L_2 \circ H$ is a lift of T_1,

$$\pi \circ L_1 = \underbrace{\pi \circ H^{-1}} \circ L_2 \circ H = h^{-1} \circ \underbrace{\pi \circ L_2} \circ H = h^{-1} \circ T_2 \circ \underbrace{\pi \circ H} =$$

$$= (h^{-1} \circ T_2 \circ h) \circ \pi = T_1 \circ \pi .$$

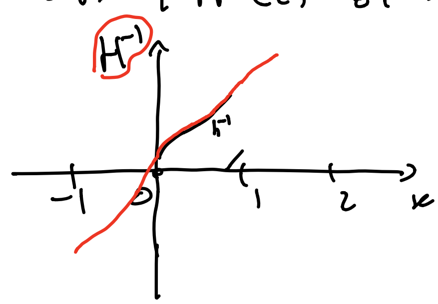
Then $$\tau(T_1) = \pi\left(\lim_{n\to+\infty} \frac{L_1^n(x) - x}{n} \right) = \pi\left(\lim_{n\to+\infty} \frac{(H^{-1}\circ L_2 \circ H)^n(x) - x}{n} \right)$$

$$= \pi\left(\lim_{n\to+\infty} \frac{(H^{-1}\circ L_2^n)(H(x)) - x}{n} \right) =$$

$(H^{-1}\circ L_2 \circ H)^n = H^{-1}\circ L_2^n \circ H$

$$= \pi\left(\lim_{n\to+\infty} \frac{(H^{-1}\circ L_2^n)(H(x)) - L_2^n(H(x))}{n} + \lim_{n\to+\infty} \frac{L_2^n(H(x)) - H(x)}{n} + \lim_{n\to+\infty} \frac{H(x) - x}{n} \right)$$

Use that $\exists\, c > 0$ s.t. $|H^{-1}(z) - z| < c \quad \forall z \in \mathbb{R}$



$$\Rightarrow \left| H^{-1} \circ L_2^n(H(x)) - L_2^n(H(x)) \right| < c \quad \forall x, \forall n.$$

$$= \pi\left(\lim_{n\to+\infty} \frac{L_2^n(H(x)) - H(x)}{n} \right) = \tau(T_2) \quad \forall x. \qquad \square$$

Rem $R_\alpha : x \mapsto \{x + \alpha\}$ is OPCH with $\tau(R_\alpha) = \{\alpha\}$.

Def Given T_1, T_2 two OPCH with principal lifts L_1 and L_2, we say that $T_1 \prec T_2$ if $L_1(x) < L_2(x) \ \forall x \in \mathbb{R}$.

Prop (i) τ is continuous as a function from $(\mathrm{Homeo}_{S^1}, \|\cdot\|_\infty)$ to $\mathbb{R}$

(ii) if $T_1 \prec T_2$ then $\tau(T_1) \leq \tau(T_2)$

(iii) if $\tau(T_1) \in \mathbb{R} \setminus \mathbb{Q}$ then for all $T_2 \succ T_1$, $\tau(T_2) > \tau(T_1)$

(iv) if $\tau(T_1) \in \mathbb{Q}$ then $\exists\, T_2$ either $T_2 \succ T_1$ or $T_2 \prec T_1$ with $\tau(T_2) = \tau(T_1)$

proof (i) $\forall \varepsilon > 0 \ \exists \tilde{\delta} > 0$ s.t. $\|L_1 - L_2\|_\infty < \tilde{\delta}$ then
$$|\tau(L_1) - \tau(L_2)| < \varepsilon$$

Fix $\varepsilon > 0$ and L_1. Let $\frac{p}{q} \in \mathbb{Q}$ s.t. $\frac{p}{q} \in (\tau(L_1), \tau(L_1) + \varepsilon)$ then:

- $\exists\, \bar{x} \in \mathbb{R}$ s.t. $L_1^q(\bar{x}) \leq \bar{x} + p$

  Assume that $L_1^q(x) > x + p \ \forall x \in \mathbb{R}$, then

$$\tau(L_1) > \frac{p}{q} \Leftarrow \begin{cases} \dfrac{L_1^{qk}(x) - x}{k} = \dfrac{\sum_{j=1}^{k} \left[L_1^q\left(L_1^{q(j-1)}(x)\right) - L_1^{q(j-1)}(x) \right]}{k} > \\ \qquad > \dfrac{\sum_{j=1}^{k} p}{k} = p \\ q \, \dfrac{L_1^{qk}(x) - x}{qk} \xrightarrow[k \to +\infty]{} q \cdot \tau(L_1) \end{cases}$$

  We got a contradiction

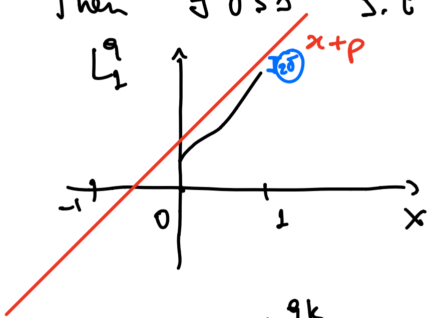
- $\forall x \in \mathbb{R}$, $L_1^q(x) < x + p$

Since $\exists\, \bar{x}$ s.t. $L_1^q(\bar{x}) \leq \bar{x}+p$ then either the claim is True or $\exists\, x_0$ s.t. $L_1^q(x_0) = x_0+p$.

If $\exists\, x_0$ s.t. $L_1^q(x_0) = x_0+p$ then $L_1^{2q}(x_0) = L_1^q(L_1^q(x_0)) =$
$= L_1^q(x_0+p) = L_1^q(x_0)+p = x_0+2p$, and in general $L_1^{kq}(x_0) = x_0+kp$. Then

$$\tau(L_1) = \lim_{k\to+\infty} \frac{L_1^{kq}(x_0)-x_0}{kq} = \frac{p}{q} \quad \text{FALSE.}$$

Then $\exists\, \delta>0$ s.t. $L_1^q(x) < x+p-\delta \;\; \forall x\in\mathbb{R}$. Then if L_2 s.t. $\|L_2^q - L_1^q\|_\infty < \delta$ we have that $L_2^q(x) \leq L_1^q(x)+\delta < x+p$



$$\frac{L_2^{qk}(x)-x}{qk} = \sum_{j=1}^{k} \frac{L_2^q(L_2^{q(j-1)}(x)) - L_2^{q(j-1)}(x)}{kq} < \frac{p}{q}$$

$$\Rightarrow \tau(L_2) \leq \frac{p}{q} < \tau(L_1)+\varepsilon$$

Now we choose $\frac{p'}{q'} \in (\tau(L_1)-\varepsilon, \tau(L_1))$ and show as before that $\exists\, \delta>0$ s.t. if L_2 satisfies $\|L_1^{q'} - L_2^{q'}\|_\infty < \delta$ then $\tau(L_2) \geq \frac{p'}{q'} > \tau(L_1)-\varepsilon$.

In conclusion, fixed $\varepsilon>0$ $\exists\, \tilde{\delta}>0$ s.t. $\|L_1 - L_2\|_\infty < \tilde{\delta}$ then $\|L_1^q - L_2^q\|_\infty < \delta$ and $\|L_1^{q'} - L_2^{q'}\|_\infty < \delta$, hence

$\tau(L_2) \in (\tau(L_1)-\varepsilon, \tau(L_1)+\varepsilon)$. $\square$ (i).

(ii) If $T_1 \prec T_2$ then $\tau(T_1) \leq \tau(T_2)$

$T_1 \prec T_2 \Leftrightarrow L_1 < L_2$ then $\displaystyle\lim_{n\to+\infty} \frac{L_1^n(x)-x}{n} \leq \lim_{n\to+\infty} \frac{L_2^n(x)-x}{n}$

$\square$ (ii)

(iii) T_1 s.t. $\tau(T_1) \in \mathbb{R} \setminus \mathbb{Q}$ then $\forall\, T_2 > T_1$, $\tau(T_2) > \tau(T_1)$.

Since $\tau(T_1) \in \mathbb{R} \setminus \mathbb{Q}$ $\forall \delta > 0$ $\exists \frac{p}{q}$ s.t.

$$\frac{p-\delta}{q} < \tau(T_1) < \frac{p}{q}$$

$$\left(\forall \delta > 0 \ \exists\, p, q \text{ with } (p,q) = 1 \text{ s.t. } \frac{1}{q} < \delta \text{ and } \left|\tau(T_1) - \frac{p}{q}\right| < \frac{1}{q^2} \right)$$

$\Rightarrow$ $\left|\tau(T_1) - \frac{p}{q}\right| < \frac{1}{q} \cdot \frac{1}{q} < \frac{\delta}{q}$ and choose $\frac{p}{q}$ an approximant of $\tau(T_1)$ with odd index.

Then $\exists\, \bar{x} \in \mathbb{R}$ s.t. $L_1^q(\bar{x}) \geq \bar{x} + p - \delta$.

Otherwise $q\,\tau(L_1) \xleftarrow{k \to +\infty} \frac{L_1^{qk}(x) - x}{k} < \frac{k(p-\delta)}{k} = p - \delta \Rightarrow \tau(L_1) < \frac{p-\delta}{q}$.

Let $L_2 > L_1$ and let $\delta_0 := \min(L_2(x) - L_1(x)) > 0$ and find $\frac{p}{q}$ s.t. $\frac{p-\delta_0}{q} < \tau(L_1) < \frac{p}{q}$. Then $\exists\, \bar{x} \in \mathbb{R}$ s.t. $L_1^q(\bar{x}) \geq \bar{x} + p - \delta_0$ and then

$$L_2^q(\bar{x}) = L_2(L_2^{q-1}(\bar{x})) \geq L_1(L_2^{q-1}(\bar{x})) + \delta_0 > L_1(L_1^{q-1}(\bar{x})) + \delta_0 \geq \bar{x} + p$$

then $\tau(L_2) = \lim_{k \to +\infty} \frac{L_2^{qk}(x) - x}{kq}$ and either $L_2^q(x) - x \geq p$ for all $x \in \mathbb{R}$ or $\exists\, x_0$ s.t. $L_2^q(x_0) = x_0 + p$. And in the first case

$$\frac{L_2^{qk}(x) - x}{kq} = \sum_j \frac{L_2^q(L_2^{q(j-1)}(x)) - L_2^{q(j-1)}(x)}{kq} \geq \frac{p}{q}$$

and in the second case $L_2^{qk}(x_0) = x_0 + kp \Rightarrow \frac{L_2^{qk}(x_0) - x_0}{kq} = \frac{p}{q}$

In both cases $\tau(L_2) \geq \frac{p}{q} > \tau(L_1)$. $\square$(iii)

Prop $T: S^1 \to S^1$ OPCH and $\tau(T) = \frac{p}{q} \in \mathbb{Q}$

(i) $\tau(T) = \frac{p}{q} \in \mathbb{Q} \iff \exists\, x \in \mathbb{R}$ which is periodic of period q

(ii) If $\tau(T) = \frac{p}{q}$ then all periodic orbits have the same period

$(p,q)=1$

proof (i) $\boxed{\Leftarrow}$ If $T^q(x) = x$ then $\exists\, p \in \mathbb{N}$ s.t.

$$L^q(x) = x + p$$

Then $$\tau(L) = \lim_{n\to+\infty} \frac{L^{qn}(x) - x}{qn} = \lim_{n\to+\infty} \frac{np}{nq} = \frac{p}{q}$$

since $$L^{qn}(x) = L^{q(n-1)}(L^q(x)) = L^{q(n-1)}(x+p) = L^{q(n-1)}(x) + p$$
$$= \dots = x + np$$

$\boxed{\Rightarrow}$ Claim: $\forall\, k \in \mathbb{Z}$ $\tau(T^k) = k\,\tau(T)$

proof: $$\tau(L^k) = \left(\lim_{n\to+\infty} \frac{L^{kn}(x) - x}{kn}\right) k = k\,\tau(L)$$

Then if $\tau(T) = \frac{p}{q}$, $\tau(T^q) = 0$. Let $L: \mathbb{R} \to \mathbb{R}$ and $\tau(L^q) = 0$, I need to prove that $\exists\, x$ s.t. $L^q(x) - x \in \mathbb{Z}$. Let's assume that L is the lift for which $L^q(0) \in [0,1)$, then if $\forall x$, $L^q(x) - x \notin \mathbb{Z}$ it means that $\forall\, x \in \mathbb{R}$, $0 < L^q(x) - x < 1$. Then $\exists\, \delta > 0$ s.t. $\delta \leq L^q(x) - x \leq 1 - \delta$, $\forall\, x \in \mathbb{R}$.

Hence $\tau(L^q) \in [\delta, 1-\delta]$, by writing

$$\delta \leq \frac{L^{qn}(x) - x}{n} = \frac{\sum_{j=1}^{n} L^q(L^{q(j-1)}(x)) - L^{q(j-1)}(x)}{n} \leq 1 - \delta$$

This is a contradiction.

(ii) $\tau(T) = \frac{p}{q} \in \mathbb{Q}$, $(p,q)=1$, then from (i) if x is a periodic point, $\exists\, r,s$ s.t. $L^s(x) = x+r$, we have

$\tau(T) = \frac{r}{s} \Rightarrow \frac{r}{s} = \frac{mp}{mq}$ for some $m \geq 1$.

Let's assume $m \geq 2$, and that s is the minimal period of x.

Then $L^q(x) > x+p$ or $L^q(x) < x+p$. In the first case $L^q(x) - p > x$,

$$L^{2q}(x) - 2p = L^q(L^q(x)) - 2p = L^q(L^q(x) - p) + p - 2p =$$
$$= L^q(L^q(x)-p) - p \underset{\substack{\downarrow \\ L^q \text{ strictly increasing}}}{>} L^q(x) - p > x$$

$L^{mq}(x) - mp > x$, this is a contradiction

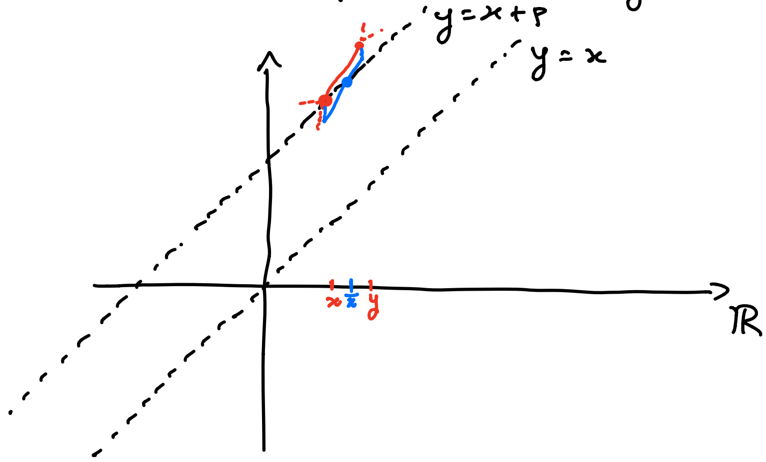
In the second case we prove that $L^{mq}(x) - mp < x$, this is a contradiction.

Hence $L^q(x) = x+p$, so x is periodic of period q.

$\square$

Prop (iv) If $\tau(T_1) \in \mathbb{Q}$ and T_1 is not conjugate to $R_{\tau(T_1)}$ then $\exists\, T_2$ either $T_2 < T_1$ or $T_2 > T_1$ s.t. $\tau(T_2) = \tau(T_1)$.

proof $\tau(T_1) = \frac{p}{q} \in \mathbb{Q}$, then $\tau(T_1^q) = 0$. Then $\exists\, x \in \mathbb{R}$ s.t. $L_1^q(x) = x+p$, but $\exists\, y \in \mathbb{R}$ s.t. $L_1^q(y) \neq y+p$.



Then $\exists\, L_2^q$ s.t. $\exists\, \bar{x} \in \mathbb{R}$

$L_2^q(\bar{x}) = \bar{x} + p$

$\Rightarrow \tau(L_2^q) = 0$

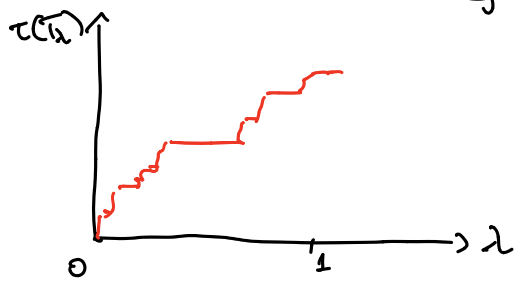
$\Rightarrow \tau(T_2) = \frac{p}{q}$.

$\square$

Theorem Let $[0,1] \ni \lambda \mapsto T_\lambda \in (\mathrm{Homeo}_{S^1}, \|\cdot\|_\infty)$ be a continuous family of OPCH which is strictly increasing ($\lambda_1 < \lambda_2 \Rightarrow T_{\lambda_1} < T_{\lambda_2}$).

Then $\lambda \mapsto \tau(T_\lambda)$ is an increasing continuous function.

If $\exists\, S \subseteq \mathbb{Q}$ which is dense in $\mathbb{R}$ and $\forall \alpha \in S$ $\nexists \lambda$ s.t. T_λ is conjugate to R_α, and $\tau(T_\lambda)$ is not constant then $\tau(T_\lambda)$ is a devil staircase ($\exists \{I_j\}$ open intervals with dense union in $[0,1]$ s.t. $\tau|_{I_j}$ is constant $\forall j$).



proof Let $I_\alpha \subset [0,1]$ is the set of λ's s.t. T_λ has rotation number $\alpha \in S \subseteq \mathbb{Q}$. Then I_α is an open interval with $|I_\alpha| > 0$.

Then we need to show that $\overline{\bigcup_\alpha I_\alpha} = [0,1]$.

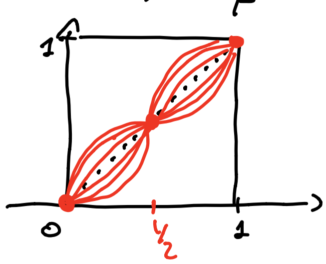
Let $\lambda \in [0,1] \setminus \overline{\bigcup_\alpha I_\alpha}$ and let $\varepsilon > 0$ s.t.

$(\lambda - \varepsilon, \lambda + \varepsilon) \subset [0,1] \setminus \overline{\bigcup_\alpha I_\alpha}$.

Then $\tau|_{(\lambda-\varepsilon, \lambda+\varepsilon)}$ is not in S hence is strictly increasing and continuous. Then $\tau(\lambda-\varepsilon, \lambda+\varepsilon)$ is a interval and must contain points of S for density. It is a contradiction.

$\square$

Example $T_\beta : S^1 \to S^1$, $T_\beta(x) = x - \beta \sin(2\pi x)$, $0 < |\beta| < \frac{1}{2\pi}$



$[0,1] \ni \alpha \mapsto R_\alpha \circ T_\beta =: T_{\alpha,\beta}(x) = \{ x + \alpha - \beta \sin(2\pi x) \}$

For all fixed β s.t. $0 < |\beta| < \frac{1}{2\pi}$ we have:

(i) $\alpha \mapsto T_{\alpha,\beta}$ is continuous

(ii) $\alpha \mapsto T_{\alpha,\beta}$ is strictly increasing

(iii) $\tau(T_{0,\beta}) = 0$, but $\exists\, \alpha$ s.t. $L_{\alpha,\beta}(x) - x \notin \mathbb{Z}$

or $0 < L_{\alpha,\beta}(x) - x < 1$.

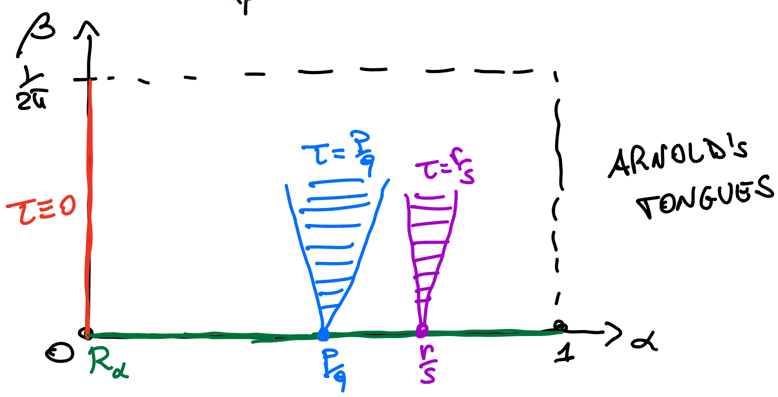
(iv) For all $\mu \in \mathbb{Q}$, $\nexists\, \alpha \in [0,1]$ s.t. $\tau(T_{\alpha,\beta}) = \mu$ and $T_{\alpha,\beta}$ is conjugate to R_μ.

proof Let $L_{\alpha,\beta}$ satisfy $L^q_{\alpha,\beta}(x) = x + p$ for all $x \in \mathbb{R}$.

Then
$$(R_{-p} \circ L^{q-1}_{\alpha,\beta}) \circ L_{\alpha,\beta} = Id$$
$$x \mapsto x+p \mapsto x$$

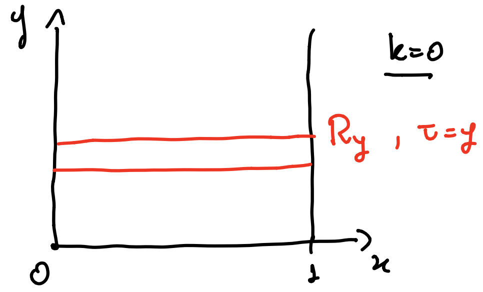
It follows that $L_{\alpha,\beta}(x) = x + \alpha - \beta \sin(2\pi x)$ is an entire function with an inverse, hence it is injective, hence by Picard's Theorem, $L_{\alpha,\beta}$ is a polynomial of degree 1. This is a contradiction.

Then $\alpha \mapsto \tau(T_{\alpha,\beta})$ is a devil staircase.



Remark Standard map

$$T(x,y) = \left(x + y - \frac{k}{2\pi}\sin(2\pi x),\ y - \frac{k}{2\pi}\sin(2\pi x)\right)$$



$T: S^1 \to S^1$ orientation preserving circle homeomorphism

$\underline{\tau(T) = \frac{p}{q} \in \mathbb{Q}}$ $\qquad\qquad$ $\underline{\tau(T) \in \mathbb{R} \setminus \mathbb{Q}}$

$\boxed{\text{Rational case}}$

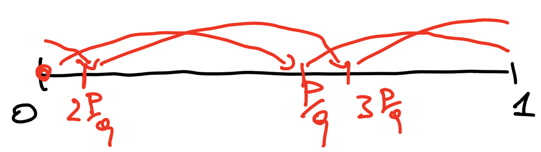
Either T is conjugate to $R_{\frac{p}{q}}$, in which case $T^q = Id$,

or not.

Prop $\tau(T) = \frac{p}{q}$ $(p,q)=1$ implies that periodic orbits are ordered in S^1 as the orbits of the rotation $R_{\frac{p}{q}}$.

proof For $R_{\frac{p}{q}}$, there are q points in S^1 ordered as

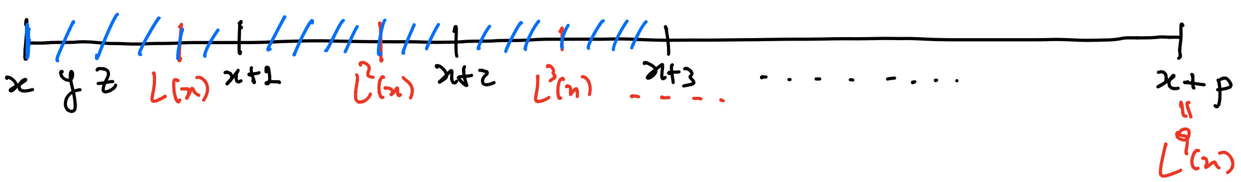
$$\left\{ 0,\ k\frac{p}{q},\ 2k\frac{p}{q}, \dots, k(q-1)\frac{p}{q} \right\}$$



$$kp \equiv 1 \pmod q$$

Let $T: S^1 \to S^1$, x a periodic point and $L: \mathbb{R} \to \mathbb{R}$ lift.

$L^q(x) = x + p$



$$\underline{A = \{ L^k(x) + m \ / \ m, k \in \mathbb{Z} \} \cap [x, x+p] = q \cdot p + 1}$$

$\underline{[L^i(x), L^{i+1}(x)]}$, $i = 0, \dots, q-1$, L sends $[L^i(x), L^{i+1}(x)]$ onto $[L^{i+1}(x), L^{i+2}(x)]$ as a homeo.

L preserves A. Hence in $[L^i(x), L^{i+1}(x))$ there are p points of A.

Let y be the point in A which is the closest to x. So $y = L^r(x)+s$ for $r,s \in \mathbb{Z}$. Then if z is the closest point to the right of y in A, we show that $z = L^r(y)+s$.

If $z \in A \cap (y, L^r(y)+s)$, then $z = L^m(x)+n$ and

$$\tilde{L}^{-1}(z) = L^{-r}(z)-s \in \left(\tilde{L}^{-1}(y), \tilde{L}^{-1}(L^r(y)+s)\right) = \left(\tilde{L}^{-1}(\tilde{L}(x)), \tilde{L}^{-1}(\tilde{L}(y))\right) = (x,y)$$

$\tilde{L}(\cdot) := L^r(\cdot)+s$ $\quad$ $\tilde{L}$ is an increasing homeo

$\tilde{L}^{-1}(\cdot) = L^{-r}(\cdot - s) = L^{-r}(\cdot) - s$

But $L^{-r}(z)-s \in A$ and $A \cap (x,y) = \emptyset$. It is a contradiction.

