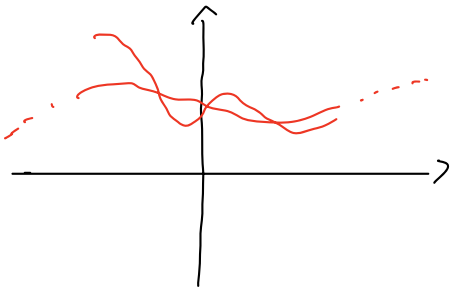
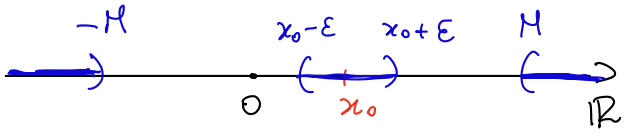
- Limiti ✓
- Derivate ~
- Integrali .



$f(x) = e^x + e^{-x}$

$f: D \subseteq \mathbb{R} \to \mathbb{R}$, $D$ dominio (naturale)

$\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ $\quad -\infty$




$+\infty$

**Def** · Sia $x_0 \in \mathbb{R}$, un INTORNO di $x_0$ è un intervallo aperto della forma $(x_0 - \varepsilon, x_0 + \varepsilon)$ con $\varepsilon > 0$.

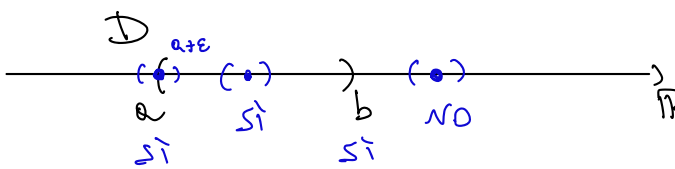
· Un INTORNO di $+\infty$ è un intervallo aperto della forma $(M, +\infty)$ con $M \gg 1$. ← molto più grande di

· Un INTORNO di $-\infty$ è un intervallo aperto della forma $(-\infty, -M)$ con $M \gg 1$.

**Def** Dato un insieme $D \subseteq \mathbb{R}$, un punto $x \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ si dice PUNTO DI ACCUMULAZIONE per $D$ se

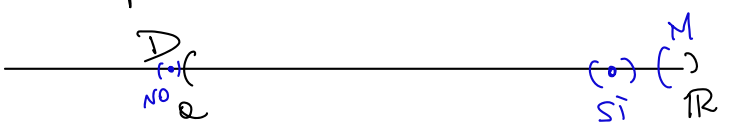
$\forall\, U$ intorno di $x$, $\exists\, x_0 \in D \cap U$, $x_0 \neq x$.

ES




$D = (a,b)$

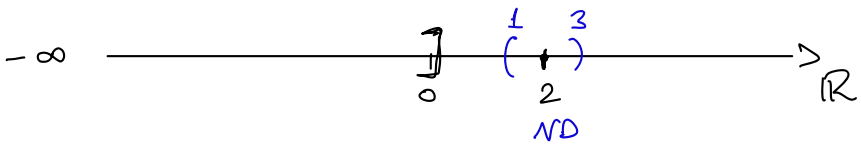
punti di acc. di $(a,b) = [a,b]$




$+\infty$ $\quad D = (a, +\infty)$

punti di acc di $(a, +\infty) = [a, +\infty) \cup \{+\infty\}$

$D = (-\infty, 0] \cup \{2\}$. Punto di acc di $D = (-\infty, 0] \cup \{-\infty\}$




$(1,3) \cap D = \{2\}$

**Def** Sia $f: D \subseteq \mathbb{R} \to \mathbb{R}$, e sia $x_0 \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ che sia punto di accumulazione per $D$ (non è necessario che $x_0 \in D$).

Si dice che "il limite di $f(x)$ per $x$ che tende a $x_0$ è $\ell$", e si scrive