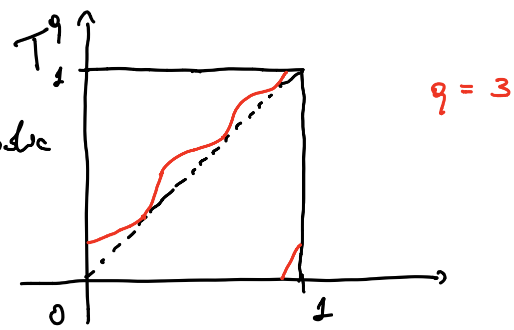
Hence $z = L^r(y)+s$, and by repeating the same argument we have $L(x) = \tilde{L}^p(x) = L^{rp}(x) + ps$, hence $rp \equiv 1 \pmod{q}$. Hence $y = L^r(x)+s$ is the analogue of $R^k_{p/q}(0)$. $\square$

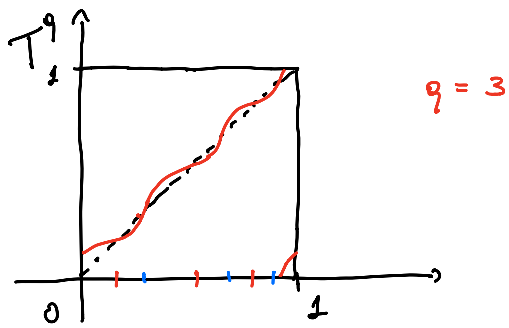
If $\tau(T) = \frac{p}{q} \in \mathbb{Q}$ all the orbits of T are:

(i) periodic orbits of period q;

or

(ii) homoclinic orbits to one periodic orbit;





(iii) heteroclinic orbits to two different periodic orbits.

Irrational rotation number

Prop If $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$ then

(i) there are no periodic orbits;

(ii) $\forall\, x, y \in S^1$, $\omega(x) = \omega(y)$;

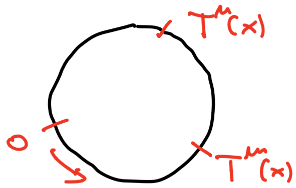
(iii) let E to be the ω-limit set, then $E = S^1$ or E is a closed set with no isolated points and empty interior (perfect set).

proof (i) follows from the fact that $\exists$ of periodic orbits implies that $\tau(T) \in \mathbb{Q}$.

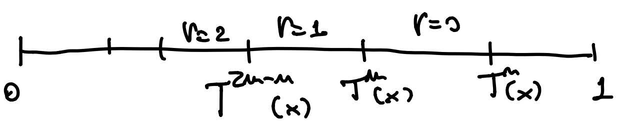
(ii) **Lemma** For all $x \in S^1$ and all $m, n \in \mathbb{N}$, $m < n$,

$$\bigcup_{r \geq 1} T^{-r}\big([T^m(x), T^n(x)]\big) = S^1$$

proof



$$\bigcup_{r \geq 1} T^{-r(n-m)}[T^m(x), T^n(x)]$$



$$T^{-(n-m)}[T^m(x), T^n(x)] = [T^{2m-n}(x), T^m(x)]$$

$\{T^{-r(n-m)}[T^m(x), T^n(x)]\}_{r \geq 1}$ are contiguous intervals.

If $\bigcup_{r \geq 1} T^{-r(n-m)}[T^m(x), T^n(x)] \neq S^1$ then the length of $T^{-r(n-m)}[T^m(x), T^n(x)]$ goes to 0. Then $\exists\, \xi \in S^1$ s.t.

$$\xi = \lim_{r \to +\infty} T^{-r(n-m)}(T^m(x)) = \lim_{r \to +\infty} T^{-r(n-m)}(T^n(x))$$

$$= \lim_{r \to +\infty} T^{(-r+1)(n-m)}(T^m(x)) = T^{n-m}\Big(\lim_{r \to +\infty} T^{-r(n-m)}(T^m(x))\Big) =$$

$= T^{m-n}(\xi)$. $\Rightarrow$ ξ is periodic. Contradiction. $\square$

Let $x \in S^1$, and let $z \in \omega(x)$, hence $\exists \{k_n\}$ s.t.

$k_n \nearrow +\infty$ and $T^{k_n}(x) \xrightarrow[n\to+\infty]{} z$.

For $y \neq x$, let $\{k_n^1\}$ and $\{k_n^2\}$ two subsequences of $\{k_n\}$ s.t. $k_n^1 \nearrow +\infty$, $k_n^2 \nearrow +\infty$, $k_n^1 < k_n^2$ $\forall n$, $(k_n^2 - k_n^1) \to +\infty$, then by the lemma

$$y \in \bigcup_{r\geq 1} T^{-r(k_n^2-k_n^1)}[T^{k_n^1}(x), T^{k_n^2}(x)] \quad \forall n.$$

hence $\forall n$ $\exists$ $r_n \geq 1$ s.t. $T^{r_n(k_n^2-k_n^1)}(y) \in [T^{k_n^1}(x), T^{k_n^2}(x)]$

$\Rightarrow$ $T^{r_n(k_n^2-k_n^1)}(y) \xrightarrow[n\to+\infty]{} z$ and $r_n(k_n^2-k_n^1) \to +\infty$.

$\Rightarrow$ $z \in \omega(y)$. $\Rightarrow$ $\omega(x) \subset \omega(y)$ $\forall x, y \in S^1$

$\Rightarrow$ $\omega(x) = \omega(y)$ $\forall x, y \in S^1$ $\square$ (ii)

(iii) E be ω-limit set, $E = \omega(x)$ $\forall x \in S^1$.

Then E is closed and $T(E) = T^{-1}(E) = E$.

<u>Claim</u> E is the minimal set with these properties wrt inclusion.

<u>proof</u> If A is closed and invariant, then $\forall x \in A$ $\mathcal{O}(x) \subset A$

hence $\omega(x) \subset \overline{\mathcal{O}(x)} \subset \overline{A} = A$ $\Rightarrow$ $E \subseteq A$.

If $E = S^1$ then $\partial E = \emptyset$, if $E \subsetneq S^1$ then $\partial E \neq \emptyset$.

<u>Claim</u> ∂E is invariant

Then $E \subseteq \partial E$, then $E = \partial E$, that is E has no interior.

Moreover if $x \in E$ then $x \in \omega(x)$, hence $\exists \{k_n\}$ s.t.

$T^{k_n}(x) \xrightarrow[n\to+\infty]{} x$ and $T^{k_n}(x) \in E$ $\forall x$. Hence E has no isolated points.

$\square$ (iii).

Theorem (POINCARÉ CLASSIFICATION) If $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$ then

if $E = S^1$ then T is top. conjugate to $R_{\tau(T)}$,

if $E \subsetneq S^1$ then $R_{\tau(T)}$ is a top. factor of T.

proof Let $L : \mathbb{R} \to \mathbb{R}$ be the lift of T, and let $\tilde{R}_{\tau(T)}$ the lift of $R_{\tau(T)}$.

$\tilde{R}_{\tau(T)} : \mathbb{R} \to \mathbb{R}$, $\tilde{R}_{\tau(T)}(x) = x + \tau(T)$.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "h"] \arrow[d, "L"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{R} \arrow[d, "\tilde{R}_{\tau(T)}"] \\ \mathbb{R} \arrow[r, "h"'] & \mathbb{R} \end{tikzcd}$$

We construct $h : \mathbb{R} \to \mathbb{R}$ surjective and continuous, and

$$h \circ L = \tilde{R}_{\tau(T)} \circ h$$

Let us fix $x \in \mathbb{R}$, and $A = \{ L^n(x) + m \ / \ n, m \in \mathbb{Z} \}$.

Let $\ell : A \to \mathbb{R}$, $\ell(L^n(x) + m) = n\,\tau(T) + m = \tilde{R}^n_{\tau(T)}(x) + m$

Lemma 1 ℓ is monotone.

proof Let n_1, m_1, n_2, m_2 s.t. $L^{n_1}(x) + m_1 < L^{n_2}(x) + m_2$.

Claim For all $p, q, r, s \in \mathbb{Z}$,

$$\mathbb{R} \ni y \longmapsto d^{r,s}_{p,q}(y) := L^r(y) + s - L^p(y) - q$$

never vanishes and is continuous.

proof If $d^{r,s}_{p,q}(y_0) = 0$ then y_0 is periodic. $\square$

So $d^{n_1,m_1}_{n_2,m_2}(x) < 0$ and this implies $d^{n_1,m_1}_{n_2,m_2}(y) < 0 \ \forall y \in \mathbb{R}$.

Hence $L^{n_1}(0) - L^{n_2}(0) < m_2 - m_1$, but

$$L^{n_1}(0) - L^{n_2}(0) = L^{n_1 - n_2}(L^{n_2}(0)) - L^{n_2}(0) < m_2 - m_1$$

than $d^{n_1-n_2, m_1}_{0, m_2}(L^{n_2}(0)) < 0 \Rightarrow L^{n_1-n_2}(y) + m_1 < y + m_2 \ \forall y$

Hence $L^{n_1-n_2}(L^{n_1-n_2}(0)) - L^{n_1-n_2}(0) < m_2 - m_1$ and

$$L^{n_1-n_2}(\quad 0 \quad) - \quad 0 \quad < m_2 - m_1$$

$\Rightarrow$ $L^{2(n_1-n_2)}(0) - 0 < 2(m_2-m_1)$. Then

$L^{k(n_1-n_2)}(0) < k(m_2-m_1)$.

If $n_1 > n_2$ then

$$\tau(T) = \lim_{k\to+\infty} \frac{L^{k(n_1-n_2)}(0) - 0}{k(n_1-n_2)} < \frac{m_2-m_1}{n_1-n_2}$$

and $\quad n_1 \tau(T) - n_2 \tau(T) < m_2 - m_1 \Leftrightarrow$

$$n_1 \tau(T) + m_1 < n_2 \tau(T) + m_2$$

$$\ell(L^{n_1}(x) + m_1) < \ell(L^{n_2}(x) + m_2)$$

If $n_1 < n_2$ then

$$\tau(T) = \lim_{k\to+\infty} \frac{L^{k(n_1-n_2)}(0) - 0}{k(n_1-n_2)} > \frac{m_2-m_1}{n_1-n_2}$$

and $\quad (n_1-n_2)\tau(T) < m_2 - m_1$. $\square$ Lemma 1.

Theorem (Poincaré classification) $\tau(T)\in\mathbb{R}\setminus\mathbb{Q}$, E ω-limit set

$E=S^1 \Rightarrow T$ is top. conjugate to $R_{\tau(T)}$

$E\subsetneq S^1 \Rightarrow R_{\tau(T)}$ is a top. factor of T

proof Fix $x\in\mathbb{R}$, $A=\{L^n(x)+m \,/\, n,m\in\mathbb{Z}\}$ L a lift of T

$\ell: A\to\mathbb{R}$, $\ell(L^n(x)+m)=n\tau(T)+m$

Lemma 1 ℓ is strictly increasing

Lemma 2 ℓ can be continuously extended to a map

$\tilde{\ell}:\mathbb{R}\to\mathbb{R}$

proof $\ell(A)=\{n\tau(T)+m \,/\, n,m\in\mathbb{Z}\}$ is dense in $\mathbb{R}$

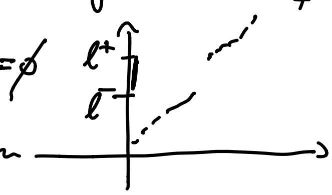
- Extend ℓ to $\bar{A}$ as a continuous function.

  Let $x\in\bar{A}\setminus A$, then either $\exists\ y_n\nearrow x$ with $y_n\in A$ and $x\in\partial A$ with A^c on its right, or $\exists\ y_n\searrow x$ with $y_n\in A$ and $x\in\partial A$ with A^c on its left, or $\exists\ y_n\nearrow x$ and $z_n\searrow x$ with $y_n, z_n\in A$.

  In the first and second case, let $\ell(x):=\lim_{n\to+\infty}\ell(y_n)$ . .

  In the third case, let $\ell^-(x):=\lim_{n\to+\infty}\ell(y_n)$ and $\ell^+(x):=\lim_{n\to+\infty}\ell(z_n)$. But if $\ell^-(x)\lneq\ell^+(x)$ then

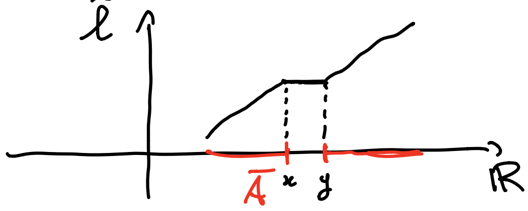
  $(\ell^-(x),\ell^+(x))\cap\ell(A)=\emptyset$

  which is a contradiction because $\ell(A)$ is dense.

- Extend ℓ to $\mathbb{R}$.

  If $\bar{A}=\mathbb{R}$ then I'm done, but if $\bar{A}\subsetneq\mathbb{R}$ then



then $\ell(x) = \ell(y)$. Hence let $\tilde{\ell}(z) := \ell(x) = \ell(y)$ $\forall z \in (x, y)$.

$\square$ Lemma 2

Lemma 3 Let $h : S^1 \to S^1$ be $h = \pi \circ \tilde{\ell}$, then $h \circ T = R_{\tau(T)} \circ h$. Moreover h is continuous and surj.

proof It is enough to show that $\tilde{\ell} \circ L = \tilde{R}_{\tau(T)} \circ \tilde{\ell}$.

Let $y \in A$, $y = L^r(x) + s$ for $r, s \in \mathbb{Z}$ then

$\tilde{\ell} \circ L(y) = \tilde{\ell}(L(L^r(x) + s)) = \tilde{\ell}(L^{r+1}(x) + s) = (r+1)\tau(T) + s$

$\tilde{R}_{\tau(T)} \circ \tilde{\ell}(y) = \tilde{R}_{\tau(T)}(\tilde{\ell}(L^r(x) + s)) = \tilde{R}_{\tau(T)}(r\tau(T) + s) = (r+1)\tau(T) + s$

If $y \notin A$, use continuity of $\tilde{\ell}$. $\square$ Lemma 3

Lemma 4 $\pi(\bar{A}) = E$, E ω-limit set

proof • $\pi(A) = \pi(\{L^n(x) + m \,/\, n, m \in \mathbb{Z}\}) = \mathcal{O}(x)$ in S^1

Then $E = \omega(x) \subset \overline{\mathcal{O}(x)} \subset \pi(\bar{A})$

• Let A be defined for $x \in \pi^{-1}(E)$, then

$\pi(\mathcal{O}_L(x)) \subset E \Rightarrow \pi(A) \subset E \Rightarrow \pi(\bar{A}) \subset \bar{E} = E$

$\square$ Lemma 4

$\square$ Poincaré classific.

If $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$ then for all $x \in S^1$:

(i) $\mathcal{O}(x)$ is dense;

(ii) $\mathcal{O}(x)$ is homoclinic to E.

Remark Invariant measures:

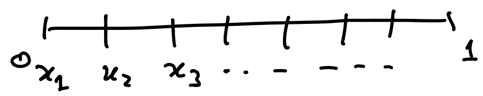
– $\tau(T) = \frac{p}{q} \in \mathbb{Q}$, all invariant measures are $\frac{1}{q}\sum_{j=0}^{q-1} \delta_{T^j(x)}$ for x periodic

- $\tau(T) \in \mathbb{R} \setminus \mathbb{Q}$, there is only one probability invariant measure

---

Theorem (DENJOY) Let T be an orientation preserving circle homeomorphism, if $\tau(T) \in \mathbb{R}\setminus\mathbb{Q}$ and T is a C^1 diffeomorphism with $T' \in BV(0,1)$, then T is top. conjugate to $R_{\tau(T)}$.

proof $BV(0,1) = \left\{ f : [0,1] \to \mathbb{R} \;/\; \sup_{\substack{\mathcal{I} \text{ finite part.} \\ \text{of } [0,1]}} \sum_{j=1}^{\#\mathcal{I}} |f(x_j) - f(x_{j-1})| < +\infty \right\}$



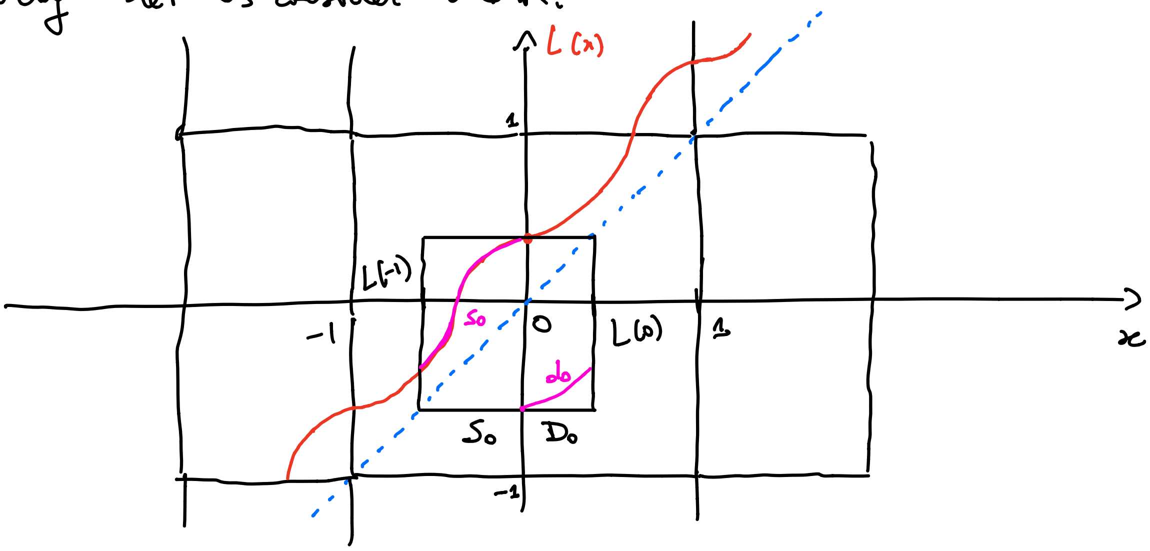
The role is to use Poinc. class. Thm, so we need to prove that all orbits are dense, $E = S^1$.

Claim If all $x \in S^1$ are positively recurrent then $E = S^1$.

proof x is positively recurrent if $x \in \omega(x)$, that is $\exists\, k_n \nearrow +\infty$ s.t. $T^{k_n}(x) \xrightarrow[n\to+\infty]{} x$.

If $E \subsetneq S^1$ then for $y \in S^1 \setminus E$ we have $y \notin E = \omega(y)$, hence y is not positively recurrent. $\square$ claim

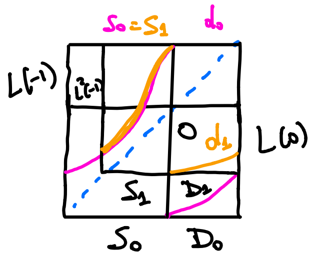
Wlog let us consider $0 \in \mathbb{R}$.



$L(-1) = L(0) - 1$, $\big[L(-1), L(0)\big]_{/L(-1)\sim L(0)} \sim S^1$

$S_0 = [L(-1), 0]$ $\qquad s_0 = L|_{S_0}$

$D_0 = [0, L(0)]$ $\qquad d_0 = (L \circ R_{-1})|_{D_0}$



$s_0(0) = L(0)$

$d_0(0) = L(-1)$

$s_0 \circ d_0(0) = d_0 \circ s_0(0) = L^2(-1)$

If $s_0 \circ d_0(0) < 0$ then $S_1 = [s_0 \circ d_0(0), 0]$, $D_1 = D_0$

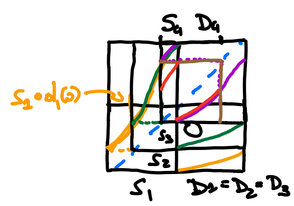
$s_1 = s_0|_{S_1}$, $d_1 = s_0 \circ d_0|_{D_1}$

$(s_1, d_1) : S_1 \cup D_1 \to S_1 \cup D_1 = [s_0 \circ d_0(0), L(0)]$

a circle homeomorphism since $s_1(s_0 \circ d_0(0)) = d_1(L(0))$

$s_1(0) = L(0)$, $d_1(0) = s_0 \circ d_0(0)$

$s_1 \circ d_1(0) = d_1 \circ s_1(0)$



$D_1 = D_2 = D_3$ $\qquad D_4 \subsetneq D_3$

$S_3 \subsetneq S_2 \subsetneq S_1$ $\qquad S_4 = S_3$

$s_3 = s_2|_{S_3}$, $s_2 = s_1|_{S_2}$ $\qquad s_4 = d_3 \circ s_3|_{S_4}$

For all $n \geq 0$, we construct a sequence (S_n, D_n, s_n, d_n) s.t.

- $(s_n, d_n) : S_n \cup D_n \to S_n \cup D_n = [d_n(0), s_n(0)]$
  circle homeomorphism
- $s_n \circ d_n(0) = d_n \circ s_n(0)$
- $s_n \leq s_{n-1}$, $d_n \geq d_{n-1}$

Theorem (Denjoy) $T \in C^{1,BV}$, $\tau(T) \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow T \sim R_{\tau(T)}$

proof · All orbits are positively recurrent $\Rightarrow E = S^1$.

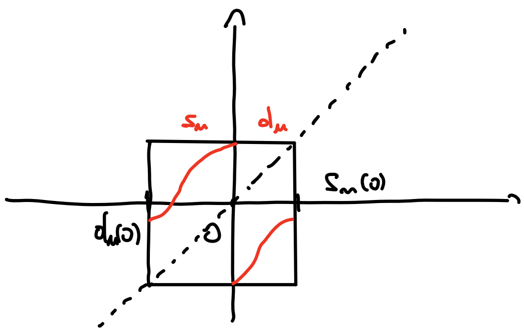
(S_n, D_n, s_n, d_n), $\sigma_n \in \{\pm\}$ s.t.

$s_n : S_n \to S_n \cup D_n$, $d_n : D_n \to S_n \cup D_n$ and together $(s_n, d_n) : S_n \cup D_n \to S_n \cup D_n$ is a circle homeo

$S_n = [d_n(0), 0]$, $D_n = [0, s_n(0)]$ with $s_n(d_n(0)) = d_n(s_n(0))$

$S_n \le S_{n-1}$

$d_n \ge d_{n-1}$



$\sigma_n = + \Leftrightarrow s_n(d_n(0)) > 0$, $\sigma_n = - \Leftrightarrow s_n(d_n(0)) < 0$

(Rem $\tau(T) \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow s_n(d_n(0)) \neq 0 \ \forall n$)

$\sigma_n = +$

$S_{n+1} = S_n$ and $D_{n+1} = [0, s_n(d_n(0))]$

$s_{n+1} = d_n \circ s_n$ $\quad$ $d_{n+1} = d_n|_{D_{n+1}}$

$\sigma_n = -$

$S_{n+1} = [s_n(d_n(0)), 0]$ and $D_{n+1} = D_n$

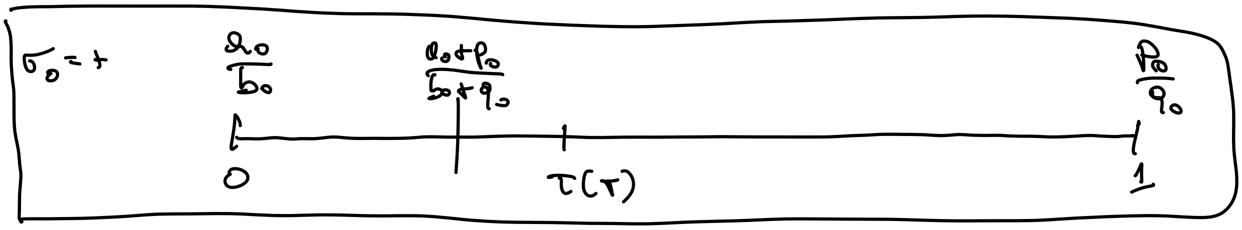
$s_{n+1} = s_n|_{S_{n+1}}$ $\quad$ $d_{n+1} = s_n \circ d_n|_{D_{n+1}}$

Lemma 1 Let $a_0 = 0$, $b_0 = 1$, $p_0 = 1$, $q_0 = 1$ and define

$$\left[\frac{a_{n+1}}{b_{n+1}}, \frac{p_{n+1}}{q_{n+1}}\right] = \begin{cases} \left[\dfrac{a_n + p_n}{b_n + q_n}, \dfrac{p_n}{q_n}\right] & \text{if } \sigma_n = + \\ \left[\dfrac{a_n}{b_n}, \dfrac{a_n + p_n}{b_n + q_n}\right] & \text{if } \sigma_n = - \end{cases} \qquad \forall n \ge 0$$

Then $s_n = L^{b_n} \circ \tilde{R}_{-a_n}$ and $d_n = L^{q_n} \circ \tilde{R}_{-p_n}$ $\forall n$

and $\left[\frac{a_n}{b_n}, \frac{p_n}{q_n}\right] \xrightarrow[n\to+\infty]{} \tau(T)$.



<u>proof</u> • <u>$n=0$</u> $s_0 = L$, $d_0 = L \circ R_{-1}$ (by def.)

If $\sigma_n = +$, $s_{n+1} = d_n \circ s_n = L^{q_n} \circ \tilde{R}_{-p_n} \circ L^{b_n} \circ \tilde{R}_{-a_n} =$

$$= L^{q_n + b_n} \circ \tilde{R}_{-(p_n+a_n)} = L^{b_{n+1}} \circ \tilde{R}_{-a_{n+1}}$$

$$d_{n+1} = d_n = L^{q_n} \circ \tilde{R}_{-p_n} = L^{q_{n+1}} \circ \tilde{R}_{-p_{n+1}}.$$

- $p_n b_n - a_n q_n = 1$ $\forall n \geq 0$ (by induction)
- $\max\{b_n, q_n\} \xrightarrow[n\to+\infty]{} +\infty$

Then $\frac{p_n}{q_n} - \frac{a_n}{b_n} = \frac{1}{q_n b_n} \xrightarrow[n\to+\infty]{} 0$.

If $\{b_n\}$ is unbounded then

$$\tau(T) = \lim_{n\to+\infty} \frac{L^{b_n}(0) - 0}{b_n} = \lim_{n\to+\infty} \frac{s_n(\tilde{R}_{a_n}(0)) - 0}{b_n} =$$

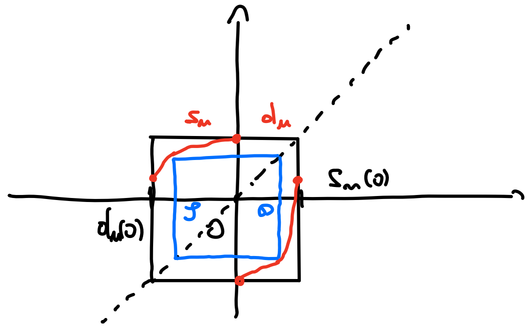
$$= \lim_{n\to+\infty} \frac{s_n(0 + a_n) - 0}{b_n} = \lim_{n\to+\infty} \frac{s_n(0) + a_n - 0}{b_n} =$$

$$= \lim_{n\to+\infty} \frac{a_n}{b_n} \quad \text{since } s_n(0) \in [-1, 1]. \qquad \square$$

Let's assume that $\mathcal{O}^+(0)$ is not positively recurrent, then

$$\bigcap_{n\geq 0} (S_n \cup D_n) =: [\mathcal{S}, \mathcal{D}]$$

with $\mathcal{S} < 0 < \mathcal{D}$.



$d_n(0) \nearrow \mathcal{S}$

$s_n(0) \searrow \mathcal{D}$

<u>Lemma 2</u> For all $n \geq 0$ we have $s_n(\mathcal{S}) \geq \mathcal{D}$ and $d_n(\mathcal{D}) \leq \mathcal{S}$.

<u>proof</u> Prove that $s_n(\mathcal{S}) \geq \mathcal{D}$. Let's assume $\exists N > 0$ s.t. $s_N(\mathcal{S}) < \mathcal{D}$, then $s_n(\mathcal{S}) \leq s_N(\mathcal{S}) < \mathcal{D}$ $\forall n \geq N$ and $s_n(d_n(0)) < s_n(\mathcal{S}) < \mathcal{D}$ $\forall n \geq N$.

But $s_n(d_n(0))$ is a boundary point of $S_{n+1} \cup D_{n+1}$, then $s_n(d_n(0)) < \mathcal{S} < 0$ $\forall n \geq N$. Hence $\sigma_n = -$ $\forall n \geq N$, and from lemma 1 the sequence $\left\{\frac{a_n}{b_n}\right\}_{n \geq N}$ is constant and equal to $\frac{a_N}{b_N}$. But $\tau(T) = \lim_{n \to +\infty} \frac{a_n}{b_n} = \frac{a_N}{b_N} \in \mathbb{Q}$ and this is a contradiction. $\square$

Then
$$\left.\begin{array}{l} \inf s_n' \xrightarrow[n \to +\infty]{} 0, \quad \inf d_n' \xrightarrow[n \to +\infty]{} 0 \\ \max\{s_n', d_n'\} \xrightarrow[n \to +\infty]{} +\infty \end{array}\right\} (*)$$

<u>Lemma 3</u> $\operatorname{var}_{S_n}(\log s_n') + \operatorname{var}_{D_n}(\log d_n') \leq \operatorname{var}_{S_{n-1}}(\log s_{n-1}') + \operatorname{var}_{D_{n-1}}(\log d_{n-1}')$

Then by lemma 3,

$$\operatorname{var}_{S_n}(\log s_n') + \operatorname{var}_{D_n}(\log d_n') \leq \operatorname{var}_{S_0}(\log s_0') + \operatorname{var}_{D_0}(\log d_0') < +\infty$$

but by $(*)$, $\mathrm{var}_{S_n}(\log s_n') + \mathrm{var}_{D_n}(\log d_n') \xrightarrow[n\to+\infty]{} +\infty$ $\square$ Thm.

Regularity of the conjugacy

Local theory $T = R_\alpha + \eta$, $\tau(T) = \alpha \in \mathbb{R}\setminus\mathbb{Q}$

- If T is analytic and α is Diophantine, then T is analytically conjugate to R_α (Arnold)

(α is Diophantine if $\exists\, c, \nu > 0$ s.t. $\alpha \in D(c,\nu)$ that is $\forall\, p \in \mathbb{Z}$, $q \in \mathbb{N}$, $\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^{\nu+2}}$)

- Let T be analytic, then T is analytically conjugate to R_α if and only if α is Brjuno (Yoccoz)

(α is Brjuno if $\sum_{n\geq 1} \frac{\log q_{n+1}}{q_n} < +\infty$, where $\left\{\frac{p_n}{q_n}\right\}$ is the sequence of fast convergents)

Rem α is Diophantine $\underset{\not\Leftarrow}{\Rightarrow}$ α is Brjuno

- There exist $T \in C^k$, $(k \geq 3)$, and α is Liouville s.t. the conjugacy is not absolutely continuous. (Herman)

Global theory $T : S^1 \to S^1$, $\tau(T) = \alpha \in \mathbb{R}\setminus\mathbb{Q}$

- There exists $\mathcal{H}$ with Dioph $\subsetneq \mathcal{H} \subsetneq$ Brjuno so that

(Yoccoz) 1995 if $\alpha \in \mathcal{H}$ and T is analytic then T is analytically conjugate to R_α. If $\alpha \notin \mathcal{H}$ there exists T analytic not analytically conj to R_α.

(Khanin Teplinsky) 2009 • $T \in C^{2+\beta}$, $\alpha \in \bigcup_{c>0} D(c,\nu)$ with $0<\nu<\beta\le 1$ and $\beta-\nu<1$, then the conjugacy is in $C^{1+\beta-\nu}$.

(Katznelson Ornstein) 1989 • $T \in C^k$, $\exists\, c,\nu>0$ t.c. $|e^{2\pi i n\alpha}-1| > \frac{c}{|n|^{\nu+1}}$ $\forall n \in \mathbb{Z}$ with $k>\nu+2 \Rightarrow$ the conjugacy is in $C^{k-1-\nu-\varepsilon}$ $\forall \varepsilon>0$

Theorem (Arnold - local analytic theory). Let $T: S^1 \to S^1$ with lift $L: \mathbb{R} \to \mathbb{R}$, $L(x) = x + \alpha + \eta(x)$ where η can be extended to an analytic function on $S_\sigma := \{ z \in \mathbb{C} \,/\, |\mathrm{Im}\, z| < \sigma \}$ with $\|\eta\|_\sigma := \sup_{S_\sigma} |\eta(z)| < +\infty$, for some $\sigma > 0$, and $\eta(x+1) = \eta(x)$ $\forall x \in \mathbb{R}$, and let $\tau(T) = \alpha$.
If $\exists\, c > 0, \nu > 0$ s.t. $\alpha \in D(c, \nu)$ then $\exists\, \varepsilon = \varepsilon(c, \nu, \sigma) > 0$ s.t. if $\|\eta\|_\sigma < \varepsilon$ then T is analytically conjugate to R_α.

Rem $T: S^1 \to S^1$, $\exists\, \mu$ T-inv. prob. measure and μ is uniquely ergodic. Then

$$\tau(T) = \lim_{n \to +\infty} \frac{L^n(x) - x}{n} = \lim_{n \to +\infty} \frac{L^n(x)}{n} \qquad \forall x \in S^1$$

If $L(x) = x + M(x)$, $L^2(x) = x + M(x) + M(L(x))$,
$L^n(x) = x + \sum_{k=0}^{n-1} M(L^k(x))$, then

$$\tau(T) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} M(L^k(x)) \underset{\substack{\uparrow \\ M \text{ cont.}}}{=} \int_{S^1} M(x)\, d\mu$$

Proof of Arnold Theorem We need to find $H: \mathbb{R} \to \mathbb{R}$ s.t.

$$L \circ H = H \circ \tilde{R}_\alpha$$

$$H(x) + \alpha + \eta(H(x)) = H(x+\alpha)$$

Let $H(x) = x + G(x)$ with G small and 1-periodic. Then

$$x + G(x) + \alpha + \eta(x + G(x)) = x + \alpha + G(x+\alpha)$$

$$G(x+\alpha) - G(x) = \eta(x + G(x))$$

Approximate it by

$$(*) \qquad G(x+\alpha) - G(x) = \eta(x)$$

$$\varphi_0(x) = \sum_{n\in\mathbb{Z}} \hat{g}_n e^{2\pi i n x} , \qquad \eta(x) = \sum_{n\in\mathbb{Z}} \hat{\eta}_n e^{2\pi i n x}$$

Put in (*) and find

$$\underbrace{\hat{g}_n e^{2\pi i n(x+\alpha)} - \hat{g}_n e^{2\pi i n x}}_{\hat{g}_n (e^{2\pi i n\alpha} - 1) e^{2\pi i n x}} = \hat{\eta}_n e^{2\pi i n x} \qquad \forall n \neq 0$$

and for $n=0$, $\hat{g}_0 - \hat{g}_0 = \hat{\eta}_0$, then use

$$(**) \qquad \varphi(x+\alpha) - \varphi(x) = \eta(x) - \int_0^1 \eta(x)\,dx$$

Then a formal solution of (**) is

$$\varphi_0(x) = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\hat{\eta}_n}{e^{2\pi i n\alpha} - 1} e^{2\pi i n x} + \hat{g}_0$$

<u>Lemma 1</u> If $\alpha \in D(c,\nu)$ then $|e^{2\pi i n\alpha} - 1| \geq 4c|n|^{-(\nu+1)}$ $\forall n \in \mathbb{Z}, n \neq 0$

<u>proof</u> $\alpha \in D(c,\nu)$ iff $\forall n \neq 0, m \in \mathbb{Z}$

$$\left|\alpha - \frac{m}{n}\right| > \frac{c}{|n|^{\nu+2}} \Rightarrow |n\alpha - m| > c|n|^{-(\nu+1)}$$

Fixed $n \in \mathbb{Z}\setminus\{0\}$, for all $m \in \mathbb{N}$ we can write

$$|e^{2\pi i n\alpha} - 1| = |e^{2\pi i m}(e^{2\pi i(n\alpha - m)} - 1)| =$$

$$|e^{2\pi i(n\alpha - m)} - 1| = 2|\sin(\pi(n\alpha - m))| \geq$$

$$\geq 4|n\alpha - m| > 4c|n|^{-(\nu+1)}$$

$|\sin(\pi x)| \geq 2|x|$ $\forall |x| < \frac{1}{2}$

And $m \in \mathbb{N}$ is chosen s.t. $|n\alpha - m| < \frac{1}{2}$ $\square$

Lemma 2 $\quad \eta$ is analytic on $S_\sigma = \{ z \in \mathbb{C} \,/\, |\mathrm{Im}\, z| < \sigma \}$ implies that $|\hat{\eta}_n| \le \|\eta\|_\sigma \, e^{-2\pi\sigma|n|} \quad \forall\, n \in \mathbb{Z}$.

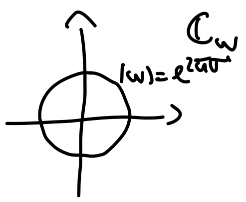
proof $\quad$ Let $\xi : D \to \mathbb{C}$ be defined as $\xi(e^{2\pi i z}) = \eta(z)$ and $D = \{ e^{-2\pi\sigma} < |w| < e^{2\pi\sigma} \}$. Then

$\|\xi\|_D = \|\eta\|_\sigma$ and $\xi(e^{2\pi i z}) = \sum_{n \in \mathbb{Z}} \hat{\eta}_n \, e^{2\pi i n z}$

Then for all $\sigma' \in (0, \sigma)$ we have

$$(n > 0) \qquad \hat{\eta}_n = \frac{1}{2\pi i} \oint_{|w| = e^{2\pi\sigma'}} \xi(w) \, \frac{dw}{w^{n+1}}$$

$$(n < 0) \qquad \hat{\eta}_n = \frac{1}{2\pi i} \oint_{|w| = e^{-2\pi\sigma'}} \xi(w) \, \frac{dw}{w^{n+1}}$$



$$w = \rho e^{2\pi i \vartheta} \qquad \left| \frac{1}{2\pi i} \int_0^1 \xi(w) \, \frac{2\pi i \rho \, e^{2\pi i \vartheta}}{\rho^{n+1} e^{2\pi i (n+1)\vartheta}} \, d\vartheta \right| =$$

$dw = \rho \, d(e^{2\pi i \vartheta}) = 2\pi i \rho \, e^{2\pi i \vartheta} \, d\vartheta$

$$= \left| \int_0^1 \xi(w) \, \frac{1}{\rho^n} \, e^{-2\pi i n \vartheta} \, d\vartheta \right| \le$$

$$\le \left(\sup |\xi(w)| \right) \rho^{-n}$$

Then $\quad |\hat{\eta}_n| \le \|\eta\|_\sigma \, e^{-2\pi|n|\sigma'} \quad \forall\, \sigma' \in (0, \sigma)$

$$\Rightarrow |\hat{\eta}_n| \le \|\eta\|_\sigma \, e^{-2\pi|n|\sigma}. \qquad \square$$

Lemma 3 $\quad$ Let $\sigma \in (0, \frac{1}{2\pi} \log 2)$. Let $\delta_0 \in (0, \frac{\sigma}{6})$ be such that $\quad 2\pi \, \Gamma(\nu+2) \, \|\eta\|_\sigma < c (2\pi\delta_0)^{\nu+3} < 2\pi\delta_0 \, \Gamma(\nu+2)$.

Then $\varphi_0(x) = \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{\hat{\eta}_m}{e^{2\pi i m\alpha}-1} e^{2\pi i m x}$ can be extended to an analytic function on $S_{\sigma-\delta_0} = \{ |\mathrm{Im}\, z| < \sigma-\delta_0 \}$, and

$$\|\varphi_0\|_{\sigma-\delta_0} \leq \frac{\Gamma(\nu+2)}{c\,(2\pi\delta_0)^{\nu+2}} \|\eta\|_\sigma < \delta_0$$

and φ_0' is analytic on $S_{\sigma-2\delta_0}$ with

$$\|\varphi_0'\|_{\sigma-2\delta_0} \leq \frac{2\pi\,\Gamma(\nu+2)}{c\,(2\pi\delta_0)^{\nu+3}} < 1$$

<u>proof</u>

$$|\varphi_0(z)| = \left| \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{\hat{\eta}_m}{e^{2\pi i m\alpha}-1} e^{2\pi i m z} \right| \overset{z\in S_{\sigma-\delta_0}}{\leq} \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{|\hat{\eta}_m|}{|e^{2\pi i m\alpha}-1|} e^{2\pi|m|(\sigma-\delta_0)}$$

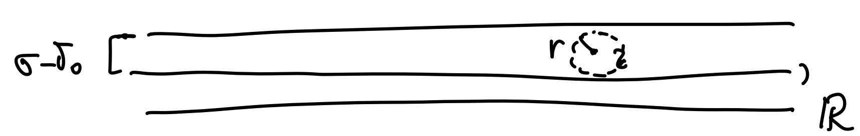
$$\leq \sum_{\substack{m\in\mathbb{Z}\\ m\neq 0}} \frac{\|\eta\|_\sigma\, e^{-2\pi|m|\sigma}}{4c\,|m|^{-(\nu+1)}} e^{2\pi|m|(\sigma-\delta_0)} = \frac{\|\eta\|_\sigma}{2c} \sum_{m\geq 1} m^{\nu+1} e^{-2\pi m\delta_0}$$

$$\leq \frac{\|\eta\|_\sigma}{2c} \frac{2}{(2\pi\delta_0)^{\nu+2}} \sum_{m\geq 1} \int_{2\pi m\delta_0}^{2\pi(m+1)\delta_0} t^{\nu+1} e^{-t}\, dt \leq$$

$$\frac{1}{2\pi\delta_0} \int_{2\pi m\delta_0}^{2\pi(m+1)\delta_0} t^{\nu+1} e^{-t}\, dt \geq \begin{cases} (2\pi m\delta_0)^{\nu+1} e^{-2\pi m\delta_0} \\ (2\pi(m+1)\delta_0)^{\nu+1} e^{-2\pi(m+1)\delta_0} \end{cases}$$

$$\leq \frac{\|\eta\|_\sigma}{c\,(2\pi\delta_0)^{\nu+2}} \underbrace{\int_0^{+\infty} t^{\nu+1} e^{-t}\, dt}_{\Gamma(\nu+2)}$$

Then $\varphi_0'(z) = \frac{1}{2\pi i} \oint_{|w-z|=r} \frac{\varphi_0(w)}{(w-z)^2}\, dw \quad \forall\, r<\delta_0$



$\square$

Prop 4 Let $H_0(x) := x + \varphi_0(x)$ (φ_0 solves $\varphi_0(x+\alpha) - \varphi_0(x) = \eta(x)$)

then H_0 is analytic on $S_{\sigma-2\delta_0}$ with an analytic inverse on $H_0(S_{2-\delta_0})$, and $S_{\sigma-3\delta_0} \subset H_0(S-2\delta_0)$.

Moreover we can write $H_0^{-1}(z) = z - \varphi_0(z) + F_0(z)$ with

$$\|F_0\|_{\sigma-4\delta_0} \leq \frac{2\pi \left(\Gamma(\nu+2)\right)^2}{C^2 (2\pi\delta_0)^{2\nu+5}} \|\eta\|_\sigma^2$$

Then $L_1 := H_0^{-1} \circ L \circ H_0$ (H should satisfy $R_\alpha = H^{-1} \circ L \circ H$)

can be written as $L_1(x) = x + \alpha + \eta_1(x)$ with

Prop 5 $\tau(L_1) = \alpha$, η_1 is analytic on $S_{\sigma-6\delta_0}$ and $\eta_1(x+1) = \eta_1(x)\ \forall x$,

Moreover $$\|\eta_1\|_{\sigma-6\delta_0} \leq \frac{16\pi \left(\Gamma(\nu+2)\right)^2}{C^2 (2\pi\delta_0)^{2\nu+5}} \|\eta\|_\sigma^2$$

Prop 6 We can define a sequence of functions $H_n(x) = x + \varphi_n(x)$ and $L_n(x) = x + \alpha + \eta_n(x)$ s.t. $L_0 = L$, sequences of constants $\delta_n = \frac{\delta_0}{1+n^2}$, $\sigma_{n+1} = \sigma_n - 6\delta_n$ $(\sigma_0 = \sigma)$, $\varepsilon_n = \|\eta\|_\sigma^{(3/2)^n}$ s.t

if $$\|\eta\|_\sigma < \left(\frac{C^2 (2\pi\delta_0)^{2\nu+5}}{16\pi\left(\Gamma(\nu+2)\right)^2}\right)^2$$ then $\forall n \geq 0$

(i) $\varphi_n(x+\alpha) - \varphi_n(x) = \eta_n(x) - \int_0^1 \eta_n(x)\,dx$

(ii) φ_n is analytic on $S_{\sigma_n-\delta_n}$ and $\|\varphi_n\|_{\sigma_n-\delta_n} \leq \frac{\Gamma(\nu+2)}{C(2\pi\delta_n)^{\nu+2}} \varepsilon_n$

(iii) $H_n^{-1}(x) = x - \varphi_n(x) + F_n(x)$ with $\|F_n\|_{\sigma_n-4\delta_n} \leq \frac{2\pi\left(\Gamma(\nu+2)\right)^2}{C^2(2\pi\delta_n)^{2\nu+5}} \varepsilon_n^2$

(iv) $L_{n+1} = H_n^{-1} \circ L_n \circ H_n$ and $\|\eta_{n+1}\|_{\sigma_{n+1}} \leq \varepsilon_{n+1}$

(v) $\lim_{n\to+\infty} \sigma_n = \sigma^* > \frac{\sigma}{2}$.

Then $L_{n+1} = (H_n^{-1} \dots \circ H_1^{-1} \circ H_0^{-1}) \circ L \circ (H_0 \circ H_1 \circ \dots H_n)$.

Let $K_n := H_0 \circ H_1 \circ \dots \circ H_{n-1}$, then

$K_1 = H_0 = x + \varphi_0(x)$, $K_2 = H_0 \circ H_1 = H_0(x + \varphi_1(x)) = x + \varphi_1(x) + \varphi_0(x + \varphi_1(x))$

$K_{N+1}(x) = x + \varphi_N(x) + \varphi_{N-1}(x + \varphi_N(x)) + \dots + \varphi_0(x + \varphi_N(x) + \dots)$

and K_N is analytic on $S_{\sigma_N - \delta_N}$. Moreover

$$| K_N(z) - z | \leq \sum_{n=0}^{N-1} \| \varphi_n \|_{\sigma_N - \delta_N} \leq \sum_{n=0}^{+\infty} \frac{\Gamma(\nu+2)}{c\,(2\pi\delta_n)^{\nu+2}} \varepsilon_n =: \Delta < +\infty$$

and

$$K_{N+1}(z) - K_N(z) = K_N(H_N(z)) - K_N(z) =$$

$$= \int_0^1 K_N'(z + s\underbrace{(H_N(z) - z)}_{\varphi_N(z)})\underbrace{(H_N(z) - z)}_{\varphi_N(z)}\, ds$$

$$\Rightarrow \quad \| K_{N+1} - K_N \|_{\sigma^*} \leq \left(1 + \frac{\Delta}{\delta_{N+1}}\right) \frac{\Gamma(\nu+2)}{c\,(2\pi\delta_{N+1})^{\nu+2}} \varepsilon_{N+1}$$

$$\Rightarrow \quad \sum_{N=0}^{+\infty} \| K_{N+1} - K_N \|_{\sigma^*} < +\infty$$

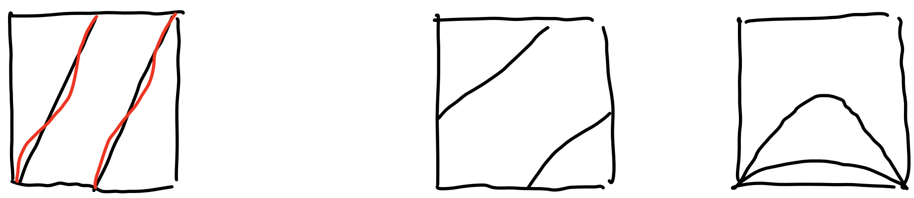
Then $K_{N+1} = K_1 + \sum_{j=1}^{N} (K_{j+1} - K_j)$, and hence $\{K_N\}$ converges uniformly to a analytic function K^* on S_{σ^*}, with K^* invertible on $K^*(S_{\sigma^*})$.

$$\lim_{N\to+\infty} (K_N^{-1} \circ L \circ K_N) = \lim_{N\to+\infty} L_{N+1} = \lim_{n\to+\infty} (\tilde{R}_\alpha + \eta_{N+1}) = \tilde{R}_\alpha$$

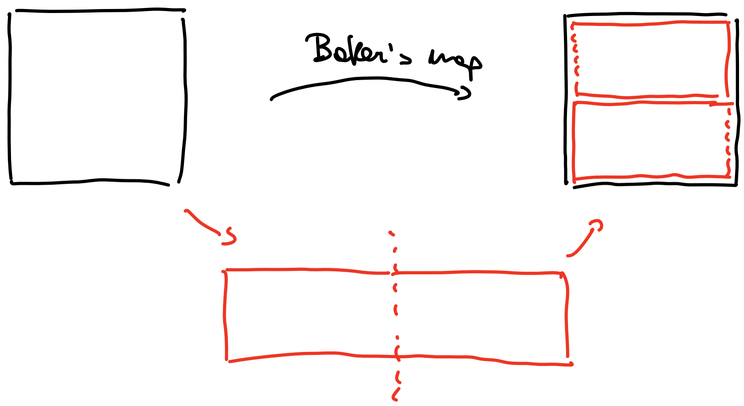
$\square$ THM.

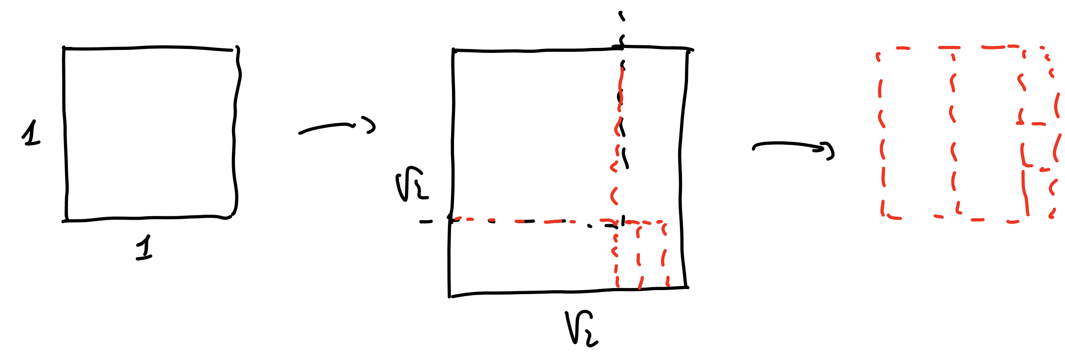
$T : X \to X$, $X = S^1 \times [a,b]$, $S^1 \times \mathbb{R}$, $S^1 \times (0,+\infty)$

1-D maps



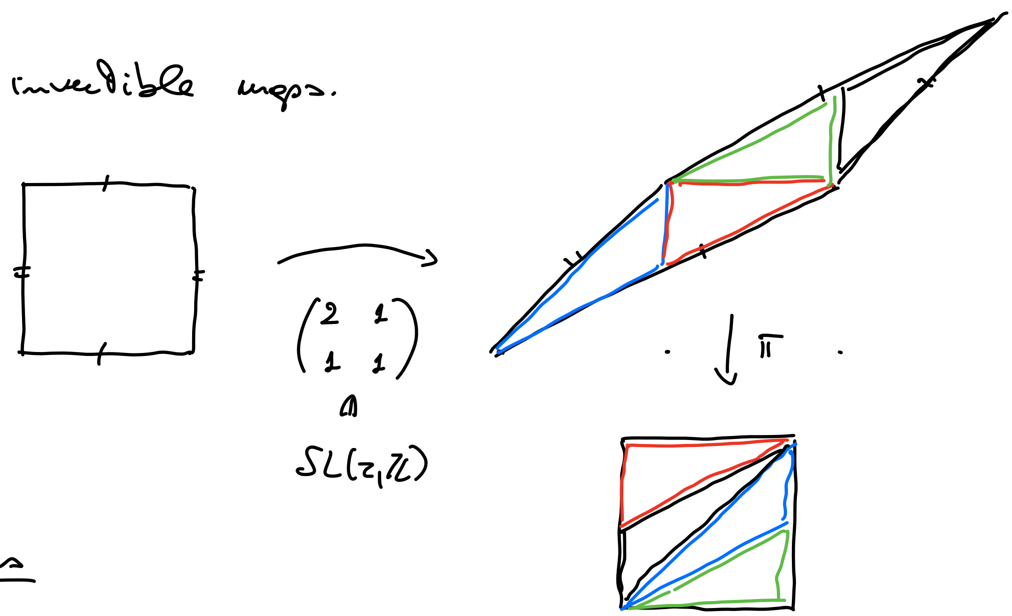
2D maps





Area-preserving and invertible maps.