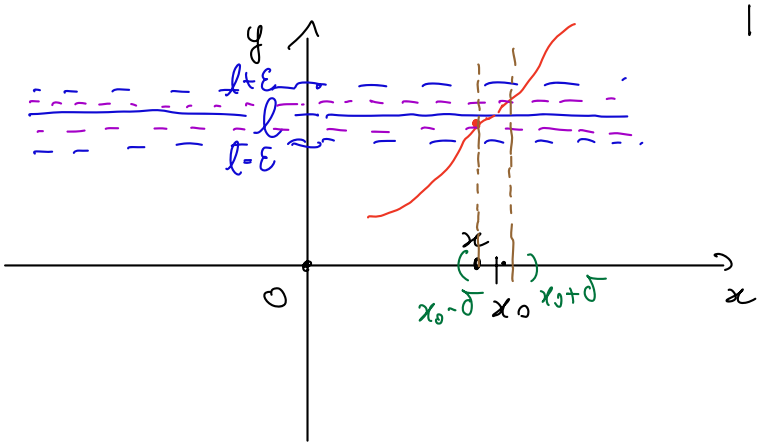
$$\lim_{x \to x_0} f(x) = \ell \in \overline{\mathbb{R}}$$

se $\forall$ $V$ intorno di $\ell$, $\exists$ $U$ intorno di $x_0$, tale che se $x \in (U \setminus \{x_0\}) \cap D$ vale $f(x) \in V$.

**ES** • $x_0 \in \mathbb{R}$, $\ell \in \mathbb{R}$

$\lim_{x \to x_0} f(x) = \ell$ $\iff$ $\forall \varepsilon > 0$ $\exists \delta > 0$ tale che se $x \in (x_0 - \delta, x_0 + \delta) \cap D$ con $x \neq x_0$, allora $f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$

$|f(x) - \ell| < \varepsilon$

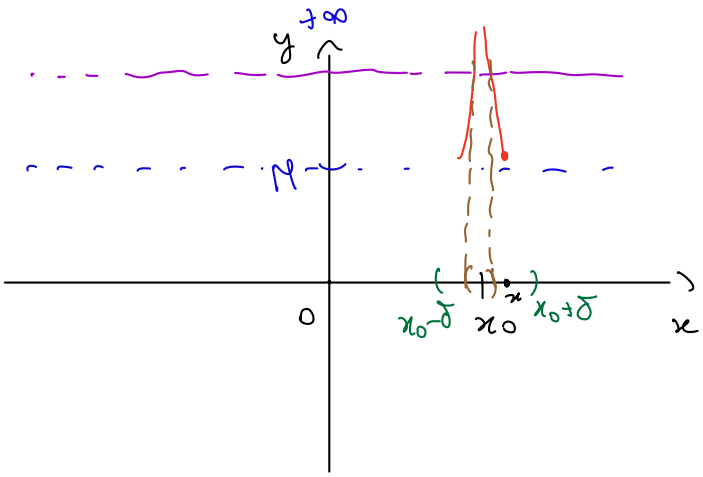



• $x_0 \in \mathbb{R}$, $\ell = +\infty$

$\lim_{x \to x_0} f(x) = +\infty$ $\iff$ $\forall M >> 1$, $\exists \delta > 0$ tale che se $x \in (x_0 - \delta, x_0 + \delta) \cap D$ con $x \neq x_0$, si ha $f(x) > M$

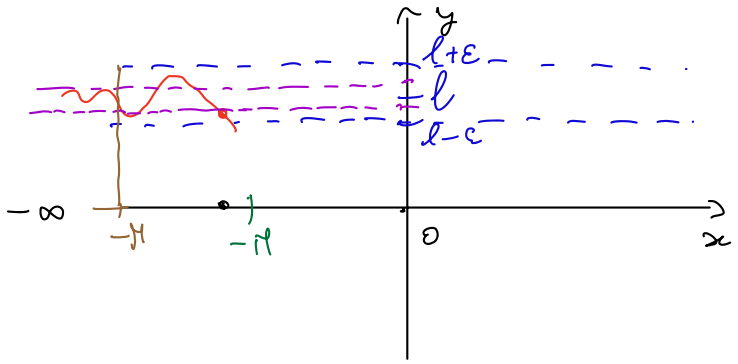
$\Downarrow$

$f(x) \in (M, +\infty)$



- $x_0 = -\infty, \quad \ell \in \mathbb{R}$

$\lim_{x \to -\infty} f(x) = \ell \iff \forall \varepsilon > 0 \ \exists M >> 1$ tale che se $x \in (-\infty, -M) \cap D$ si ha $f(x) \in (\ell - \varepsilon, \ell + \varepsilon)$ $(\iff |f(x) - \ell| < \varepsilon)$

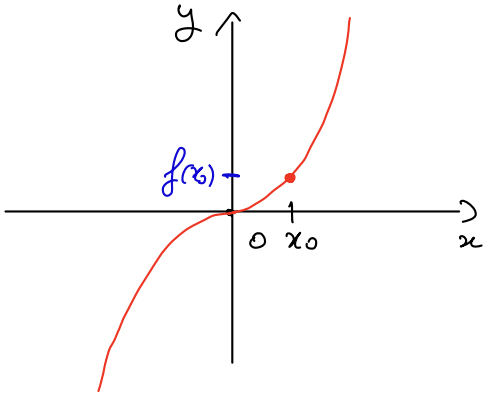



OSS Per tutte le funzioni elementari vale che se $x_0 \in D$ allora $\lim_{x \to x_0} f(x) = f(x_0)$.