Toral automorphisms are continuous.

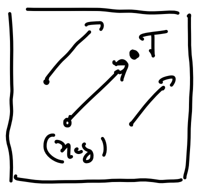


Hyperbolic dynamics

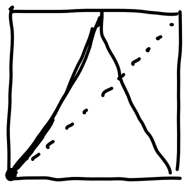
Regular $\longmapsto$ Mixed phase-space Hamiltonian

Stickyness $\longleftarrow$ Fully chaotic

Toral translations $\quad (x,y) \overset{T}{\longmapsto} (x,y) + (\omega_1, \omega_2) \pmod{\mathbb{Z}^2}$

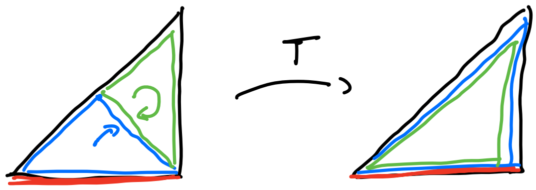


Stickyness



1D

Farey map



2D

<u>Standard map</u> $\quad T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$

$$T(x,y) = \left(x + y - \frac{k}{2\pi} \sin(2\pi x) ,\ y - \frac{k}{2\pi} \sin(2\pi x) \right) , \quad k \in \mathbb{R}$$

It is continuous and invertible.

$$JT(x,y) = \begin{pmatrix} 1 - k\cos(2\pi x) & 1 \\ -k\cos(2\pi x) & 1 \end{pmatrix} \qquad \det JT(x,y) = 1 \quad \forall (x,y)$$

$\Rightarrow$ area is preserved

<u>Bouncing balls</u> $\quad T : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$, gravity law

$f : \mathbb{R} \to \mathbb{R}$, 1-periodic , $f \in C^2$

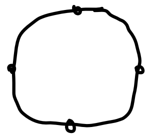
$$T(t,v) = \left(t + 2\frac{v}{g} ,\ v + 2\dot{f}\left(t + 2\frac{v}{g}\right) \right) \qquad \boxed{g = 2}$$

$$JT(t,v) = \begin{pmatrix} 1 & 1 \\ 2\ddot{f}(t+v) & 1 + 2\ddot{f}(t+v) \end{pmatrix} \qquad \det JT(t,v) = 1$$

$\Rightarrow$ area is preserved

Billiard map $\quad \Omega \subset \mathbb{R}^2$ compact and strictly convex

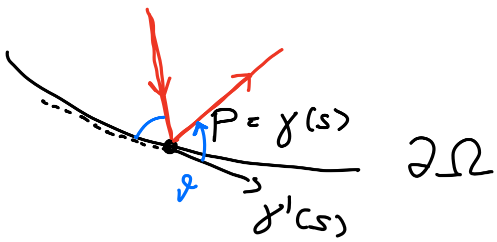
ES $\quad \Omega = \{x^4 + y^4 \leq 1\}$



strictly convex
but curvature vanishes

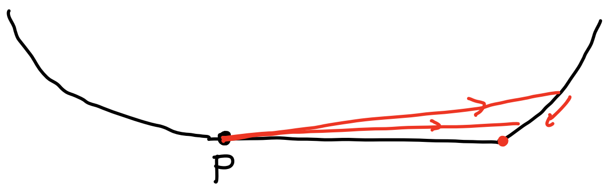
$\partial\Omega \in C^3$, $\quad |\partial\Omega| = 1$ and $\gamma : [0,1] \to \mathbb{R}^2$ parametrization of $\partial\Omega$

$\|\gamma'(s)\| = 1 \quad \forall s$, $\gamma(0) = \gamma(1)$



$\vartheta \in [0, \pi]$

Rem

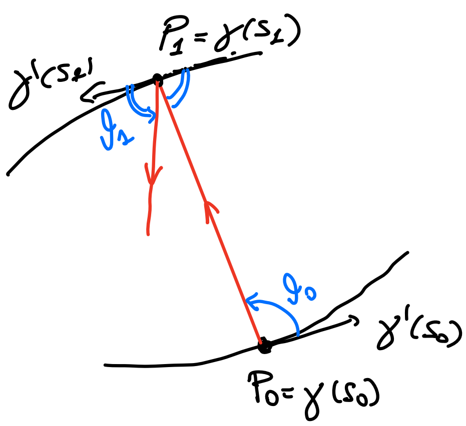


$T : S^1_s \times [0,\pi]_\vartheta \longrightarrow S^1_s \times [0,\pi]_\vartheta$ continuous and invertible

Area-preserving?

Proposition $\quad T$ preserves the measure $d\mu = \sin\vartheta \, ds \, d\vartheta$

proof



$P_0 = \gamma(s_0) = (x(s_0), y(s_0))$

$P_1 = \gamma(s_1) = (x(s_1), y(s_1))$

$h : [0,1] \times [0,1] \to \mathbb{R} \qquad h(s_0, s_1) = d(\gamma(s_0), \gamma(s_1)) = d(P_0, P_1)$

$$h(s_0, s_1) = \sqrt{(x(s_1) - x(s_0))^2 + (y(s_1) - y(s_0))^2}$$

$$\frac{\partial}{\partial s_0} h(s_0,s_1) = \frac{1}{2h(s_0,s_1)} \cdot \left[-2\,(x(s_1)-x(s_0))\,x'(s_0) - 2\,(y(s_1)-y(s_0))\,y'(s_0)\right]$$

$$= -\left\langle \frac{\overrightarrow{P_0P_1}}{h(s_0,s_1)}, \gamma'(s_0)\right\rangle = -\langle \underbrace{\widehat{\overrightarrow{P_0P_1}}}_{\text{unit vector}}, \gamma'(s_0)\rangle = -\cos\vartheta_0$$

$$\frac{\partial}{\partial s_1} h(s_0,s_1) = \langle \widehat{\overrightarrow{P_0P_1}}, \gamma'(s_1)\rangle = \cos\vartheta_1$$

$$dh = -\cos\vartheta_0\, ds_0 + \cos\vartheta_1\, ds_1$$

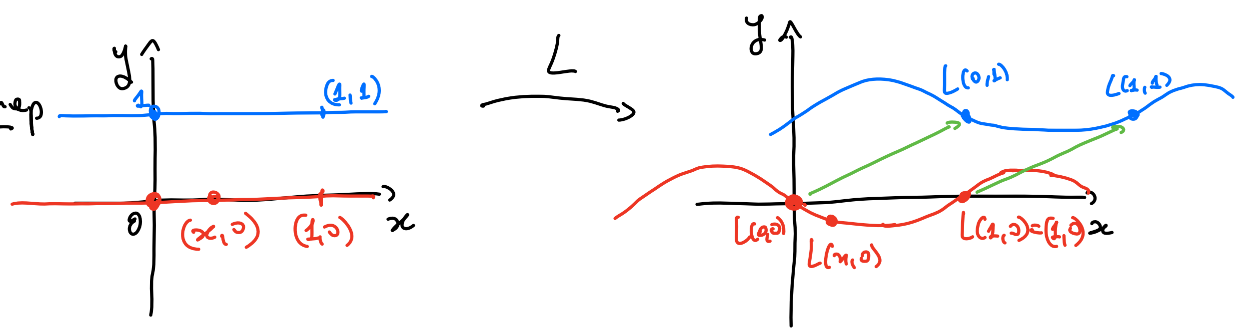
$$0 = d^2h = \sin\vartheta_0\, d\vartheta_0 \wedge ds_0 - \sin\vartheta_1\, d\vartheta_1\wedge ds_1 \Rightarrow \sin\vartheta\, d\vartheta\wedge ds \text{ is an invariant 2-form} \qquad \square$$

Def Let $x = t$, $y = -\cos\vartheta$. Then $(x,y)\in S^1_x \times [-1,1]_y$. They are called Birkhoff coordinates

Prop The billiard map in Birkhoff coordinates preserves the area.

Given $T: S^1\times(a,b)$ let $L: \mathbb{R}\times(a,b)$ to be a lift of T, that is if $\pi: \mathbb{R}\times(a,b)\to S^1\times(a,b)$ is the projection $\pi(x,y) = (\{x\}, y)$ then $T\circ\pi = \pi\circ L$.

Standard map



$$\{y=0\} \qquad L(x,0) = \left(x - \frac{k}{2\pi}\sin(2\pi x),\ -\frac{k}{2\pi}\sin(2\pi x)\right)$$

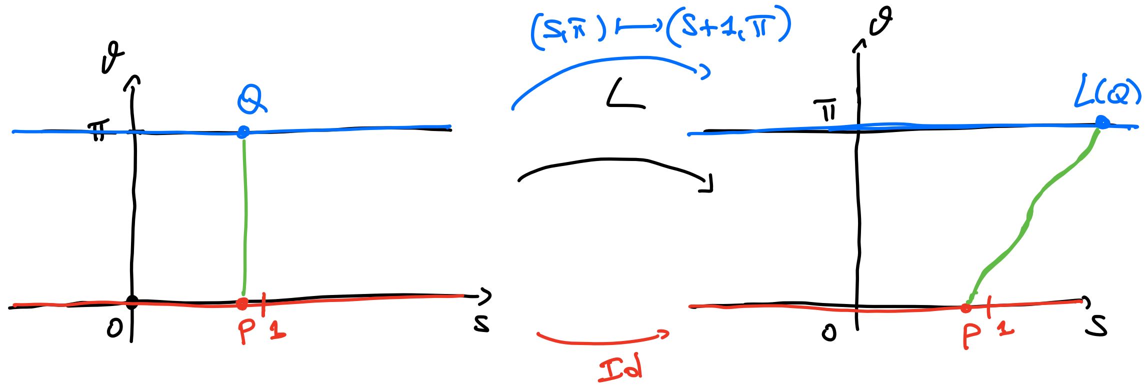
$$\{y=1\} \qquad L(x,1) = \left(x+1-\frac{k}{2\pi}\sin(2\pi x),\ 1-\frac{k}{2\pi}\sin(2\pi x)\right)$$

$L(1,1) = (2, 1)$

$L(x,1) = L(x,0) + (1,1)$

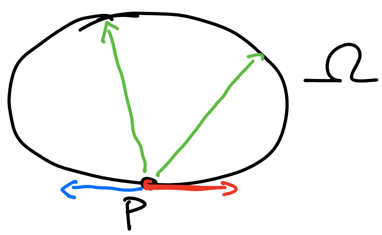
$L(x+1, y) = L(x,y) + (1,0)$

<u>Billiard map</u>

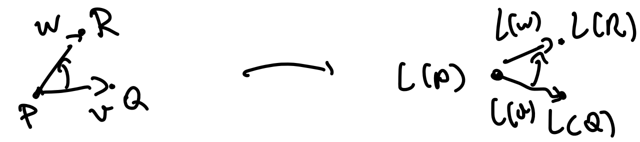


$L(s,0) = (s,0)$

$L(s,\pi) = (s+1,\pi)$



$L : \mathbb{R}\times[a,b] \to \mathbb{R}\times[a,b]$, homeo, L preserves orientation



Theorem (Poincaré - Birkhoff) Let L be an order-preserving homeo of a $X = \mathbb{R}\times[a,b]$, with $a,b\in\mathbb{R}$, $L(x,y) = (L_1(x,y), L_2(x,y))$, assume that

(lift) (i) $L(x+1,y) = L(x,y) + (1,0)$ $\forall (x,y)\in X$

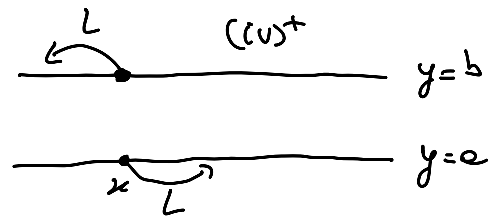
(*) (ii) L preserves the area

(iii) $L_2(x,a) = a$, $L_2(x,b) = b$ $\forall x$

(**) (iv) $(L_1(x,a) - x)(L_1(x,b) - x) < 0$ $\forall x$

$(iv)^+$ $L_1(x,a) > x$, $L_1(x,b) < x$

$(iv)^-$ $L_1(x,a) < x$, $L_1(x,b) > x$



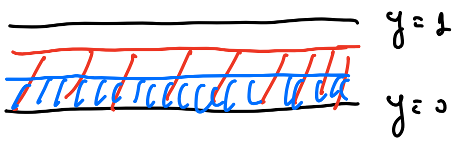
Then L admits at least two fixed points P, Q which are not equivalent, i.e. $\forall k\in\mathbb{Z}$ $P+(k,0)\neq Q$

(If P is fixed, $L(P+(k,0)) = L(P) + (k,0) = P+(k,0)$).

Remark • $L(x,y) = (x+y-\frac{1}{2}, y^2)$ on $\mathbb{R}\times[0,1]$

satisfies (i), (iii), (iv) $L(x,0) = (x-\frac{1}{2},0)$

$L(x,1) = (x+\frac{1}{2},1)$



L does not satisfy (ii)

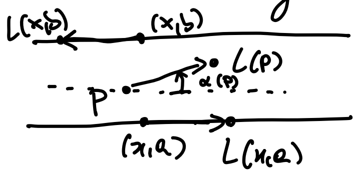
$L(x_0,y_0) = (x_0,y_0) \iff (x_0+y_0-\frac{1}{2}, y_0^2) = (x_0,y_0) \Rightarrow y_0 = 0,1$

$\Rightarrow x_0 \pm \frac{1}{2} = x_0$ impossible.

• $L(x,y) = (x+c, y)$ $\forall c>0$, does not satisfy (iv) and $\nexists$ fixed points.

proof Let's assume that L has no fixed points on $X = \mathbb{R} \times [a,b]$, and L satisfies (i)-(ii)-(iii)-(iv)$^+$.

Then let $\alpha : X \to \mathbb{R}$, $\alpha(P) = \text{angle}(\overrightarrow{PL(P)}, x \text{ axis})$ in anti-clockwise direction.



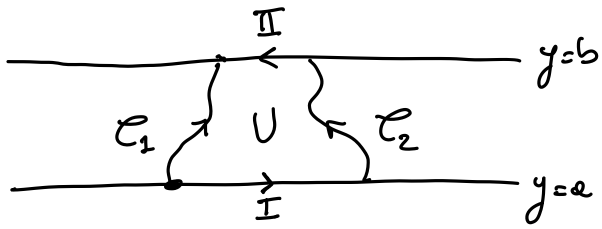
We fix that $\alpha(x,a) = 0$.

We introduce $j(L) := \alpha|_{\{y=b\}}$, then if $\mathcal{C}$ is a simple curve from $\{y=a\}$ to $\{y=b\}$ then $j(L) = \int_{\mathcal{C}} d\alpha$

We know that $j(L) = \pi \pmod{2\pi}$

__Lemma 1__ $\int_{\mathcal{C}} d\alpha$ does not depend on the choice of $\mathcal{C}$.

proof



$$\int_{\mathcal{C}_1} d\alpha - \int_{\mathcal{C}_2} d\alpha = \int_{\mathcal{C}_1} d\alpha - \int_{\mathcal{C}_2} d\alpha - \int_{\mathrm{I}} d\alpha - \int_{\mathrm{II}} d\alpha =$$

$$= - \int_{\underbrace{\mathrm{I} \cup \mathcal{C}_2 \cup \mathrm{II} \cup (-\mathcal{C}_1)}_{= \partial^+ U}} d\alpha = - \iint_U d^2\alpha = 0 \qquad \square$$

$d\alpha = \omega$

STOKES THM

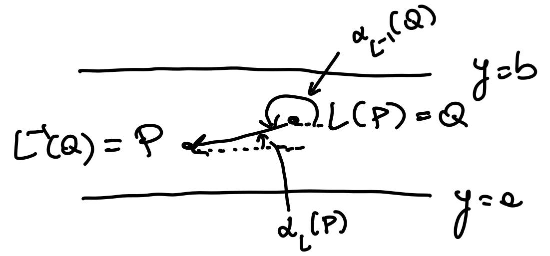
ω 1-form

$$\int_{\partial^+ U} \omega = \iint_U d\omega$$

__Lemma 2__ (i) $j(L^{-1}) = j(L)$

(ii) Let $\rho : X \to X$, $\rho(x,y) = (-x,y)$, then $\rho^{-1} L \rho$ satisfies (i)-(ii)-(iii)-(iv)$^-$ of the thm, has no fixed points, and $j(\rho^{-1}L\rho) = -j(L)$.

__proof__ (i)



$\Rightarrow \alpha_{L^{-1}}(Q) = \alpha_L(P) + \pi$

$$j(L^{-1}) = \int_{\mathcal{C}} d\alpha_{L^{-1}} = \int_{L^{-1}(\mathcal{C})} d\alpha_L = j(L)$$

(ii) • $\rho^{-1}L\rho$ satisfies:

(i) $\rho^{-1}L\rho(x+1,y) = \rho^{-1}L(-x-1,y) =$

$\rho^{-1}(L(-x,y) + (-1,0)) = \rho^{-1}((L_1(-x,y), L_2(-x,y)) + (-1,0))$

$= (-L_1(-x,y) + 1, L_2(-x,y)) =$

$= \rho^{-1}L\rho(x,y) + (1,0)$

(ii) $\rho^{-1}L\rho$ preserves the area

(iii) $\rho^{-1}L\rho$ preserves $\{y=a\}$ and $\{y=b\}$

(iv) $(\rho^{-1}L\rho(x,a))_1 - x = (\rho^{-1}L(-x,a))_1 - x =$

$= (\rho^{-1}(L_1(-x,a), L_2(-x,a)))_1 - x = -L_1(-x,a) - x \overset{\tilde{x}=-x}{=}$

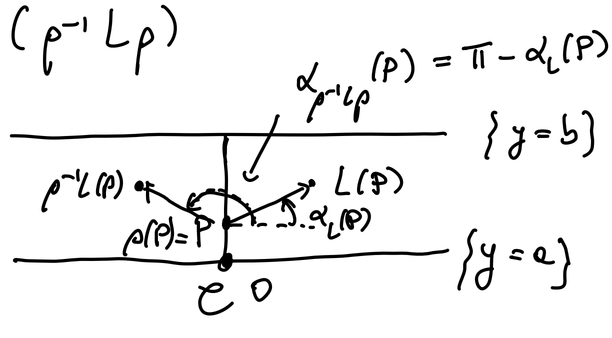
$= -L_1(\tilde{x},a) + \tilde{x} = -(L_1(\tilde{x},a) - \tilde{x}) < 0$

$(\rho^{-1}L\rho(x,b))_1 - x = -(L_1(-x,b) + x) > 0$

• $\rho^{-1}L\rho$ has no fixed points. If $\rho^{-1}L\rho(P) = P = (x_0,y_0)$

$\Rightarrow \begin{array}{l} -L_1(-x_0,y_0) = x_0 \\ L_2(-x_0,y_0) = y_0 \end{array} \Rightarrow (-x_0,y_0)$ is a fixed point for L_1.

• $j(\rho^{-1}L\rho)$



$\mathcal{C} = \{x=0\}$

$$j(\rho^{-1}L\rho) = \int_{\mathcal{C}} d\alpha_{\rho^{-1}L\rho} = \int_{\mathcal{C}} d\alpha_{\rho^{-1}L} = \int_{\mathcal{C}} d(\pi - \alpha_L) = -\int_{\mathcal{C}} d\alpha_L$$

$$= -j(L) \qquad \square \text{ L.2.}$$

<u>Lemma 3</u> If L satisfies (i)-(ii)-(iii)-(iv)$^+$ then
$$j(L) = \pi.$$

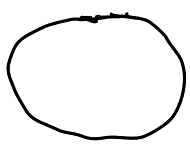
Using lemmas 1, 2, 3, $\rho^{-1}L^{-1}\rho$ satisfies (i)-(ii)-(iii)-(iv)$^+$ and has no fixed points, hence

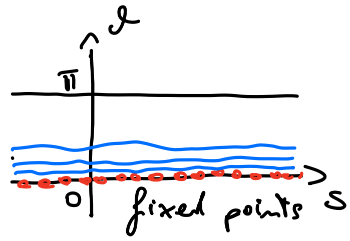
$$\pi \overset{L.3}{=} j(\rho^{-1}L^{-1}\rho) \overset{L.2}{=} -j(L^{-1}) \overset{L.2}{=} -j(L) \overset{L.3}{=} -\pi \quad \text{. Contradiction.}$$

Then L has at least one fixed point.

(To be concluded next time).

<u>Billiard map</u>





Lazutkin Thm, if $\partial\Omega$ is smooth enough and curvature does not vanish, there is a family of invariant curves accumulating to the boundary $\{\vartheta = 0\}$ which are homotopically non trivial.

<u>Standard map</u>

$$T_k(x,y) = \left(x + y - \frac{k}{2\pi}\sin(2\pi x),\ y - \frac{k}{2\pi}\sin(2\pi x) \right)$$

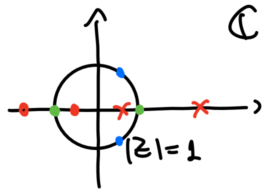
$$\boxed{k \geq 0.}$$

$$\text{Fixed points} = \{(0,0),\ (\tfrac{1}{2},0)\}$$

$$JT_k(x,y) = \begin{pmatrix} 1 - k\cos(2\pi x) & 1 \\ -k\cos(2\pi x) & 1 \end{pmatrix}$$

$\det JT_k(x,y) = 1 \implies$

- $\lambda, \frac{1}{\lambda}$ are real eigenvalues with $|\lambda| > 1$ (hyperbolic case)
- $\lambda, \bar{\lambda} \in \mathbb{C}$ with $|\lambda| = 1$ (elliptic case)
- $\lambda \in \{\pm 1\}$ (parabolic case)



$$JT_k(0,0) = \begin{pmatrix} 1-k & 1 \\ -k & 1 \end{pmatrix} \qquad tr = 2-k$$

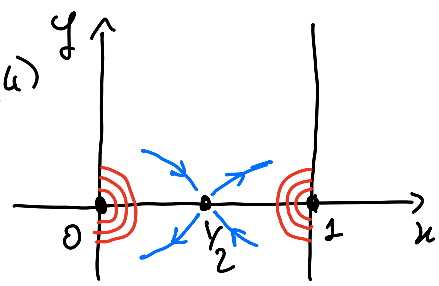
$$\lambda_\pm = \frac{2-k \pm \sqrt{k(k-4)}}{2}$$

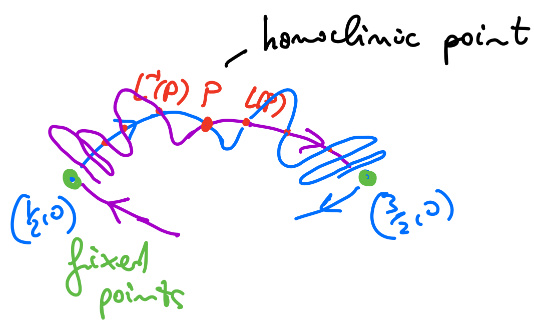
$k \in (0,4) \to$ elliptic case , $k > 4 \to$ hyperbolic case

$$JT_k(\tfrac{1}{2},0) = \begin{pmatrix} 1+k & 1 \\ k & 1 \end{pmatrix} \qquad tr = 2+k$$

$$\lambda_\pm = \frac{2+k \pm \sqrt{k(k+4)}}{2} \qquad \text{hyperbolic case } \forall k > 0$$

$k \in (0,4)$





$L : \mathbb{R}\times[a,b] \to \mathbb{R}\times[a,b]$ order preserving homeo, $L = (L_1, L_2)$

(i) $L(x+1,y) = L(x,y) + (1,0)$

(ii) L is area preserving

(iii) $L_2(x,a) = a$, $L_2(x,b) = b$ $\forall x$

(iv)$^+$ $L_1(x,a) - x > 0$, $L_1(x,b) - x < 0$ (twist)

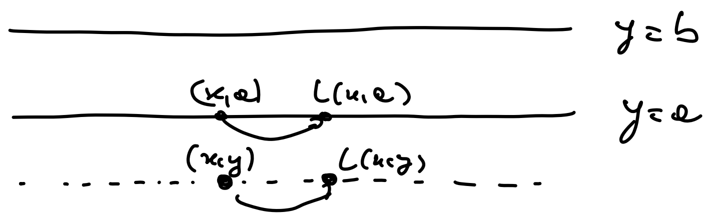
If L has no fixed points, $j(L) = \int_{\mathcal{C}} d\alpha$, $\alpha(x,a) = 0$ $\forall x$

($\alpha(P)$ = angle $(\overrightarrow{P\,L(P)}$, x axis), $\mathcal{C}$ is a simple curve connecting $\{y=a\}$ and $\{y=b\}$)

Then $j(L^{-1}) = j(L)$, $j(\rho^{-1}L\rho) = -j(L)$ where $\rho(x,y) = (-x,y)$.

<u>Lemma 3</u> Under assumption (i)-(iv)$^+$ we have $j(L) = \pi$

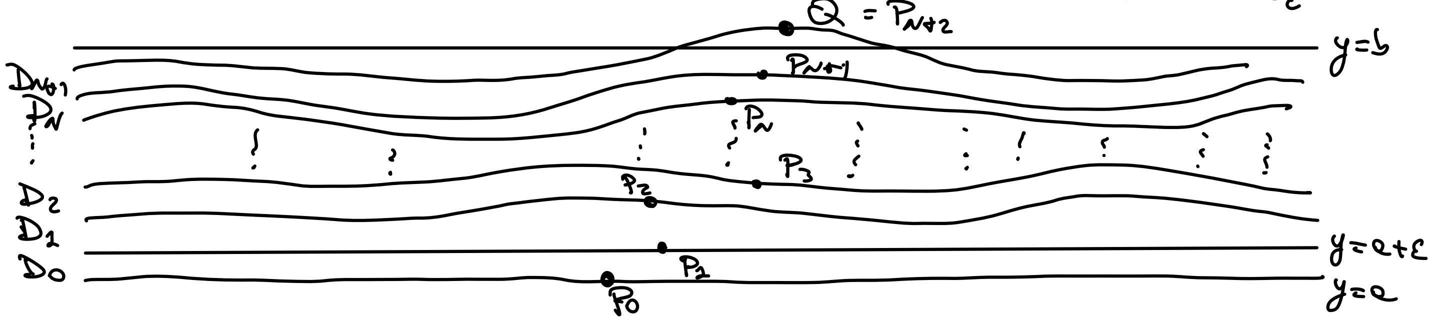
<u>proof</u> Let extend $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L_1(x,y) = L_1(x,a)$ $\forall y \le a$, $L_2(x,y) = y$ $\forall y \le a$, and $L(x,y) = (L_1(x,b), y)$ $\forall y \ge b$.