ES $f(x) = x^3$

$D = \mathbb{R}$

$Acc(D) = \overline{\mathbb{R}}$




$\lim_{x \to +\infty} x^3 = +\infty \iff \forall M >> 1 \ \exists K >> 1$ tale che se

$x \in (K, +\infty) \cap D$ allora $x^3 \in (M, +\infty)$

$\Updownarrow$ $\qquad$ $\Updownarrow$

$x > K \qquad x^3 > M$

$x^3 > M \iff x > \sqrt[3]{M} = K$

$\lim_{x \to -\infty} x^3 = -\infty \iff \forall M >> 1 \ \exists K >> 1$ Vale che

se $x < -K$ allora $x^3 < -M$

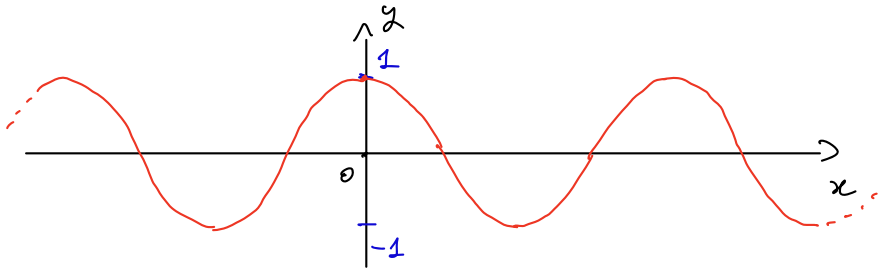
$x^3 < -M \iff x < \sqrt[3]{-M} = -K$

ES $\alpha$ pari, $\lim_{x \to +\infty} x^\alpha = \lim_{x \to -\infty} x^\alpha = +\infty$

$\alpha$ dispari, $\lim_{x \to +\infty} x^\alpha = +\infty$, $\lim_{x \to -\infty} x^\alpha = -\infty$.

**ES** $f(x) = \cos x$ , $D = \mathbb{R}$, $Acc(D) = \overline{\mathbb{R}}$

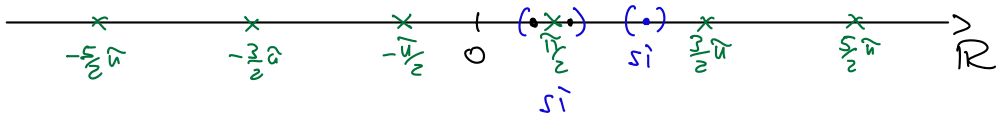
- $x_0 \in \mathbb{R}$, $\lim_{x \to x_0} \cos x = \cos x_0$.
- $x_0 = +\infty$, $\lim_{x \to +\infty} \cos x = ?$ non esiste



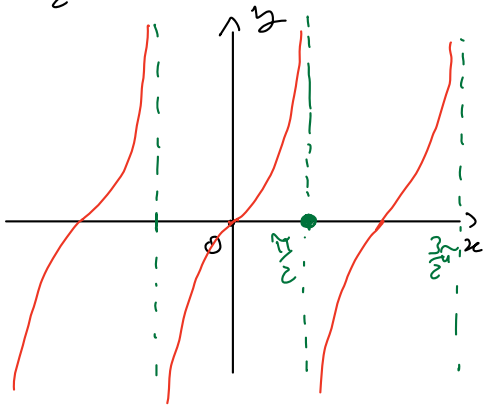

$\lim_{x \to -\infty} \cos x$ non esiste

**ES** $f(x) = \tan x$ , $D = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$

$Acc(D) = \overline{\mathbb{R}}$




- $x_0 \in D$ , $\lim_{x \to x_0} \tan x = \tan x_0$
- $x_0 = +\infty$
  $x_0 = -\infty$ $\lim_{x \to +\infty} \tan x$ , $\lim_{x \to -\infty} \tan x$ non esistono
- $x_0 = \frac{\pi}{2}$ $\lim_{x \to \frac{\pi}{2}} \tan x = ?$ non esiste