Let $\min_{\mathbb{R}\times[a,b]} d(P, L(P)) > 0$ and choose $\varepsilon < \frac{1}{2} \min_{\mathbb{R}\times[a,b]} d(P, L(P))$.

Define $\tilde{L}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ by $\tilde{L}_\varepsilon(x,y) = L(x,y) + (0,\varepsilon)$

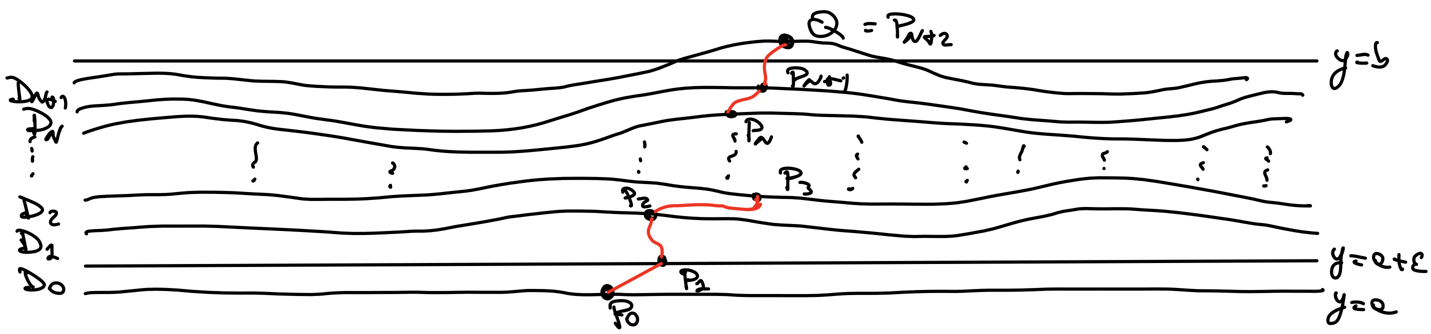
Then $\tilde{L}_\varepsilon$ has no fixed points and it is well defined $\alpha_{\tilde{L}_\varepsilon}$.



$D_0 = \{ a \le y < a+\varepsilon \}$, $D_j := \tilde{L}_\varepsilon^j (D_0)$

If $j<0$, $\tilde{L}_\varepsilon^{-1}(D_0) = \{ a-\varepsilon \le y < a \}$, $\tilde{L}_\varepsilon^j(D_0) = \{ a+j\varepsilon \le y < a+(j+1)\varepsilon \}$
then $\{D_j\}_{j<0}$ are disjoint.
In particular $\{D_j\}_{j\in\mathbb{Z}}$ are disjoint, and $\tilde{L}_\varepsilon$ is area preserving.
Then $\exists N>0$ s.t. $D_0, D_1, D_2, \dots, D_N \subset \mathbb{R}\times[a,b]$ and D_{N+1} is not. There exist $Q \in \overline{D_{N+1}}$, $Q \notin \mathbb{R}\times[a,b]$, and Q is the point with largest y-component in $\overline{D_{N+1}}$.

Then let $P_0 \in \{y=a\}$, $P_j = \tilde{L}_\varepsilon^j(P_j) \in D_j$, with $Q = \tilde{L}_\varepsilon^{N+2}(P_0)$.



Let $\gamma : [0,\varepsilon] \to \mathbb{R}^2$ be $\gamma(t) = (1-\frac{t}{\varepsilon}) P_0 + \frac{t}{\varepsilon} P_1$, and let
$\gamma(t+\varepsilon j) = \tilde{L}_\varepsilon^j(\gamma(t))$ $\forall\, t\in[0,\varepsilon]$, for $j=1,\dots,N+1$.
So that $\gamma : [0, \varepsilon(N+2)] \to \mathbb{R}^2$

$[0,\varepsilon]$, $[\varepsilon, 2\varepsilon]$, $[2\varepsilon, 3\varepsilon]$, ... , $[\varepsilon(N+1), \varepsilon(N+2)]$
$\tilde{L}_\varepsilon(\gamma|_{[0,\varepsilon]})$ $\tilde{L}_\varepsilon(\gamma|_{[\varepsilon,2\varepsilon]})$ $\tilde{L}_\varepsilon^{N+1}(\gamma|_{[0,\varepsilon]})$

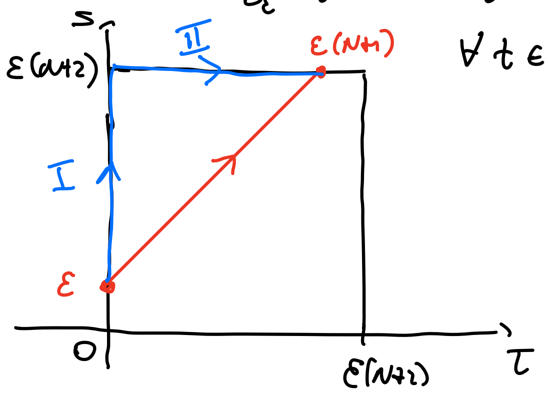
$\gamma(\varepsilon(N+2)) = \tilde{L}_\varepsilon^{N+1}(\gamma(\varepsilon)) =$
$= \tilde{L}_\varepsilon^{N+1}(P_1) = P_{N+2} = Q$

Let then consider $\displaystyle\int_{\gamma([0,\varepsilon(N+1)])} d\alpha_{\tilde{L}_\varepsilon}$.

Let $\beta : [0, \varepsilon(N+2)] \times [0, \varepsilon(N+2)] \to \mathbb{R}$ to be

$$\beta(\tau, s) := \text{angle} (\overrightarrow{\gamma(\tau)\gamma(s)} , x \text{ axis})$$

so that $\alpha_{\tilde{L}_\varepsilon}(\gamma(t)) = \text{angle}\left(\overrightarrow{\gamma(t)\,\tilde{L}_\varepsilon(\gamma(t))}, x \text{ axis}\right) = \beta(t, t+\varepsilon)$



$\forall t \in [0, \varepsilon(N+1)]$

$$\int_{\gamma[0,\varepsilon(N+1)]} d\alpha_{\tilde{L}_\varepsilon} = \underline{\underline{\int_0^{\varepsilon(N+1)}}} d\beta(t, t+\varepsilon) =$$

$$= \int_{I} d\beta + \int_{II} d\beta$$

$\beta|_I = \beta(0, s) = \text{angle}\left(\overrightarrow{P_0\,\gamma(s)}, x \text{ axis}\right)$

$\Rightarrow \int_I d\beta \in (0, \pi)$ for ε small enough

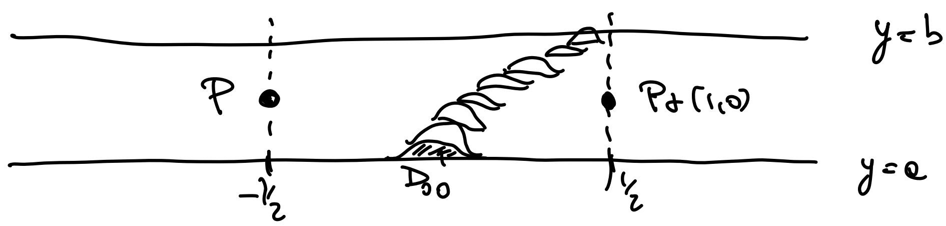
$\beta|_{II} = \beta(\tau, \varepsilon(N+2)) = \text{angle}\left(\overrightarrow{\gamma(\tau)\,Q}, x \text{ axis}\right)$

$\Rightarrow \int_{II} d\beta \in (0, \pi)$ for ε small enough

Then for ε small enough, $\int_{\gamma[0,\varepsilon(N+1)]} d\alpha_{\tilde{L}_\varepsilon} \in (0, 2\pi)$.

This implies that $j(L) \in (0, 2\pi) \Rightarrow j(L) = \pi$ $\square$ L3.

Then we have proved by contradiction that L has a fixed point.
Let P be this fixed point, $P = (-\frac{1}{2}, y_0)$.



If there are no fixed points for L in $(-\frac{1}{2}, \frac{1}{2}) \times [a,b]$, we can define $j(L)$ by using $\mathcal{C} \subset (-\frac{1}{2}, \frac{1}{2}) \times [a,b]$. We can repeat everything, and in L3 we set $\tilde{L}_\varepsilon(x,y) = L(x,y) + (0, \varepsilon\chi(x))$ where χ is a continuous 1-periodic function, $\int_0^1 \chi(x)\,dx > 0$, $\chi(x) \in [0,1]$

and $\chi(x) = 0$ for $x \in [-\frac{1}{8}, \frac{1}{8}]$. Then $D_0 = \{ -\xi \le x \le \xi, \, a \le y < a + \varepsilon \chi(x) \}$

$\square$

<u>Corollary</u> Let L satisfy the assumptions of Poincaré – Birkhoff Thm, and let $c_1, c_2 \in \mathbb{R}$ s.t. $c_1 < c_2$ and

$$L_1(x,b) - x \le c_1 < c_2 \le L_1(x,a) - x \quad \forall x.$$

Then $\forall \frac{p}{q} \in (c_1, c_2)$, $(p,q) = 1$, L has a periodic point P of type (p,q), that is $L^q(P) = P + (p,0)$.

<u>proof</u> Let $R_k : \mathbb{R} \times [a,b] \circlearrowleft$ be $R_k(x,y) = (x+k, y)$ $\forall k \in \mathbb{Z}$.

Then $\tilde{L} := R_{-p} \circ L^q$ satisfies (i)–(iv) of P-B. Thm.

(i) $\tilde{L}(x+1, y) = (R_{-p} \circ L^q)(x+1, y) = L^q(x - p + 1, y) =$

$= L^q(x,y) + (-p+1, 0) = (R_{-p} \circ L^q)(x,y) + (1,0)$.

(ii) ok

(iii) ok

(iv)$^+$ $(\tilde{L}(x,a))_1 - x = (R_{-p} \circ L^q(x,a))_1 - x = (L^q(x,a))_1 - x - p$

$$= \sum_{j=1}^{q} \underbrace{\left[L_1(L^{j-1}(x,a)) - (L^{j-1}(x,a))_1 \right]}_{c_2 \le L_1(Q) - Q_1 \text{ with } Q = (Q_1, a)} - p \ge$$

$\ge q \cdot c_2 - p > 0$

$(\tilde{L}(x,b))_1 - x < 0$.

Hence there exists P s.t. $R_{-p} \circ L^q(P) = P \Rightarrow L^q(P) = P + (p,0)$.

Moreover let $\frac{p}{q} \neq \frac{p'}{q'} \in (c_1, c_2)$ and P, P' satisfy

$$L^q(P) = R_p(P), \quad L^{q'}(P') = R_{p'}(P').$$

Then P and P' are not equivalent.

If $\exists\, k \in \mathbb{Z}$ s.t. $P = R_k(P')$ then

$$L^q(P') = L^q(R_{-k}(P)) = R_{-k}(L^q(P)) = R_{-k}(R_p(P)) =$$
$$= R_p(P')$$
$$L^{q'}(P') = R_{p'}(P')$$

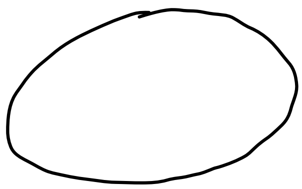
These imply that

$$L^{qq'}(P') = \begin{cases} (L^q)^{q'}(P') = R_p^{q'}(P') = R_{pq'}(P') \\ \qquad\qquad\qquad\qquad\qquad\qquad \shortparallel \\ (L^{q'})^q(P') = R_{p'}^q(P') = R_{p'q}(P') \end{cases}$$

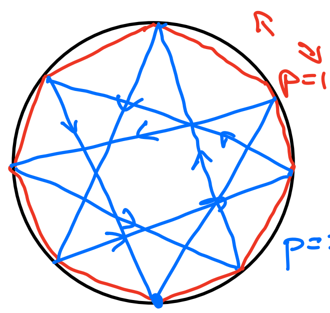
$\Rightarrow \quad pq' = p'q \quad$ FALSE. $\square$

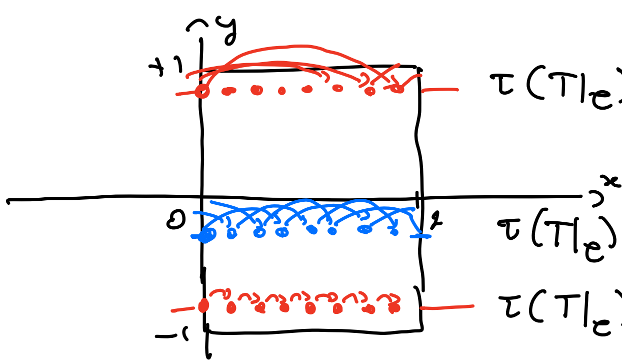
<u>Ex</u> Billiard map $T : S^1_x \times [-1,1]_y \circlearrowleft$ satisfies (i)-(iv) and $L_1(x,-1) - x = 0$, $L_1(x,1) - x = 1$

Then $\forall\, \frac{p}{q} \in (0,1)$ there are two periodic points of type (p,q).



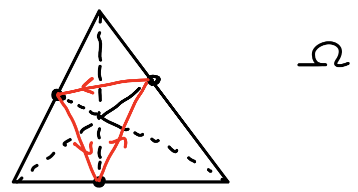
Ω circle

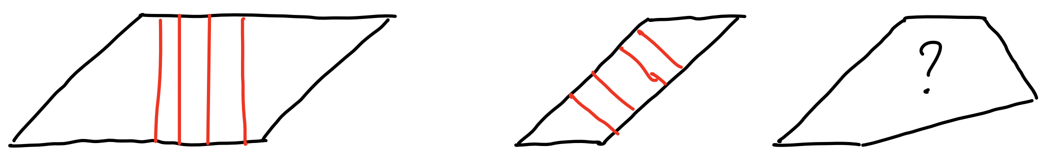




<u>Rem</u>

- If Ω is a triangle with all acute angles then there is a 3-period orbit (Fagnano)



- There are periodic orbits if Ω is a triangle with all angles $\leq 100°$.
- If Ω is a triangle with irrational angles (not in $\mathbb{Q}\pi$) and not in the previous cases, it is not known whether one single periodic orbit exists.
- 

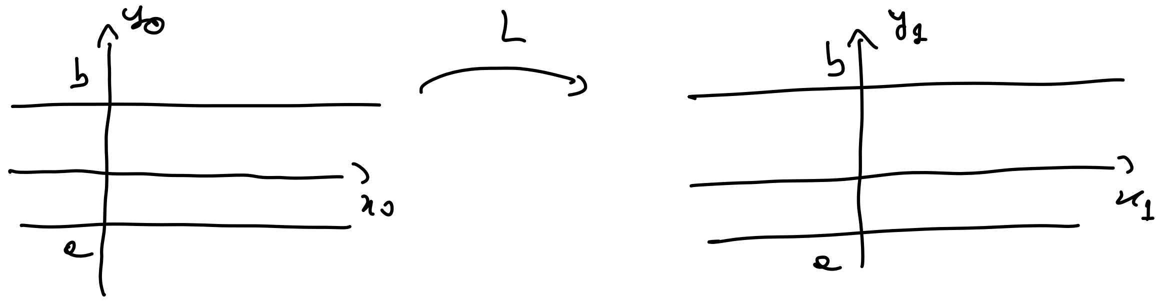
TWIST MAPS

• Aubry-Mather Theory

$L: \mathbb{R}\times(a,b)\to\mathbb{R}\times(a,b)$, $a\in\mathbb{R}\cup\{-\infty\}$, $b\in\mathbb{R}\cup\{+\infty\}$

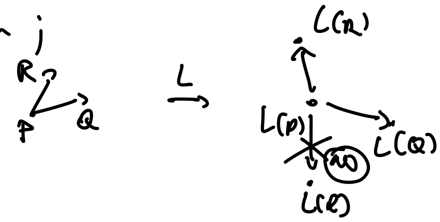
$L(x,y)=\big(L_1(x,y),L_2(x,y)\big)$, given $(x_0,y_0)\in\mathbb{R}\times(a,b)$ let $\{(x_n,y_n)\}_{n\in\mathbb{Z}}=\mathcal{O}(x_0,y_0)$, $(x_n,y_n)=L^n(x_0,y_0)$.

$(x_1,y_1)=L(x_0,y_0)$, $x_1=x_1(x_0,y_0)$, $y_1=y_1(x_0,y_0)$



<u>Def</u> L is a twist map if:

(i) L is a C^1 diffeomorphism;

(ii) L preserves orientation ();

(iii) $L(x+1,y)=L(x,y)+(1,0)$ $\forall(x,y)$ (L is the lift of $T: S^1\times(a,b)\circlearrowleft$)

(iv) $\lim_{y\to a^+} L_2(x,y)=a$, $\lim_{y\to b^-} L_2(x,y)=b$ (preservation of "boundaries")

(v) If $a\in\mathbb{R}$, $\exists\ \omega_-\in\mathbb{R}$ s.t. $L_1(x,a)=x+\omega_-$ $\forall x$

If $b\in\mathbb{R}$, $\exists\ \omega_+\in\mathbb{R}$ s.t. $L_1(x,b)=x+\omega_+$ $\forall x$

If $a=-\infty$, $\exists\ \omega_-\in\mathbb{R}$ s.t. $\omega_-=\lim_{y\to a^+}(L_1(x,y)-x)$ $\forall x$

If $b = +\infty$, $\exists\, \omega_+ \in \mathbb{R}$ s.t. $\omega_+ = \lim_{y \to b^+} (L_1(x,y) - x)$ $\forall x$

(vi) $\frac{\partial L_1}{\partial y}(x,y) > 0$ $\forall (x,y)$ (twist)

(vii) L is exact symplectic, that is $\exists\, h(s,t) \in C^2$ in $\{\omega_- < t - s < \omega_+\}$ s.t.

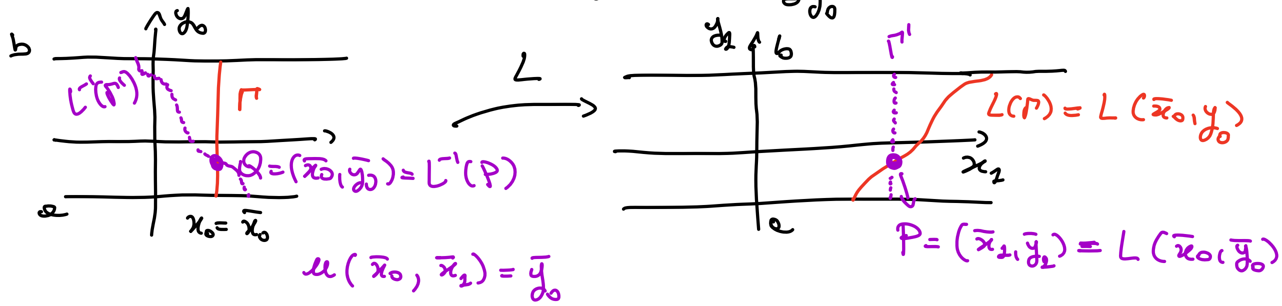
$$y_0(x_0,x_2) = y_0 = -\frac{\partial h}{\partial s}(x_0,x_2) \text{ and } y_2(x_0,x_2) = y_2 = \frac{\partial h}{\partial t}(x_0,x_2)$$

(or $dh(x_0,x_2) = y_2\, dx_2 - y_0\, dx_0$)

h is called the generating function of L

($\Rightarrow L$ is area preserving)

Rem (I) Twist $x_2 = x_2(x_0,y_0)$ $\qquad \frac{\partial x_2}{\partial y_0} > 0$



$\forall x_0$, $y_0 \mapsto x_2(x_0,y_0)$ is increasing, then there exist its inverse, $x_2 \mapsto u(x_0,x_2)$ s.t.

$$u(x_0, x_2(x_0,y_0)) = y_0 \quad \forall x_0.$$

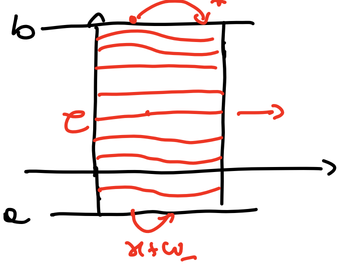
Then $y_2 = y_2(x_0,y_0)$ can be written as $y_2 = v(x_0,x_2)$ with $v(x_0,x_2) = y_2(x_0, u(x_0,x_2))$.

(II) L is twist then L^2 is not necessarily twist

(III) $\frac{\partial x_2}{\partial y_0} > 0 \Rightarrow \omega_- < \omega_+$

$x + \omega_+$

EX



$\tau(L|_e) \in [\omega_-, \omega_+]$
$\downarrow$
home of the circle

(IV) Let $y_0 = u(x_0, x_1)$, $y_1 = v(x_0, x_1)$

From (V) + (VI) , $u, v : \{\omega_- < x_1 - x_0 < \omega_+\} \to \mathbb{R}$

and $u(x_0, x_1(x_0, y_0)) = u(x_0, L_1(x_0, y_0)) = y_0$ $\quad (*)$

$v(x_0, x_1) = L_2(x_0, u(x_0, x_1)) = y_1$ $\quad (**)$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x_0}(x_0, x_1) + \frac{\partial u}{\partial x_1}(x_0, x_1)\frac{\partial L_1}{\partial x_0}(x_0, y_0) = 0} \qquad \frac{d}{dx_0}(*).$$

$$\frac{\partial u}{\partial x_1}(x_0, x_1) \cdot \frac{\partial L_1}{\partial y_0}(x_0, y_0) = 1 \qquad \frac{d}{dy_0}(*)$$

$$\boxed{\frac{\partial u}{\partial x_1}(x_0, x_1) = \frac{1}{\frac{\partial L_1}{\partial y_0}(x_0, u(x_0, x_1))}}$$

$$\boxed{\frac{\partial v}{\partial x_0}(x_0, x_1) = \frac{\partial L_2}{\partial x_0}(x_0, u) + \frac{\partial L_2}{\partial y_0}(x_0, u)\frac{\partial u}{\partial x_0}(x_0, x_1)} \qquad \frac{d}{dx_0}(**)$$

Then
$$\frac{\partial v}{\partial x_0}(x_0, x_1) + \frac{\partial u}{\partial x_1}(x_0, x_1) = \frac{\partial L_2}{\partial x_0} + \frac{\partial L_2}{\partial y_0}\frac{\partial u}{\partial x_0} + \left(\frac{\partial L_1}{\partial y_0}\right)^{-1} =$$

$$= \frac{\partial L_2}{\partial x_0} + \frac{\partial L_2}{\partial y_0}\left(-\frac{\partial L_1}{\partial x_0}\frac{\partial u}{\partial x_1}\right) + \left(\frac{\partial L_1}{\partial y_0}\right)^{-1} =$$

$$= \frac{\partial L_2}{\partial x_0} - \frac{\partial L_2}{\partial y_0}\frac{\partial L_1}{\partial x_0}\left(\frac{\partial L_1}{\partial y_0}\right)^{-1} + \left(\frac{\partial L_1}{\partial y_0}\right)^{-1} =$$

$$= \left(\frac{\partial L_1}{\partial y_0}\right)^{-1}\left[1 + \frac{\partial L_2}{\partial x_0}\frac{\partial L_1}{\partial y_0} - \frac{\partial L_2}{\partial y_0}\frac{\partial L_1}{\partial x_0}\right] =$$

$$= \left(\frac{\partial L_1}{\partial y_0}\right)^{-1}\left[1 - \det JL\right] \qquad JL = \begin{pmatrix} \frac{\partial L_1}{\partial x_0} & \frac{\partial L_1}{\partial y_0} \\ \frac{\partial L_2}{\partial x_0} & \frac{\partial L_2}{\partial y_0} \end{pmatrix}$$

If we assume that L is area preserving, that is $\det JL = 1$,

then $\frac{\partial v}{\partial x_0} + \frac{\partial u}{\partial x_1} = 0 \quad \forall (x_0, x_1)$, that is the vector field

$F(x_0, x_1) = (-u, v)$ on $\{ \omega_- < x_1 - x_0 < \omega_+ \}$ has

$\mathrm{curl}(F) \equiv 0$. Then F is conservative, that is $\exists\, h \in C^2$

s.t. $-u(x_0, x_1) = \frac{\partial h}{\partial x_0}$, $v(x_0, x_1) = \frac{\partial h}{\partial x_1}$.

$$-y_0 = \frac{\partial h}{\partial x_0} \qquad\qquad y_1 = \frac{\partial h}{\partial x_1}$$

Then L is "locally exact symplectic".

L is exact symplectic if $\boxed{h(x_0 + 1, x_1 + 1) = h(x_0, x_1)}$.

$$\boxed{\mathrm{curl}(F) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \qquad F = (F_1, F_2) \qquad \oint_\Gamma F \cdot ds = \iint_U \mathrm{curl}(F)\, dx\, dy}$$

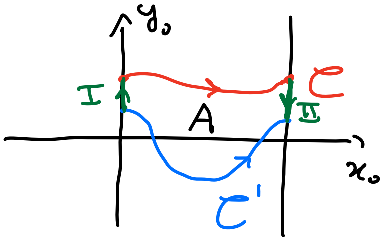
(V) Let $T : S^1 \times (a,b) \circlearrowleft$

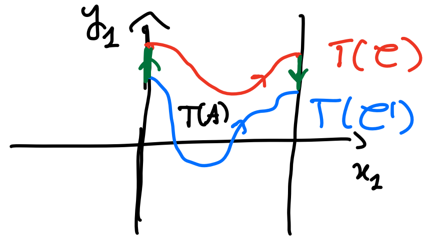
<u>Def</u> A simple closed curve $\mathcal{C}$ on $S^1 \times (a,b)$ is called ROTATIONAL if it is not contractible and goes once around the cylinder

<u>Def</u> The <u>net flux</u> of T is the number ϕ_T defined as follows: let $\mathcal{C}$ be a rotational curve and $T(\mathcal{C})$ its image. Then

$$\phi_T := \int_{T(\mathcal{C})}^{+} y_1\, dx_1 - \int_{\mathcal{C}}^{+} y_0\, dx_0$$

$\int_{\mathrm{II}} y\, dx = \int_{\mathrm{I}} y\, dx = 0$





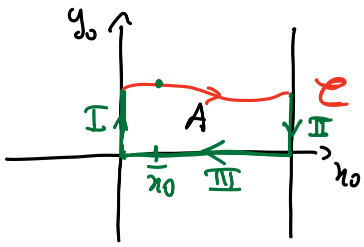
- ϕ_T does not depend on $\mathcal{C}$. Given $\mathcal{C}, \mathcal{C}'$ two vol. curves then

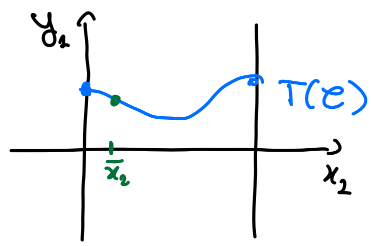
$$\left(\int_{T(\mathcal{C})} y_1\, dx_1 - \int_{\mathcal{C}} y_0\, dx_0\right) - \left(\int_{T(\mathcal{C}')} y_1\, dx_1 - \int_{\mathcal{C}'} y_0\, dx_0\right) =$$

$$= \left(\int_{T(\mathcal{C})} y_1\, dx_1 - \int_{T(\mathcal{C}')} y_1\, dx_1\right) - \left(\int_{\mathcal{C}} y_0\, dx_0 - \int_{\mathcal{C}'} y_0\, dx_0\right)$$

$$= -\iint_{T(A)} \underbrace{d(y_1\, dx_1)}_{= dy_1 \wedge dx_1} + \iint_A \underbrace{d(y_0\, dx_0)}_{= dy_0 \wedge dx_0} = \underbrace{-\mathrm{Area}(T(A)) + \mathrm{Area}(A)}_{=0}$$

- $\int^{+}_{\mathcal{C}} y_0\, dx_0 =$

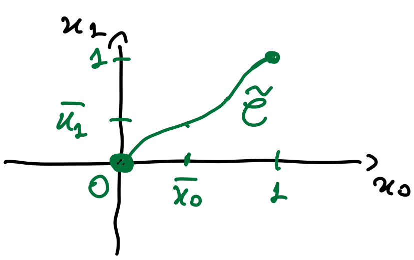




$$= \int^{+}_{\mathcal{C}} y_0\, dx_0 + \int_{\mathrm{I} \cup \mathrm{II} \cup \mathrm{III}} y_0\, dx_0 = -\mathrm{Area}(A).$$

- $$\phi_T = \int_{T(\mathcal{C})} y_1\, dx_1 - \int_{\mathcal{C}} y_0\, dx_0 = \int_{\mathcal{C}} \left(T^*(y_1\, dx_1) - y_0\, dx_0\right) =$$

$$= \int_{\mathcal{C}} \left(y_1(x_0, y_0)\, dx_1(x_0, y_0) - y_0\, dx_0\right) = \int_{\tilde{\mathcal{C}} \subset \mathbb{R}^2_{(x_0, x_1)}} dh(x_0, x_1) =$$



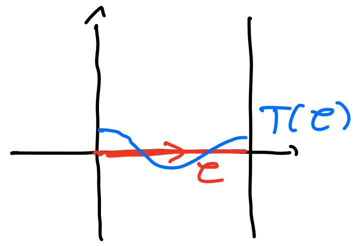
$$= h(1,1) - h(0,0) = 0$$

$T(\bar{x}_0, \bar{y}_0) = (\bar{x}_1, \bar{y}_1)$

Rem If $\mathcal{C}$ is a RIC, rotational invariant circle, then the set below $\mathcal{C}$ is invariant.

Rem $\mathcal{C} = \{y=0\}$, then $T(\mathcal{C}) \cap \mathcal{C} \neq \emptyset$.

$\int_{\mathcal{C}} y_0 \, dx_0 = 0$ then $\phi_T = 0 = \int_{T(\mathcal{C})} y_1 \, dx_1$



Rem $L(x,y) = (x+y, y+\frac{1}{2})$ is not a twist map.

$L : \mathbb{R} \times (a,b) \circlearrowleft$ twist maps s.t. $L = (L_1, L_2)$, $\frac{\partial L_1(x,y)}{\partial y} > 0$,

and $\exists\, h : \{ \omega_- < x_1 - x_0 < \omega_+ \} \to \mathbb{R}$ s.t. $y_0 = -\partial_1 h(x_0, x_1)$ and $y_1 = \partial_2 h(x_0, x_1)$ satisfy $L(x_0, y_0) = (x_1, y_1)$.

h is called generating function.

If h is given s.t. $h(x_0+1, x_1+1) = h(x_0, x_1)$, we then define L which is a twist map if $\frac{\partial^2 h}{\partial x_0 \partial x_1} < 0$.

Since $\frac{\partial x_1}{\partial y_0} > 0 \Rightarrow y_0 = y_0(x_0, x_1)$ satisfies $\frac{\partial y_0}{\partial x_1} > 0 \Rightarrow$

$$\frac{\partial}{\partial x_1}\left(-\partial_1 h\right) = -\frac{\partial^2 h}{\partial x_1 \partial x_0} > 0 .$$

__Examples__

1. Standard map

$$\begin{cases} x_1 = x_0 + y_0 + g(x_0) \\ y_1 = y_0 + g(x_0) \end{cases}, \quad g \text{ 1-periodic } \left[g(x) = -\frac{k}{2\pi} \sin(2\pi x) \right]$$

$$y_0 = -\partial_1 h(x_0, x_1), \quad y_1 = \partial_2 h(x_0, x_1)$$

$$\Downarrow$$

$$\partial_1 h(x_0, x_1) = -\left(x_1 - x_0 - g(x_0)\right) \Rightarrow h(x_0, x_1) = \tfrac{1}{2}(x_1 - x_0)^2 + \int g(x_0)\, dx_0 + c(x_1)$$

Then $y_1 = \partial_2 h(x_0, x_1) = x_1 - x_0 + c'(x_1)$

$$\Rightarrow \quad y_0 + g(x_0) = x_1 - x_0 + c'(x_1) \Rightarrow c'(x_1) = 0 .$$

The generating function is $h(x_0, x_1) = \frac{1}{2}(x_1 - x_0)^2 + \int g(x_0)\, dx_0$

If $g \in C^1$ then $h \in C^2$ and

- $h(x_0+1, x_1+1) = \frac{1}{2}(x_1-x_0)^2 + \int g(x_0+1)\,dx_0 = h(x_0, x_1)$
- $\dfrac{\partial^2 h}{\partial x_0 \partial x_1} = \dfrac{\partial}{\partial x_0}(x_1 - x_0) = -1 < 0$

For the standard map $h(x_0, x_1) = \frac{1}{2}(x_1-x_0)^2 + \frac{k}{4\pi^2}\cos(2\pi x_0)$

2. Bouncing balls in the gravity field $(g = 1)$

$$\begin{cases} t_1 = t_0 + v_0 \\ v_1 = v_0 + 2\dot{f}(t_0+v_0) \end{cases}, \quad f \in C^2 \text{ 1-periodic.}$$

$h(t_0, t_1)$ s.t. $\quad \partial_1 h(t_0, t_1) = -v_0, \quad \partial_2 h(t_0, t_1) = v_1$

$$\Downarrow$$

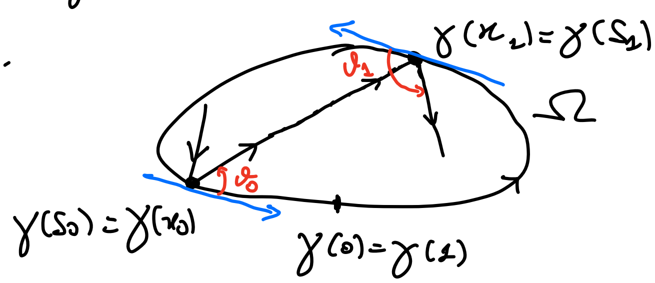
$$h(t_0, t_1) = \tfrac{1}{2}(t_1 - t_0)^2 + c(t_1)$$

$$v_1 = v_0 + 2\dot{f}(t_1) = \partial_2 h(t_0, t_1) \Rightarrow c'(t_1) = 2\dot{f}(t_1)$$

Then $\quad h(t_0, t_1) = \frac{1}{2}(t_1 - t_0)^2 + 2f(t_1)$

3. Billiard map in a strictly convex domain Ω with $\partial\Omega \in C^2$.

In Birkhoff coordinates, (x, y) with $x = s$ the parameter describing $\partial\Omega$, $y = -\cos\vartheta \in [-1, 1]$ with ϑ the angle of impact.



$L : \mathbb{R} \times [-1, 1] \circlearrowleft, \quad L(x, -1) = (x, -1), \quad L(x, 1) = (x+1, 1)$

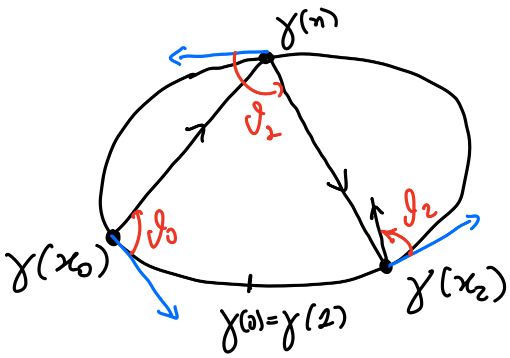
L is area-preserving.

Twist. $\frac{\partial L_1}{\partial y} > 0 \iff \frac{\partial x_1}{\partial y_0} > 0$

Generating function. If $g(s_0, s_1) = d(\gamma(s_0), \gamma(s_1))$ then

$$\frac{\partial g}{\partial s_0} = -\cos\vartheta_0 \quad \text{and} \quad \frac{\partial g}{\partial s_1} = \cos\vartheta_1$$

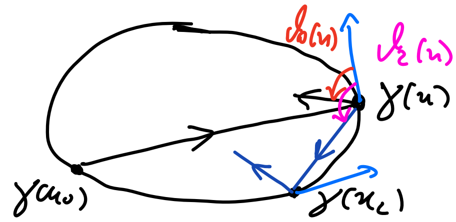
$h(x_0, x_1) = -d(\gamma(x_0), \gamma(x_1))$ is a generating function.



Find ϑ_0 s.t. the ball hits the boundary at some point $\gamma(x) \neq \gamma(x_2)$, and arrives at $\gamma(x_2)$.

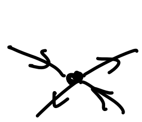
Let $H(x) := h(x_0, x) + h(x, x_2)$, then

$$H'(x) = \underbrace{\partial_2 h(x_0, x)}_{= -\cos\vartheta_0(x)} + \underbrace{\partial_1 h(x, x_2)}_{= \cos\vartheta_2(x)}$$



hence $\gamma(x)$ is the bouncing point to go from $\gamma(x_0)$ to $\gamma(x_2)$ if $H'(x) = 0$. The x is a critical point for the length of the path from $\gamma(x_0)$ to $\gamma(x_2)$.

4. Birkhoff normal form



<u>Thm</u> Let $T: \mathbb{R}^2_x \to \mathbb{R}^2_x$ be a diffeomorphism s.t. $T(\underline{0}) = \underline{0}$, $\det JT(\underline{0}) = 1$ and Eigenv. $JT(\underline{0}) = \{\lambda, \bar{\lambda}\}$, $\lambda = e^{2\pi i\alpha}$.

If $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and $n\alpha \notin \mathbb{Q}$ $\forall$ $n = 1, \dots, q-1$ then

$\exists\ U(\underline{0}) \subset \mathbb{R}^2_x$, $V(\underline{0}) \subset \mathbb{R}^2_y$ and $\phi : U(\underline{0}) \to V(\underline{0})$ s.t.
ϕ is a symplectic diffeomorphism and $\tilde{T} = \phi \circ T \circ \phi^{-1} : V(\underline{0}) \to \mathbb{R}^2_y$
can be written in polar coordinates as

$$\tilde{T}(\vartheta, r) = \left(\vartheta + \alpha + p(r^2) + O(r^{2m}) \,,\ r + O(r^{2m}) \right)$$

with $2m+1 < q-1$ and $p(t) = a_1 t + a_2 t^2 + \dots + a_m t^m$.

$$(\vartheta, r_0) \mapsto (\vartheta + \alpha + p(r_0^2), r_0) + \text{h.o.t.}$$

Then if $\exists\, a_j \neq 0$ then $\dfrac{\partial \tilde{T}_1}{\partial r} = 2r\, p'(r^2) \neq 0$ at some r_0 small enough.

hence $\tilde{T}$ is a twist map in a neighbourhood of $\{r = r_0\}$.

<u>Def</u> Let L be a twist map with generating function h, we call <u>action</u> of a sequence $\{x_M, \dots, x_m\}$ the function

$$W(x_M, \dots, x_m) := \sum_{k=M}^{m-1} h(x_k, x_{k+1})$$

<u>Prop</u> Let $\{x_M, \dots, x_m\}$ be the projections on $\mathbb{R}$ of points of an orbit of L, then $W(x_M, \xi_{M+1}, \dots, \xi_{m-1}, x_m)$ as a function of $(\xi_{M+1}, \dots, \xi_{m-1})$ has a critical point at $(x_{M+1}, \dots, x_{m-1})$.
And viceversa.

<u>proof</u>

$$W(x_M, \xi_{M+1}, \dots, \xi_{m-1}, x_m) = h(x_M, \xi_{M+1}) + \sum_{k=M+1}^{m-2} h(\xi_k, \xi_{k+1}) + h(\xi_{m-1}, x_m)$$

$$\frac{\partial W}{\partial \xi_k} = \partial_2 h(\xi_{k-1}, \xi_k) + \partial_1 h(\xi_k, \xi_{k+1}) \quad \begin{cases} \xi_M = x_M & k = M+1 \\ \xi_{k+1} = x_m & k = m-1 \end{cases}$$

Then $\dfrac{\partial W}{\partial \xi_k}(x_{M+1}, \dots, x_{m-1}) = 0 \ \forall k \iff \partial_2 h(x_{k-1}, x_k) + \partial_1 h(x_k, x_{k+1}) = 0 \ \forall k$

If we let $y_k := -\partial_1 h(x_k, x_{k+1})$ and $Y_k := \partial_2 h(x_{k-1}, x_k)$

then $y_k = Y_k$ and $\{(x_k, y_k)\}_{k=n}^{m}$ is an orbit of L. $\square$

Thm (Poincaré - Birkhoff) Let L be a twist map, $L : \mathbb{R} \times (a,b) \circlearrowleft$ and $\omega_+ := \lim_{y \to b^-} (L_1(x,y) - x)$, $\omega_- := \lim_{y \to a^+} (L_1(x,y) - x)$.

Then for all $\frac{p}{q} \in (\omega_-, \omega_+)$, $(p,q) = 1$, there exists a periodic orbit of type (p,q), that is $\exists\, (x_0, y_0)$ s.t. $L^q(x_0,y_0) = (x_0 + p, y_0)$.

proof • Let us consider the function

$W(x_0, \xi_1, \dots, \xi_{q-1}, x_0+p)$ of variables $(x_0, \xi_1, \dots, \xi_{q-1})$

$$W(x_0, \xi_1, \dots \xi_{q-1}, x_0+p) = h(x_0, \xi_1) + \sum_{k=1}^{q-2} h(\xi_k, \xi_{k+1}) + h(\xi_{q-1}, x_0+p)$$

If $(x_0, x_1, \dots, x_{q-1})$ is a critical point then we find the orbit (x_0, y_0), $(x_1, y_1), \dots, (x_{q-1}, y_{q-1})$, (x_0+p, y_q) where

$$y_0 = -\partial_1 h(x_0, x_1), \quad y_q = \partial_2 h(x_{q-1}, x_0+p)$$

Consider $h(x_{q-1}, x_0+p) + h(x_0+p, x_1+p) = h(x_{q-1}, x_0+p) + h(x_0, x_1)$

then $y_q = -\partial_1 h(x_0+p, x_1+p) = -\partial_1 h(x_0, x_1) = y_0$.

• Let $(x_0, \xi_1, \dots, \xi_{q-1})$ be in $0 \le x_0 \le \xi_1 \le \xi_2 \le \dots \le \xi_{q-1} \le x_0+p \le 1+p$

hence $W|_{\text{compact set}}$, so that W admits a minimum.

• We need to show that the point of minimum, $\{x_0, x_1, \dots, x_{q-1}\}$ satisfies $x_0 < x_1 < x_2 < \dots < x_{q-1} < x_0+p$.

If by contradiction $\exists\, k$ s.t. $x_{k-1} < x_k = x_{k+1} < x_{k+2}$ then

$$y_k = \partial_2 h(x_{k-1}, x_k) = -\partial_1 h(x_k, x_{k+1})$$
$$y_{k+1} = \partial_2 h(x_k, x_{k+1}) = -\partial_1 h(x_{k+1}, x_{k+2})$$
$$\Downarrow$$

$$0 = \partial_2 h(x_{k-1}, x_k) - \partial_2 h(x_k, x_{k+1}) + \partial_1 h(x_k, x_{k+1}) - \partial_1 h(x_{k+1}, x_{k+2})$$
$$\Downarrow \ x_k = x_{k+1}$$

$$0 = \partial_2 h(x_{k-1}, x_k) - \partial_2 h(x_k, x_k) + \partial_2 h(x_{k+1}, x_{k+1}) - \partial_2 h(x_{k+1}, x_{k+2})$$

$$\Downarrow h \in C^2$$

$$0 = \underbrace{\partial_1(\partial_2 h)(\eta_k, x_k)}_{<0} \underbrace{(x_{k-1} - x_k)}_{<0} + \underbrace{\partial_2(\partial_1 h)(x_{k+1}, \eta_{k+1})}_{<0} \underbrace{(x_{k+1} - x_{k+2})}_{<0}$$

This is absurd. $\square$

Rem If $\{x_0, x_1, \dots, x_{q-1}\}$ is a minimum point for $W(\xi_0, \xi_1, \dots, \xi_{q-1}, \xi_0 + p)$ with $x_0 \in [0,1]$, then $\forall k \in [1, \dots, q-1]$ $\exists j_k \in \mathbb{Z}$ s.t. $x_k + j_k \in [0,1]$ and the point

$$P_k = \{x_k + j_k, x_{k+1} + j_k, \dots, x_{q-1} + j_k, x_0 + p + j_k, x_1 + p + j_k, \dots, x_{k-1} + p + j_k\}$$

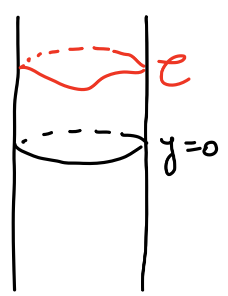
satisfies $W(P_k, x_k + p + j_k) = W(x_0, x_1, \dots, x_{q-1}, x_0 + p)$.

Then we expect W to have min-max critical points.

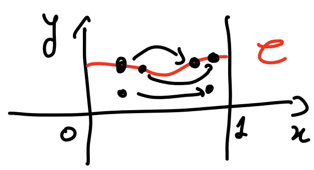
One can prove that minima give hyperbolic periodic points, whether min-max give elliptic or hyperbolic periodic points.

Rotational Invariant Circles

$\mathcal{C} \subset S^1 \times (a,b)$ s.t. $T(\mathcal{C}) = \mathcal{C}$ and $\mathcal{C}$ is a closed simple curve which is not contractible.



$$T(\{y \le \mathcal{C}\}) = \{y \le \mathcal{C}\}$$



Let $T_0 : S^1 \times (a,b) \circlearrowleft$ be a twist map, $T_0(x,y) = (x + \alpha(y), y)$ with $\alpha'(y) > 0$ $\forall y$.

Consider $T = T_0 + (f(x,y), g(x,y))$, so that remains a twist map.

<u>Theorem</u> Let's assume that $T_0 \in C^k$, with $k > 3$, and $f, g \in C^k$.

Then for all $\omega \in D(c,\nu)$ with $0 < \nu < \frac{k-3}{2}$, $\left(|\omega - \frac{p}{q}| > \frac{c}{q^{\nu+2}} \ \forall p,q \right)$

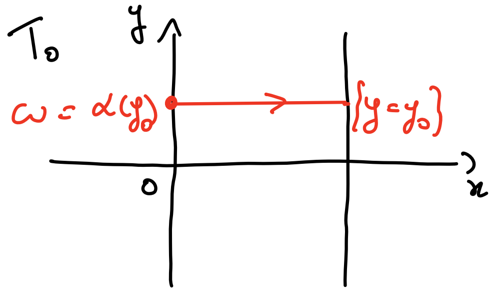
$\exists \, \varepsilon > 0$ s.t. if $|f|_{C^k} + |g|_{C^k} < c^2 \varepsilon$, there exists a rotational invariant circle $\mathcal{C}_\omega$ with the following properties:

(i) $\mathcal{C}_\omega$ is the graph of a Lipschitz function $\phi : S^1 \to \mathbb{R}$

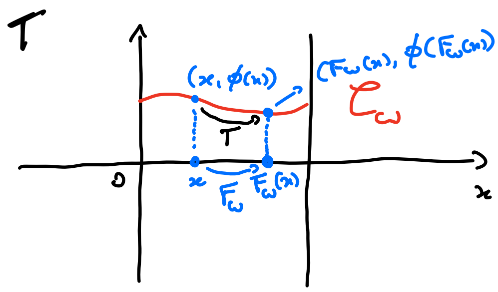
(ii) $\exists \; F_\omega : S^1 \to S^1$ homeo of the circle with $\tau(F_\omega) = \omega$ s.t.

$$\mathcal{C}_\omega \ni (x, \phi(x)) \longmapsto T(x, \phi(x)) = (F_\omega(x), \phi(F_\omega(x))) \in \mathcal{C}_\omega ,$$

and F_ω is conjugated to R_ω.



$$(x, y_0) \xmapsto{T_0} (x + \omega, y_0)$$

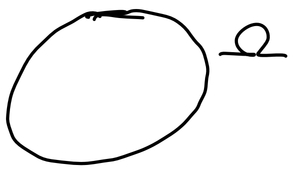


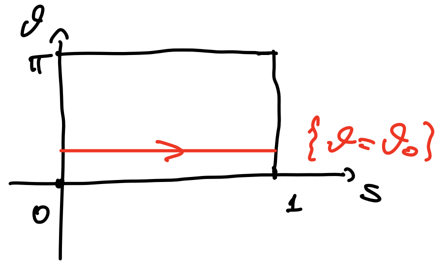
$$|T - T_0|_{C^k} < \varepsilon \, c^2$$

<u>Rem</u>

- Herman proved that $k = 3$ is sharp
- $m(D(c,\nu)) \xrightarrow[c \to 0]{} 1 \quad \forall \nu > 0 \Rightarrow$ "rotational invariant circles which persist for small enough pert. have positive Lebesgue measure"
- For the standard map it is known that $\mathcal{C}_\omega$ for $\omega = \frac{\sqrt{5}-1}{2}$ exists for $k \leq 0.91$. We know that for $k > 0.9716\ldots$ there are no rotational invariant circles (we will prove that this is true for $k \geq 4/3$).

Ex Billiard map.





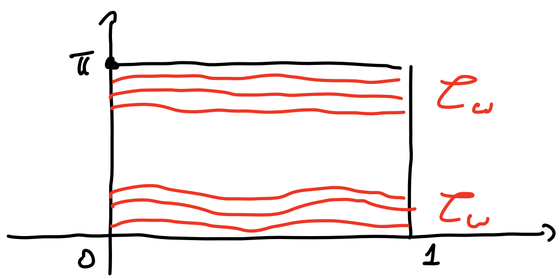
- The integrable case is $\Omega =$ circle

$$T(s, \vartheta_0) = (s + \alpha(\vartheta_0), \vartheta_0)$$

- Lazutkin in the 70's introduced a set coordinates (X, Y) so that for all Ω strictly convex with $\partial\Omega \in C^6$ and curvature strictly positive, the map can be written as

$$\begin{cases} X_1 = X_0 + Y_0 + f(X_0, Y_0)\, Y_0^3 \\ Y_1 = Y_0 + g(X_0, Y_0)\, Y_0^4 \end{cases} \qquad \text{with } f, g \in C^2$$

Then applying the theorem, Lazutkin proved the existence of rotational invariant circles for T close to the boundaries of the strip, $\{\vartheta = 0\}$ and $\{\vartheta = \pi\}$



This is not a foliation

<u>Birkhoff conjecture</u> There exists a neighbourhood of $\{\vartheta = 0\}$ and $\{\vartheta = \pi\}$ which is foliated by rotational invariant circles if and only if Ω is an ellipse.

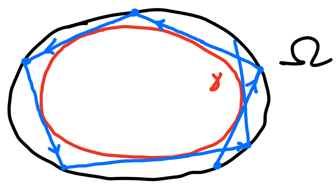
- If $S^1 \times [0, \pi]$ is foliated by rot. inv. circles then $\Omega =$ circle (Bialy)
- "If Ω is an ellipse of small eccentricity and Ω_ε is

a close enough set, then Ω_ε has a neighb. foliated by inv. circles if and only if it is an ellipse"

(Kaloshin - Sorrentino)

<u>Rem</u> • We will prove that if $\partial\Omega$ has one point with vanishing curvature then there are no rotational inv. circles.

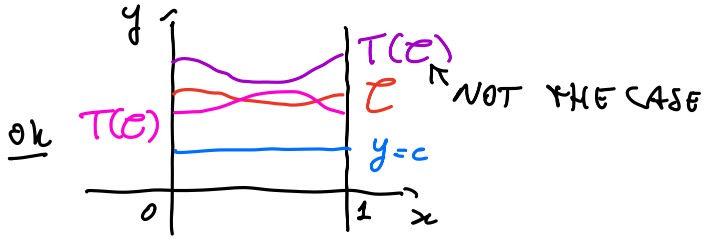
•



γ closed convex curve "caustic"

$T: S^1\times(a,b) \circlearrowleft$ twist map (positive $\frac{\partial x_1}{\partial y_0}>0$)

Let $h: D\subset\mathbb{R}^2\to\mathbb{R}$ be the generating function, so that $y_0=-\partial_1 h(x_0,x_1)$, $y_1=\partial_2 h(x_0,x_1)$, and we assume that $h(x_0+1,x_1+1)=h(x_0,x_1)$ (vanishing net flux). Then if $\mathcal{C}$ is a rotational circle, $T(\mathcal{C})\cap\mathcal{C}\neq\emptyset$.



Birkhoff's Theory

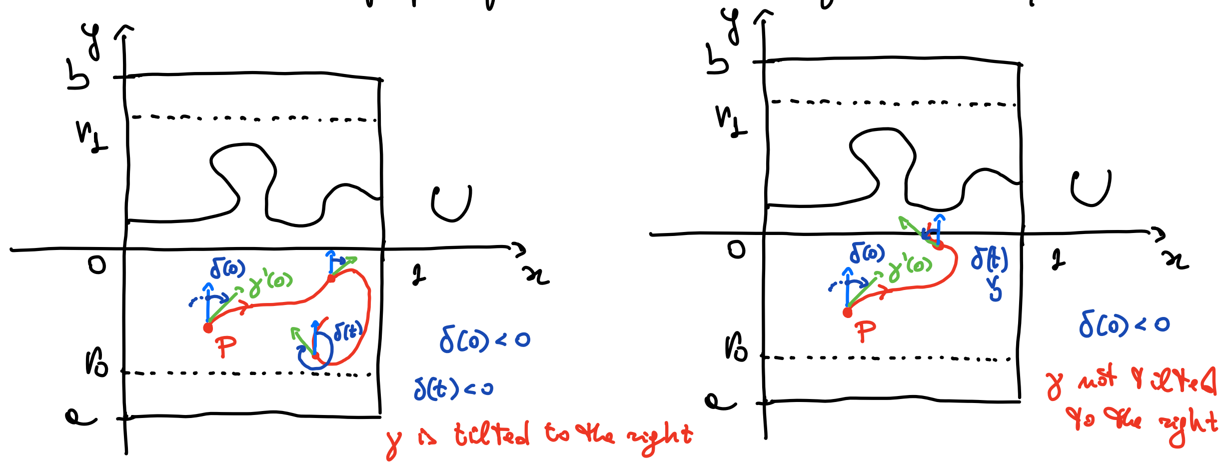
Thm Let $U\subset S^1\times(a,b)$ be an invariant set which is open and satisfies:

(i) $\exists\, r_0<r_1$ in (a,b) s.t. $S^1\times\{y<r_0\}\subset U\subset S^1\times\{y<r_1\}$;

(ii) U is homeomorphic to $S^1\times(a,b)$;

(iii) $cl(\mathring{U}) = cl(U)$

Then ∂U is the graph of a continuous function $\varphi: S^1\to\mathbb{R}$.

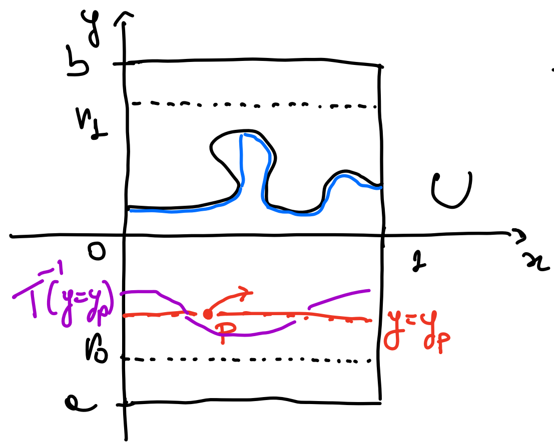


proof Let us consider the family of curves $\gamma:\mathbb{R}\to\mathbb{R}^2$ s.t. $\gamma(0)=P$ and $\|\gamma'(t)\|\neq 0$ $\forall\, t$. Let $\delta(t)$ be the angle between $\gamma'(t)$ and the positive y-axis in the anti-clockwise direction.

We consider curves γ as before for which $\delta(0) \in [-\frac{\pi}{2}, 0]$.

We say that such a curve γ is "tilted to the right" if $\delta(t) \in (-\pi, 0)$.

Then let $W^R := \{ (x,y) \in U \,;\, \exists\, \gamma$ tilted to the right for which $\gamma(t) = (x,y)$ for some $t \}$



If $W^R \subsetneqq U$ then

$\partial W^R \subset \partial U \cup \{\text{vertical segments}\}$.

Then $T^{-1}(W^R) \cap (W^R)^c \neq \emptyset$,

because T is twist.

$$\text{But } \operatorname{Area}\{T^{-1}(W^R \cap \{y > y_p\})\} \overset{\substack{T \text{ area-pres.}\\ \downarrow}}{=} \operatorname{Area}\{W^R \cap \{y > y_p\}\}$$

$$\parallel \leftarrow \text{net flux is zero}$$

$$\operatorname{Area}\{T^{-1}(W^R) \cap \{y > y_p\}\}$$

$$\Rightarrow \quad W^R = T^{-1}(W^R) \quad , \text{ a contradiction.}$$

Repeating the argument with curves tilted to the left, we obtain that ∂U has no lobes, hence ∂U is the graph of a continuous function. $\square$

Recall that $\mathcal{C}$ is a Rotational Invariant Circle (RIC) if $T(\mathcal{C}) = \mathcal{C}$ and $\mathcal{C}$ is a rotational circle.

<u>Theorem</u> Let $T : S^1 \times (a,b) \circlearrowleft$ be a twist map with

$\dfrac{\partial x_1}{\partial y_0} \geq k > 0$ for all (x_0, y_0) and $(x_1, y_1) = T(x_0, y_0)$. Then:

(i) all RIC are graphs of continuous functions;

(ii) all RIC are graphs of Lipschitz functions;

(iii) let $\mathcal{C}_1$ and $\mathcal{C}_2$ be RIC, and $\varphi_1 : S^1 \to \mathbb{R}$ and $\varphi_2 : S^1 \to \mathbb{R}$ be the funct. s.t. $\mathcal{C}_i = \text{graph}(\varphi_i)$, $i = 1, 2$. Then if $\varphi_1 < \varphi_2$ then $\tau(T|_{\mathcal{C}_1}) < \tau(T|_{\mathcal{C}_2})$;

(iv) For all $\alpha, \beta \in (a, b)$, the set of RIC in $S^1 \times [\alpha, \beta]$ is compact in the sup norm for the Lipschitz functions;

(v) $\{ \tau(T|_{\mathcal{C}}) : \mathcal{C}$ is a RIC in $S^1 \times [\alpha, \beta] \}$ is compact in $\mathbb{R}$;

(vi) let $r_0 < r_1$ in (a, b) be s.t. $T^n \{ y < r_0 \} \subset \{ y < r_1 \}$ for all $n \in \mathbb{Z}$, then there exist $\mathcal{C}$, a RIC, in $\{ r_0 < y < r_1 \}$;

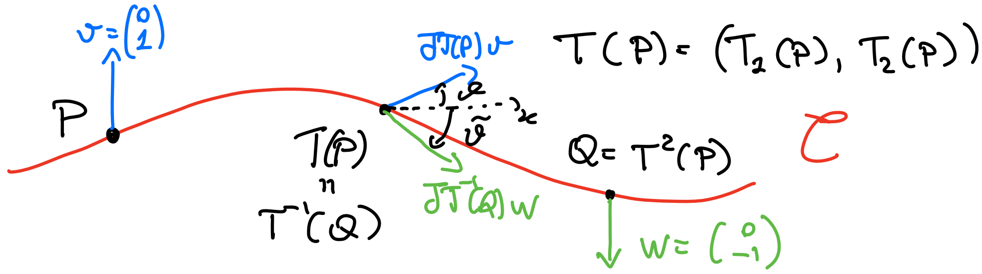
(vii) let $\mathcal{C}_1$ and $\mathcal{C}_2$ be RIC with $\tau(T|_{\mathcal{C}_1}) < \tau(T|_{\mathcal{C}_2})$, and assume that there is no RIC with rotation number $\tau \in (\tau(T|_{\mathcal{C}_1}), \tau(T|_{\mathcal{C}_2}))$. Then $\forall \varepsilon > 0$ $\exists$ P, $n \in \mathbb{Z}$ s.t. $d(P, \mathcal{C}_1) < \varepsilon$ and $d(T^n(P), \mathcal{C}_2) < \varepsilon$.

proof (i) RIC are graphs of cont. funct. $\varphi : S^1 \to \mathbb{R}$.

Let $U = \{ (x, y) \in S^1 \times (a, b) \,/\, (x, y)$ is below $\mathcal{C}$, a RIC$\}$

then $\partial U = \mathcal{C}$ is the graph of a cont. funct. by the previous thm.

(ii)



$$JT(P) = \begin{pmatrix} \frac{\partial T_1}{\partial x}(P) & \frac{\partial T_1}{\partial y}(P) \\ \frac{\partial T_2}{\partial x}(P) & \frac{\partial T_2}{\partial y}(P) \end{pmatrix} \qquad JT^{-1}(Q) = (JT(T(P)))^{-1} =$$

$$= \begin{pmatrix} \frac{\partial T_2}{\partial y}(T(P)) & -\frac{\partial T_1}{\partial y}(T(P)) \\ -\frac{\partial T_2}{\partial x}(T(P)) & \frac{\partial T_1}{\partial x}(T(P)) \end{pmatrix}$$

$$JT(P)\,v = \begin{pmatrix} \frac{\partial T_1}{\partial y}(P) \\ \frac{\partial T_2}{\partial y}(P) \end{pmatrix} \qquad \tan\vartheta = \frac{\partial T_2}{\partial y}(P) \cdot \left(\frac{\partial T_1}{\partial y}(P)\right)^{-1} \leq k^{-1} \left\| \frac{\partial T_2}{\partial y} \right\| = S_+$$

$$JT^{-1}(Q)\,w = \begin{pmatrix} \frac{\partial T_1}{\partial y}(T(P)) \\ -\frac{\partial T_1}{\partial x}(T(P)) \end{pmatrix} \qquad \tan\tilde{\vartheta} = -\frac{\partial T_1}{\partial x}(T(P)) \left(\frac{\partial T_1}{\partial y}(T(P))\right)^{-1} \geq$$

$$\geq -k^{-1} \left\| \frac{\partial T_1}{\partial x} \right\| = S_-$$

Then $\varphi : S^1 \to \mathbb{R}$ is a Lipschitz function

(iii) If $\varphi_1 < \varphi_2$ then $\tau(T|_{\mathcal{C}_1}) < \tau(T|_{\mathcal{C}_2})$.

$\mathcal{C} = \text{graph}(\varphi)$ is a RIC $\Rightarrow$ $\forall\,(x,y) \in \mathcal{C}$ we have

$(x,y) = (x, \varphi(x)) \longmapsto T(x,y) = (F(x), \varphi(F(x)))$ with

$F : S^1 \to S^1$ a homeo of the circle. Hence $\tau(T|_{\mathcal{C}_2}) = \tau(F)$.

Let F_1 and F_2 be the two circle homeos, then

$$F_2(x) - F_1(x) = T_1(x, \varphi_2(x)) - T_1(x, \varphi_1(x)) =$$

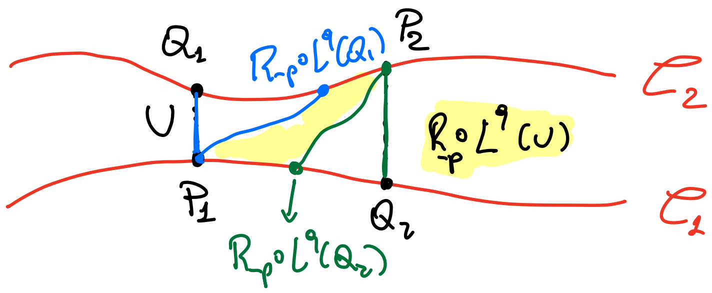
$$= \int_0^1 \frac{d}{ds} T_1(x, \varphi_1(x) + s(\varphi_2(x) - \varphi_1(x)))\, ds =$$

$$= \int_0^1 \frac{\partial T_1}{\partial y}(x, \varphi_1 + s(\varphi_2 - \varphi_1)) \cdot (\varphi_2(x) - \varphi_1(x))\, ds \geq k \cdot \underbrace{(\varphi_2(x) - \varphi_1(x))}_{>0}$$

$\Rightarrow$ $F_2 > F_1$ as circle homeos.

Then, if $\tau(F_1) \in \mathbb{R} \setminus \mathbb{Q}$ then $\tau(F_2) > \tau(F_1)$. If $\tau(F_1) = \frac{p}{q} \in \mathbb{Q}$, then $\tau(F_2) \geq \tau(F_1)$, and assume $\tau(F_2) = \tau(F_1)$. Then $\exists\, P_1 \in \mathcal{C}_1$, $P_2 \in \mathcal{C}_2$ s.t. the lift L of T satisfies $L^q(P_1) = P_1 + p$, $L^q(P_2) = P_2 + p$.

Let us consider the map $(x,y) \mapsto (R_{-p} \circ L^q)(x,y) := L^q(x,y) - (p,0)$

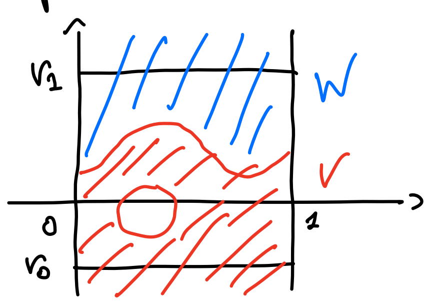


Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. If $x_1 = x_2$ then we have $R_{-p} \circ L^q(\overline{P_1 P_2})$ has endpoints in P_1 and P_2 and is tilted to the right. A contradiction with $x_1 = x_2$. Then let $x_2 > x_1$.

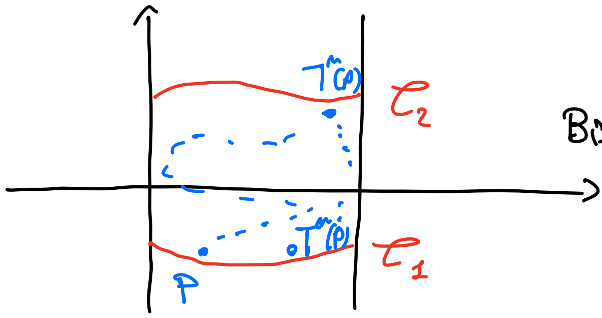
We know that $\mathrm{Area}(U) = \mathrm{Area}(R_{-p} \circ L^q(U))$, and it is a contradiction with the picture.

(iv)-(v) They follow from the continuity of φ, T and τ.

(vi) Let $T^n\{y < r_0\} \subset \{y < r_1\}$ for all $n \in \mathbb{Z}$, then let $V := \bigcup_{n \in \mathbb{Z}} T^n \{y < r_0\}$. We have that $V \subset \{y < r_1\}$ and W, the connected component of V^c contain $\{y \geq r_1\}$ is a closed invariant set, for which W^c satisfies the assumpt. of the prev. thm. Hence ∂W^c is a RIC.



(vii) Follows from (vi) by contradiction.



Birkoff Region of Instability.

$\square$

Aubry - Mather Theory $\quad T: S^1 \times (a,b) \circlearrowleft$

There is a set $(\omega_-, \omega_+) \subset \mathbb{R}$, we know that $\forall\, p, q$ s.t. $(p,q)=1$ and $\frac{p}{q} \in (\omega_-, \omega_+)$ $\exists$ two periodic orbits of type (p,q). (Poincaré - Birkhoff theorem). We know properties of RIC, $\mathcal{C}$, with $\tau(T|_{\mathcal{C}}) \in (\omega_-, \omega_+)$ <u>if</u> they exist (existence is guaranteed for Diophantine numbers $\omega \in (\omega_-, \omega_+)$ for small perturbations of integrable maps).

Let $h: D \subset \mathbb{R}^2 \to \mathbb{R}$ be the generating function of T. We have introduced the <u>action</u>

$$W(\xi_m, \xi_{m+1}, \dots, \xi_n) := \sum_{k=m}^{n-1} h(\xi_k, \xi_{k+1})$$

<u>Prop</u> A sequence $\{x_0, x_1, \dots, x_{q-1}\}$ is the projection on $\mathbb{R}$ of the points of a type (p,q) - periodic orbit for T and its lift L if and only if the sequence is a critical point for

$$(\xi_0, \xi_1, \dots, \xi_{q-1}) \longmapsto W(\xi_0, \xi_1, \dots, \xi_{q-1}, \xi_0 + p).$$

<u>Thm</u> (Poincaré - Birkhoff) The function $W(\xi_0, \dots, \xi_{q-1}, \xi_0 + p)$ has a minimum point in $\{\xi_0, \dots, \xi_{q-1} \,/\, 0 \le \xi_0 \le \xi_1 \le \dots \le \xi_{q-1} \le \xi_0 + p \le p+1\}$. Then W has at least another critical point which is a minmax.

$$\longleftrightarrow$$

<u>Def</u> Given a sequence $\{x_k\}_{k \in \mathbb{Z}} \subset \mathbb{R}$, we call <u>rotation number</u> $\omega := \lim_{n \to +\infty} \frac{x_n - x_0}{n}$.

<u>Def</u> A segment $\{x_m, x_{m+1}, \dots, x_n\}$ is a <u>minimizing point</u> if

$$W(x_m, x_{m+1}, \dots, x_n) \le W(\xi_m, \dots, \xi_n) \quad \forall \{\xi_m, \xi_{m+1}, \dots, \xi_n\} \text{ s.t.}$$

$\xi_m = x_m, \ \xi_n = x_n$.

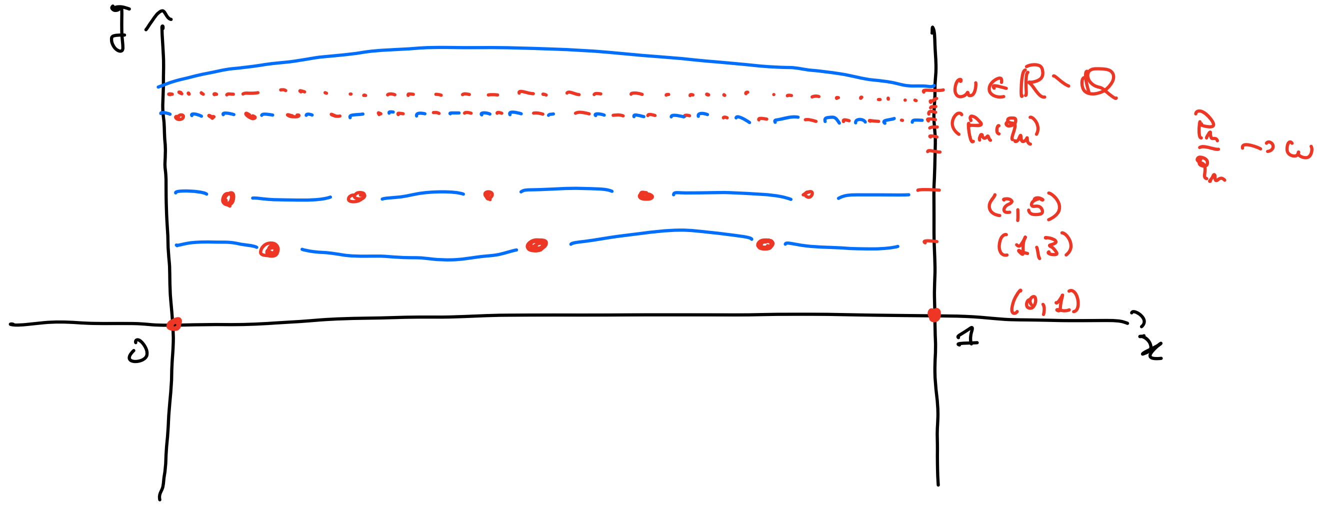
Prop If $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ is an orbit of L then each segment $\{x_m, \dots, x_n\}$ is a critical point of $(\xi_{m+1}, \dots, \xi_{n-1}) \mapsto W(x_m, \xi_{m+1}, \dots, x_n)$

And viceversa.

Def A sequence $\{x_k\}_{k\in\mathbb{Z}}$ is <u>minimal</u> if each segment $\{x_m, \dots, x_n\}$ is a minimizing point.

Def A sequence $\{x_k\}_{k\in\mathbb{Z}}$ is <u>monotone</u> if $x_k < x_j + p$ implies $x_{k+1} < x_{j+1} + p$.

Prop
- A minimal sequence is monotone
- For a monotone sequence, the rotation number exists and it is continuous wrt pointwise convergence of sequences.
- The sequences $\{x_k\}_{k\in\mathbb{Z}}$ associated to the periodic orbits of type (p, q) which have minimizing segments are minimal, and their rotation number is $\omega = \frac{p}{q} \in \mathbb{Q}$.
- For $\omega \in \mathbb{R} \setminus \mathbb{Q}$, let $\left\{\frac{p_n}{q_n}\right\} \subset \mathbb{Q}$ be a seq. satisfying $\frac{p_n}{q_n} \to \omega$. Then using the minimal periodic orbits of type (p_n, q_n), we show the existence of a recurrent invariant set which is a minimal orbit and has rotation number $\omega \in (\omega_-, \omega_+)$.
- A minimal orbit of T leaves on the graph of a Lipschitz function.



Summary • If $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ is a minimal orbit which is recurrent and has rotation number $\omega \in \mathbb{R}\setminus\mathbb{Q}$, then its closure M_ω in $S^1\times(a,b)$ can be a RIC or a Cantor set.

• If $\omega \in \mathbb{Q}$, the set M_ω of minimal orbits with rotation number ω, is given by periodic orbits and heteroclinic orbits.

$\longleftrightarrow$

CHAOS IN TWIST MAPS $\quad T: S^1\times(a,b) \circlearrowleft$

• Topological entropy $\quad h_{top}(T)$

Def Fixed $\varepsilon>0$, $m\in\mathbb{N}$, a set of points $\{P_k\}$ in $S^1\times(a,b)$ is called $\underline{(m,\varepsilon)\text{-separated}}$ if $\max\limits_{j=0,\dots,m} d(T^j(P_k), T^j(P_h)) > \varepsilon$ for all $k\neq h$.

$$h_{top}(T) = \lim_{\varepsilon\to 0}\left(\lim_{m\to+\infty} \frac{1}{m}\log \max\left\{\#(S_{m,\varepsilon}) : S_{m,\varepsilon} \text{ is a } (m,\varepsilon)\text{-separated set}\right\}\right)$$

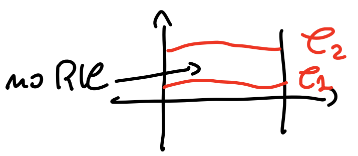
Def T is chaotic if $h_{top}(T)>0$.

• Metric entropy (or Kolmogorov-Sinai entropy) of an invariant measure.

In general there are ways to prove that $h_{top}(T) > 0$, but it is not known whether the metric entropy of m is positive. (In an informal way, the open problem is to show that the set of chaotic orbits has positive Lebesgue measure).

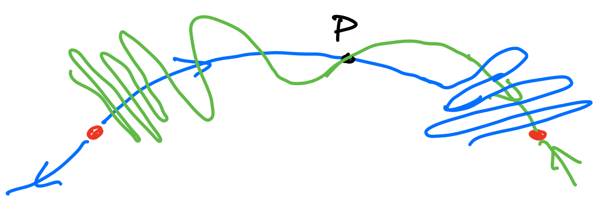
- Thm (Forni 96) If there exists $\omega \in \mathbb{R} \setminus \mathbb{Q}$ s.t. M_ω is not a RIC, then there exists a Borel probability invariant measure for T with positive metric entropy.

- Thm (Angenent 90) If $h_{top}(T) = 0$ then $\exists$ a RIC with rotation number ω, for all $\omega \in (\omega_-, \omega_+)$.



$\Rightarrow h_{top}(T) > 0.$

Rem Angenent proved that if $\exists\, \omega \in (\omega_-, \omega_+)$ for which $\mathcal{C}_\omega$ does not exist, then there is an invariant set $\Sigma \subset S^1 \times (a,b)$ on which T is top. conjugate to the shift map. It implies by a result of Katok that there exists a transversal homoclinic point.



- Gelfreich 1999. He studied the existence of the transversal homoclinic point to $(0,0)$ for the standard map for all $k \neq 0$.

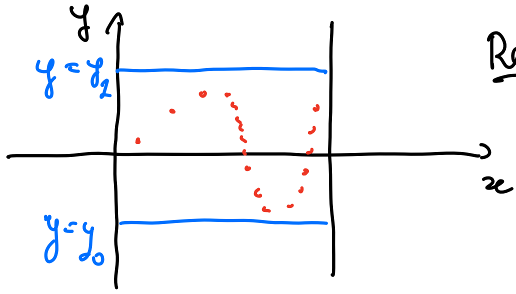
Converse KAM and properties of RIC

$T: S^1 \times \mathbb{R} \circlearrowleft$

1) **Climbing orbits**

Show the existence of orbits $\{(x_k, y_k)\}_{k\in\mathbb{Z}}$ such that

$$\limsup_{k\to+\infty} y_k = \pm\infty \implies \nexists \text{ RIC}.$$



Rem $\exists \{(x_k, y_k)\}_{k\in\mathbb{Z}}$ s.t.

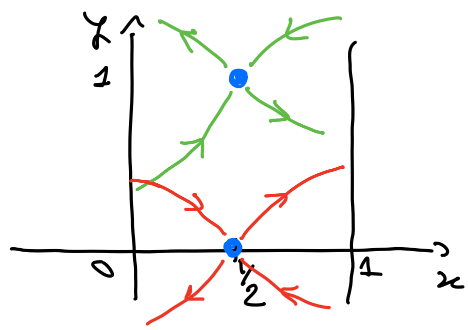
$\inf y_k = y_0$, $\sup y_k = y_1$.

may not be enough.

Thm (Pustylnikov) For the bouncing ball problem with $\alpha = 1$ there is an open set of analytic motions of the racket, for which there exist unbounded orbits.

2) **Heteroclinic connections** (obstruction criterion)

$$\begin{cases} x_1 = x_0 + \overbrace{y_0 - \frac{k}{2\pi}\sin(2\pi x_0)}^{y_1} \\ y_1 = y_0 - \frac{k}{2\pi}\sin(2\pi x_0) \end{cases}$$



$$\begin{cases} \dot{x} = y = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{k}{2\pi}\sin(2\pi x) = -\frac{\partial H}{\partial x} \end{cases} \qquad H(x,y) = \frac{1}{2}y^2 - \frac{k}{4\pi^2}\cos(2\pi x)$$

$P = (\frac{1}{2}, 0)$ $\qquad \{H(x,y) = H(\frac{1}{2},0)\} = \{H(x,y) = \frac{k}{4\pi^2}\} =$

$$= \left\{ y(x) = \pm\sqrt{\frac{k}{2\pi^2}(1+\cos(2\pi x))} \right\}$$

since $\max_{x\in[0,1]} |y(x)| = \sqrt{\frac{k}{\pi^2}}$, if $\sqrt{\frac{k}{\pi^2}} \geq \frac{1}{2}$ there is no

RIC for the standard map. It gives the value $k_0 = \frac{\pi^2}{4} \sim 2.46\ldots$

<u>Criterion</u> If there exist P_1 per. point of type (m_1, n_1) and P_2 per. point of type (m_2, n_2) s.t. $W^u(P_1) \cap W^s(P_2) \neq \emptyset$ then $\nexists$ RIC $\mathcal{C}_\omega$ with rot. number $\omega \in \left(\frac{m_1}{n_1}, \frac{m_2}{n_2}\right)$.

3. <u>Residues criterion</u> (Greene, 1978)

Let P be a periodic point of type (p,q), so that $T^q(P) = P$, then $R(P) := \frac{1}{4}\left(2 - \operatorname{tr} JT^q(P)\right)$ is the <u>residue</u> at P.

Since $\det JT^q(P) = 1$ then for the eigenvalues of JT^q we have the following possibilities:

- $\lambda, \frac{1}{\lambda} \in \mathbb{R}$ s.t. $\lambda > 0 \Rightarrow \operatorname{tr} JT^q = \lambda + \frac{1}{\lambda} \geq 2$

$$\Rightarrow \begin{array}{llll} \lambda, \frac{1}{\lambda} \neq +1, & \text{then} & \underline{R(P) \in (-\infty, 0)} & (\text{hyperbolic}) \\ \lambda = \frac{1}{\lambda} = +1, & \text{"} & \underline{R(P) = 0} & (\text{parabolic}) \end{array}$$

s.t. $\lambda < 0 \Rightarrow \operatorname{tr} JT^q = -(\lambda + \frac{1}{\lambda}) < 0$

$$\Rightarrow \begin{array}{llll} \lambda ; \frac{1}{\lambda} \neq -1, & \text{then} & \underline{R(P) \in (1, +\infty)} & (\text{hyperbolic}) \\ \lambda = \frac{1}{\lambda} = -1, & \text{"} & \underline{R(P) = 1} & (\text{parabolic}) \end{array}$$

- $\lambda, \bar{\lambda} \in S^1 \setminus \{\pm 1\} \Rightarrow \operatorname{tr} JT^q = 2\,\mathrm{Re}(\lambda) \in (-2, 2)$

$$\Rightarrow \underline{R(P) \in (0, 1)} \qquad (\text{elliptic})$$

<u>Conjecture (Greene)</u> Let $\frac{p_n}{q_n} \to \omega \in \mathbb{R} \setminus \mathbb{Q}$, with $(p_n, q_n) = 1$, and let $\{P_n\}$ a sequence of periodic points of type (p_n, q_n).

Then $\mu(\omega) := \lim_{m\to+\infty} \frac{1}{q_m} \log R(P_m)$ exists and

if $\mu(\omega) \leq 0$ then $\exists$ RIC $\mathcal{C}_\omega$ with rot. number ω,
if $\mu(\omega) > 0$ then $\nexists$ RIC $\mathcal{C}_\omega$ " " " " .

Numerical results on the standard map support this conjecture, and $\mu(\omega) > \mu\left(\frac{\sqrt{5}-1}{2}\right)$ $\forall\, \omega \neq \frac{\sqrt{5}-1}{2}$.

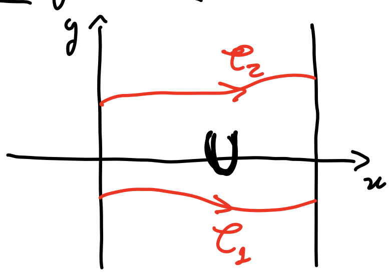
Thm (MacKay 1992)

$$\limsup_{m\to+\infty} \frac{1}{q_m} \log |R(P_m)| \leq \sup \{ \text{Lyapunov exponents of } \mathcal{M}_\omega \}$$

where $\mathcal{M}_\omega$ are invariant sets with rot. numb. ω.

Thm (Arnaud - Berger, 2016) If the periodic points $\{P_m\}$ are minimizing and $\limsup_{m\to+\infty} |R(P_m)|^{1/q_m} > 1$, then $\nexists$ RIC $\mathcal{C}_\omega$.

4. Boundary RIC



- A RIC $\mathcal{C}$ is called boundary RIC if there exist an open set U on one side of $\mathcal{C}$ (that is $\mathcal{C} \in \partial U$) s.t. U does not contain RIC.

If $\mathcal{C}_\omega$ is a boundary RIC with $\omega \in \mathbb{R} \setminus \mathbb{Q}$, then the statistical properties of the coefficient $\{a_k\}$ of the continued fraction expansion of ω, $\omega = [a_1, a_2, \dots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$,

do not correspond to the known distribution of the coefficients $\{a_k\}$ for the real numbers.

Is it true that an isolated RIC has noble rotation number?
(ω is noble if $\omega = [a_1, a_2, \dots]$ satisfies that $a_k = 1 \ \forall k \geq k_0$).

5. <u>Critical function</u> Let T be analytic and let $\mathcal{C}_\omega$ be a RIC s.t. $T|_{\mathcal{C}_\omega}$ is analytically conjugate to R_ω.
Then we know that $\exists\, u, v : S^1 \to \mathbb{R}$ and can be analytically extended to a strip, s.t. $\mathcal{C}_\omega = \{ (t + u(t), v(t)) \}_{t \in S^1}$ and $(x_1, y_1) = (t + u(t), v(t))$ implies $T^{-1}(x_1, y_1) = (x_0, y_0) = (t - \omega + u(t-\omega), v(t-\omega))$, and $(x_2, y_2) = T(x_1, y_1) = (t + \omega + u(t+\omega), v(t+\omega))$.

Then if $h(x_0, x_1)$ is the generating function of T, we have that

$$\underbrace{\partial_2 h(x_0, x_1)}_{= y_1} + \underbrace{\partial_1 h(x_1, x_2)}_{= -y_1} = 0 .$$

Hence for the standard map, $h(x_0, x_1) = \frac{1}{2}(x_1 - x_0)^2 + \frac{k}{4\pi^2}\cos(2\pi x_0)$,

$$(x_1 - x_0) + \left(-(x_2 - x_1) - \frac{k}{2\pi}\sin(2\pi x_1) \right) = 0$$

so that if these points leave in $\mathcal{C}_\omega$ we have

$$2(t + u(t)) - (t - \omega + u(t-\omega)) - (t + \omega + u(t+\omega)) = \frac{k}{2\pi}\sin(2\pi(t + u(t)))$$

$$(*) \qquad u(t+\omega) - 2u(t) + u(t-\omega) = -\frac{k}{2\pi}\sin(2\pi(t + u(t)))$$

Let's look for solutions of $(*)$ of the form

$$u(t) = \sum_{n=1}^{+\infty} k^n \sum_{m=-n}^{n} u_{m,n}\, e^{2\pi i m t}$$

<u>Def</u> Let $\rho(\omega) := \inf_{t \in [0,1]} \left(\limsup_{n \to +\infty} |u_n(t)|^{\frac{1}{n}} \right)^{-1}$ where

$$u_n(t) = \sum_{m=-n}^{n} u_{m,n}\, e^{2\pi i m t} \quad \text{for a solution of } (*).$$

$\sum k^n u_n(t)$ is called the <u>Lindstedt series</u>.

<u>Def</u> For $\omega \in \mathbb{R} \setminus \mathbb{Q}$, the critical function $K_c(\omega)$ is given by

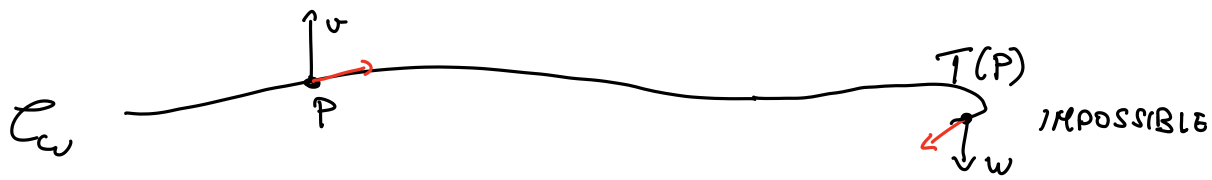
$$K_c(\omega) := \sup \{ k \in (0,+\infty) \ / \ \mathcal{C}_\omega \text{ RIC exists for the stand. map } T_k \text{ and is analytic} \}$$

<u>Prop</u> $K_c(\omega) \geq \rho(\omega)$.

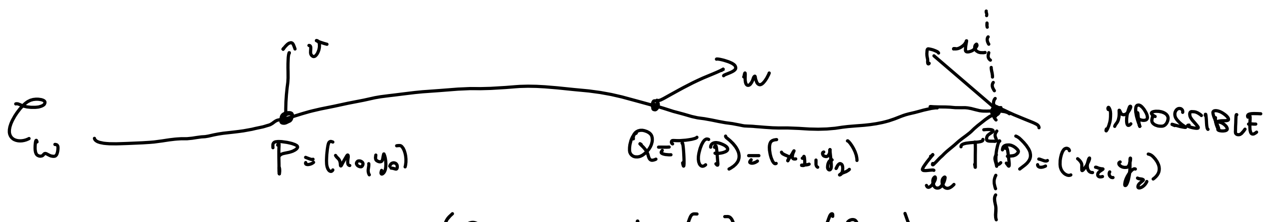
Converse KAM , $T : S^1 \times \mathbb{R} \circlearrowleft$

7. Lipschitz condition

Let $\mathcal{C}_\omega$ be a RIC for T, and let $P \in \mathcal{C}_\omega$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in T_P(S^1 \times \mathbb{R})$



$w = JT(P)(v)$



$$W = JT(P)\, v = \begin{pmatrix} \frac{\partial x_1}{\partial x_0} & \frac{\partial x_1}{\partial y_0} \\ \frac{\partial y_1}{\partial x_0} & \frac{\partial y_1}{\partial y_0} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_0} \\ \frac{\partial y_1}{\partial y_0} \end{pmatrix} > 0$$

Then the x-component of $JT^2(P)v$ cannot be negative.

$$u = JT(Q)\, w = JT^2(P)\, v = \begin{pmatrix} \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial y_1} \end{pmatrix} \begin{pmatrix} \frac{\partial x_1}{\partial y_0} \\ \frac{\partial y_1}{\partial y_0} \end{pmatrix}$$

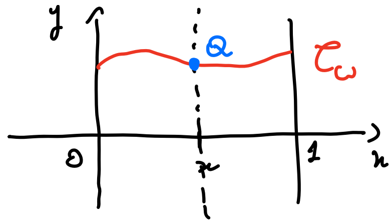
Then $$\frac{\partial x_2}{\partial x_1} \cdot \frac{\partial x_1}{\partial y_0} + \frac{\partial x_2}{\partial y_1} \frac{\partial y_1}{\partial y_0} \geq 0 .$$

Prop The standard map T_k has no RICs for $k > 2$.

proof $$\begin{cases} x_1 = x_0 + y_0 - \frac{k}{2\pi} \sin(2\pi x_0) \\ y_1 = y_0 - \frac{k}{2\pi} \sin(2\pi x_0) \end{cases}$$

$$(1 - k\cos(2\pi x_1)) \cdot (1) + (1) \cdot (1) = 2 - k\cos(2\pi x_1)$$

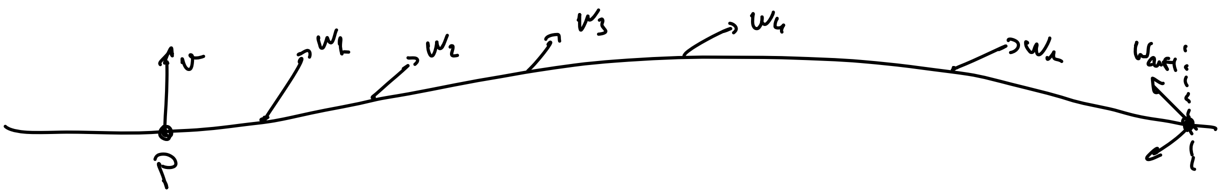
Then, if $k>2$, $\exists\, x \in [0,1]$ s.t. $2-k\cos(2\pi x)<0$.



IMPOSSIBLE

$\square$

If $v=\binom{0}{1}\in T_p(S^1\times\mathbb{R})$, and $w_k = DT^k(p)\,v$, $k\ge 1$. If the x-component of w_k is positive for all $k=1,\dots,n$, and the x-component of w_{n+1} is negative, then there are no RIC.



IMPOSS.

The standard map has no RIC for $k>\frac{63}{64}$.

8. Generating function (J. Mather)

Prop If $k>\frac{4}{3}$ then the standard map has no RIC.

proof The standard map has generating function

$$h(x_0,x_1) = \frac{1}{2}(x_1-x_0)^2 + \frac{k}{4\pi^2}\cos(2\pi x_0)$$

Let $\mathcal{C}_\omega$ be a RIC and (x_0,y_0), (x_1,y_1), (x_2,y_2) be points of an orbit on $\mathcal{C}_\omega$. Then

$y_1 = \partial_2 h(x_0,x_1) = -\partial_1 h(x_1,x_2)$ implies

$$x_1 - x_0 = +(x_2-x_1) + \frac{k}{2\pi}\sin(2\pi x_1)$$

$$2x_1 - x_0 - x_2 - \frac{k}{2\pi}\sin(2\pi x_1) = 0$$

Since $T|_{\mathcal{C}_\omega}$ is a Lipschitz circle homeo $F: S^1\to S^1$,

$x_2 = F(x_1)$ and $x_0 = F^{-1}(x_1)$. Hence

$$2x_1 - F^{-1}(x_1) - F(x_1) - \frac{k}{2\pi}\sin(2\pi x_1) = 0$$

By differentiating wrt x_1,

$$(*) \qquad 2 - k\cos(2\pi x_1) = F'(x_1) + (F^{-1})'(x_1) = F'(x_1) + \frac{1}{F'(F^{-1}(x_1))} > 0$$

Let $C = \max\left\{ \sup_{S^1} F', \sup_{S^1} \frac{1}{F'} \right\}$, and write

$L := \sup_{S^1} F'$, $\ell := \inf_{S^1} F'$, so that $C = \max\left\{ L, \frac{1}{\ell} \right\}$.

Then from $(*)$ we get that

$$2+k = \sup_{S^1}\left(2 - k\cos(2\pi x_1)\right) \geq \max\left\{ \sup F' + \inf \frac{1}{F'}, \inf F' + \sup \frac{1}{F'} \right\} =$$

$$= \max\left\{ L + \frac{1}{L}, \ell + \frac{1}{\ell} \right\} \geq C + \frac{1}{C}$$

$$2-k = \inf_{S^1}\left(2 - k\cos(2\pi x_1)\right) \geq \inf F' + \inf \frac{1}{F'} = \ell + \frac{1}{L} \geq \frac{2}{C}$$

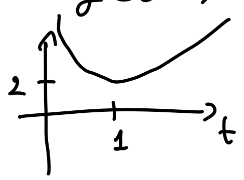
$$\ell \geq \frac{1}{C}, \quad \frac{1}{L} \geq \frac{1}{C}$$

Then by the previous prop. we can assume $k<2$ to have $R \geq C$, hence

$2-k \geq \frac{2}{C}$ and $2+k \geq C + \frac{1}{C}$, hence $C \geq \frac{2}{2-k} \geq 1$

$$2+k \geq g(C) \geq g\left(\frac{2}{2-k}\right) = \frac{2}{2-k} + \frac{2-k}{2}$$

$g(t) = t + \frac{1}{t}$



$$\iff \quad 4 - k^2 \geq 2 + \frac{(2-k)^2}{2} = 2 + 2 + \frac{1}{2}k^2 - 2k$$

$$\iff \quad \frac{3}{2}k^2 - 2k \leq 0 \iff k\left(\frac{3}{2}k - 2\right) \leq 0 \implies k \leq \frac{4}{3} \qquad \square$$

Theorem Let $\Omega \subset \mathbb{R}^2$ be a C^3 compact and strictly convex domain, if there is a point $P \in \partial\Omega$ at which the curvature vanishes, then the billiard ball map has no RIC.

Ex $\Omega = \{x^4 + y^4 \leq 1\}$

proof Let $\gamma : [0,1] \to \mathbb{R}^2$ be the arc-length parameterisation of $\partial\Omega$, and let d be the Euclidean distance.

$h(s_0, s_1) = -d(\gamma(s_0), \gamma(s_1))$ is the generating function for $T : S^1 \times [-1,1] \circlearrowleft$, $(s, \xi) \in T$, $\xi = -\cos\vartheta$.

Let $\varphi(t,u,w) := d(\gamma(t), \gamma(u)) + d(\gamma(u), \gamma(w))$, then

$$\frac{\partial\varphi}{\partial u}(t,u,w) = \frac{\partial}{\partial u}\left(\sqrt{(x(t)-x(u))^2 + (y(t)-y(u))^2}\right) + \frac{\partial}{\partial u}\left(\sqrt{(x(u)-x(w))^2 + (y(u)-y(w))^2}\right)$$

$$= \frac{-\dot{x}(u)(x(t)-x(u)) - \dot{y}(u)(y(t)-y(u))}{d(\gamma(t), \gamma(u))} + \frac{\dot{x}(u)(x(u)-x(w)) + \dot{y}(u)(y(u)-y(w))}{d(\gamma(u), \gamma(w))}$$

If s_0, s_1, s_2 are the s-components of three consecutive points of an orbit, then $\partial_2 h(s_0, s_1) + \partial_1 h(s_1, s_2) = 0$ implies that

$$\boxed{\frac{\partial\varphi}{\partial u}(s_0, s_1, s_2) = 0} \left(= -\partial_2 h(s_0, s_1) - \partial_1 h(s_1, s_2)\right)$$

In addition if $P = \gamma(s_2)$ is the point at which the curvature vanishes, then

$$\frac{\partial^2\varphi}{\partial u^2}(s_0, s_1, s_2) = \left[-\frac{\left[-\dot{x}(u)(x(t)-x(u)) - \dot{y}(u)(y(t)-y(u))\right]^2}{d(\gamma(t), \gamma(u))^3} + \right.$$

$$\left. + \frac{-\ddot{x}(u)(x(t)-x(u)) + \dot{x}^2(u) - \ddot{y}(u)(y(t)-y(u)) + \dot{y}^2(u)}{d(\gamma(t), \gamma(u))} + \dots \right]\Bigg|_{\substack{t=s_0 \\ u=s_2 \\ w=s_2}} =$$

$$\ddot{\gamma}(s_1) = (\ddot{x}(s_1), \ddot{y}(s_1)) = (0,0) \quad , \quad \|\dot{\gamma}(s)\|^2 = 1$$

$$\frac{\dot{x}(u)(x(u)-x(t)) + \dot{y}(u)(y(w)-y(t))}{d(\gamma(t),\gamma(u))} = <\dot{\gamma}(u), \widehat{\gamma(t)\gamma(u)}>$$

$$= \left[- \frac{<\dot{\gamma}(u), \widehat{\gamma(t)\gamma(u)}>^2}{d(\gamma(t),\gamma(u))} + \frac{\dot{x}^2(u)+\dot{y}^2(u)}{d(\gamma(t),\gamma(u))} + \cdots \right]\Bigg|_{\substack{t=s_0\\u=s_1\\w=s_2}} =$$

$$= \frac{1 - <\dot{\gamma}(s_1), \widehat{\gamma(s_0)\gamma(s_1)}>^2}{d(\gamma(s_0),\gamma(s_1))} + \frac{1 - <\dot{\gamma}(s_1), \widehat{\gamma(s_1)\gamma(s_2)}>^2}{d(\gamma(s_1),\gamma(s_2))} > 0$$

So we have shown that $\frac{\partial \varphi}{\partial u}(s_0,s_1,s_2)=0$, $\frac{\partial^2 \varphi}{\partial u^2}(s_0,s_1,s_2)>0$, hence we can apply the Implicit Function theorem to prove that there exist a $U(s_0,s_2)$, $V(s_1)$ and $\eta : U(s_0,s_2) \to V(s_1)$ which is diff and satisfies $\eta(s_0,s_2)=s_1$ and $\frac{\partial \varphi}{\partial u}(t, \eta(t,w), w)=0$ $\forall (t,w) \in U(s_0,s_2)$

Moreover $$\frac{\partial \eta}{\partial t}(s_0,s_2) = - \frac{\frac{\partial^2 \varphi}{\partial t \partial u}(s_0,s_1,s_2)}{\frac{\partial^2 \varphi}{\partial u^2}(s_0,s_1,s_2)} \quad , \quad \frac{\partial \eta}{\partial w}(s_0,s_2) = - \frac{\frac{\partial^2 \varphi}{\partial w \partial u}(s_0,s_1,s_2)}{\frac{\partial^2 \varphi}{\partial u^2}(s_0,s_1,s_2)}$$

$$\frac{\partial^2 \varphi}{\partial t \partial u}(s_0,s_1,s_2) = \frac{\partial^2}{\partial t \partial u}\Big(d(\gamma(t),\gamma(u)) + d(\gamma(u),\gamma(w))\Big)\Bigg|_{\substack{t=s_0\\u=s_1\\w=s_2}} =$$

$$= \frac{\partial^2}{\partial t \partial u} d(\gamma(t),\gamma(u))\Bigg|_{\substack{t=s_0\\u=s_1\\w=s_2}} = - \frac{\partial}{\partial s_1} \cos\vartheta_0(s_0,s_1) = \sin\vartheta_0 \cdot \frac{\partial \vartheta_0}{\partial s_1} =$$

$$\frac{\partial^2}{\partial s_0 \partial s_1} h(s_0,s_1) = \frac{\partial}{\partial s_1}\left(\partial_1 h(s_0,s_1)\right) = \frac{\partial}{\partial s_1}(-\delta_0)$$

$$= \sin\vartheta_0 \frac{1}{\frac{\partial s_1}{\partial \vartheta_0}} > 0$$

$$\frac{\partial^2 \varphi}{\partial w \partial u}(s_0,s_1,s_2) = \frac{\partial^2}{\partial w \partial u} d(\gamma(u),\gamma(w))\Bigg|_{\substack{t=s_0\\u=s_1\\w=s_2}} > 0$$

Here $\frac{\partial \eta}{\partial t}(s_0, s_2) < 0$, $\frac{\partial \eta}{\partial w}(s_0, s_2) < 0$.

If s_0, s_1, s_2 are the s-components of points of an orbit on a RIC, there is an increasing Lipschitz circle homeo $F: S^1 \to S^1$ s.t. $s_2 = F(s_1)$ and $s_0 = F^{-1}(s_1)$, s.t. $0 = \frac{\partial \ell}{\partial u}(s_0, F(s_0), s_2) = \frac{\partial \ell}{\partial u}(s_0, F^{-1}(s_2), s_2)$

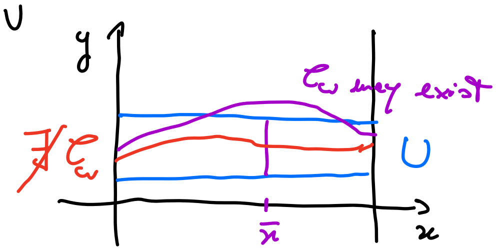
This implies that $\frac{\partial \eta}{\partial t}(s_0, s_2) = F'(s_0) < 0$, $\frac{\partial \eta}{\partial w}(s_0, s_2) = (F^{-1})'(s_2) < 0$, and this is a contradiction. $\square$

$h(x_0, x_1)$, $T(x_0, y_0) = (x_1, y_1)$, $T(x_1, y_1) = (x_2, y_2)$

$$x_1 \longmapsto \partial_2 h(x_0, x_1) + \partial_1 h(x_1, x_2)$$

$$a(x_1) := \partial_{22} h(x_0, x_1) + \partial_{11} h(x_1, x_2)$$

__Idea__ If one shows that, for $T : S^1_x \times \mathbb{R}_y \circlearrowleft$ twist map with generating function h, there exists a set $U = S^1_x \times (a, b)$ for which $\min_{x \in S^1} a(x)\big|_U < 0$, then there are no RIC contained in U.



$a\big|_U(\bar{x}) < 0$

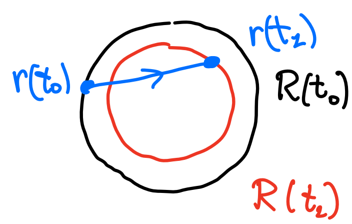
For bouncing balls and for the case of the billiard map inside a "breathing" circle $\Omega_t = \{ P \in \mathbb{R}^2 : d(P, O) \le R(t) \}$ with R periodic, one can prove that $h_{top} > 0$.

A "good" generating function in the breathing circle case is

$$h(t_0, t_1) = \int_{t_0}^{t_1} L(r(t; t_0, t_1), \dot{r}(t; t_0, t_1))\, dt$$

where t_0 and t_1 are the times of two consecutive bouncings on $\partial\Omega_t$, $r(t; t_0, t_1)$ is the solution of the following Dirichlet problem

$$\begin{cases} \ddot{r} = 0 \\ |r(t_0)| = R(t_0) \\ |r(t_1)| = R(t_1) \\ |r(t)| < R(t) \quad \forall\, t \in (t_0, t_1) \end{cases}$$



and L is the Lagrangian of the problem.