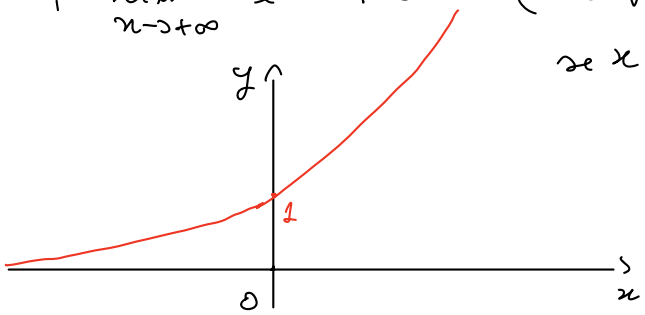

**ES** • $f(x) = e^x$, $D = \mathbb{R}$, $Acc(D) = \overline{\mathbb{R}}$

$x_0 \in D = \mathbb{R}$, $\lim_{x \to x_0} e^x = e^{x_0}$

$x_0 = +\infty$, $\lim_{x \to +\infty} e^x = +\infty$

$\left( \Leftrightarrow \forall M >> 1 \ \exists K >> 1 \text{ t.c. se } x > K \text{ allora } e^x > M \right)$

$K = \log M$

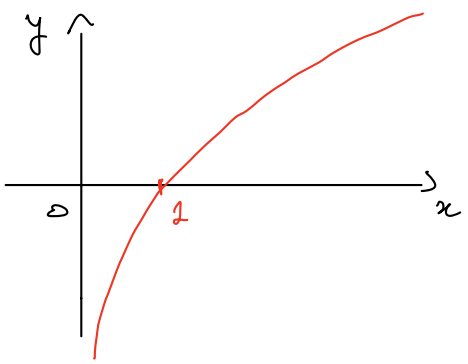



$x_0 = -\infty$, $\lim_{x \to -\infty} e^x = 0$

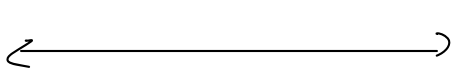
- $f(x) = \log x$, $D = (0, +\infty)$, $Acc(D) = [0, +\infty) \cup \{+\infty\}$

$x_0 \in (0, +\infty) = D$, $\lim_{x \to x_0} \log x = \log x_0$

$x_0 = +\infty$, $\lim_{x \to +\infty} \log x = +\infty$




$x_0 = 0$, $\lim_{x \to 0} \log x \left( = \lim_{x \to 0^+} \log x \right) = -\infty$



Aritmetica su $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $(x \in \mathbb{R}, \pm\infty)$

$x + x' \in \mathbb{R}$, $x + (+\infty) = +\infty$, $x + (-\infty) = -\infty$

$(+\infty) + (+\infty) = +\infty$, $(+\infty) - (+\infty)$ non è definito

$(-\infty) + (-\infty) = -\infty$, $(-\infty) - (-\infty)$ non è definito

$(+\infty) + (-\infty)$ non è definito

$x \cdot x' \in \mathbb{R}$, $x > 0$ $\quad x \cdot (+\infty) = +\infty$, $\quad x \cdot (-\infty) = -\infty$

$x < 0$ $\quad x \cdot (+\infty) = -\infty$, $\quad x \cdot (-\infty) = +\infty$

$+ (-\infty) = -\infty$, $\quad - (+\infty) = -\infty$

$(+\infty) \cdot (+\infty) = +\infty$, $\quad (-\infty) \cdot (-\infty) = +\infty$, $\quad (+\infty) \cdot (-\infty) = -\infty$

$0 \cdot (+\infty)$, $0 \cdot (-\infty)$ non sono definiti

$x' \neq 0$, $\frac{x}{x'} \in \mathbb{R}$, $\quad x \in \mathbb{R}$, $\frac{x}{+\infty} = x \cdot \frac{1}{+\infty}$

$\frac{1}{+\infty} = 0^+$, $\quad \frac{1}{-\infty} = 0^-$

$\frac{1}{0}$, $\frac{0}{0}$, $\frac{+\infty}{+\infty} = +\infty \cdot \frac{1}{+\infty}$, $\quad \frac{+\infty}{-\infty}$, $\quad \frac{-\infty}{-\infty}$ non sono definiti

<u>ES</u> $f(x) = e^x + e^{-x}$, $\quad \lim_{x \to +\infty} (e^x + e^{-x}) = \lim_{x \to +\infty} e^x + \lim_{x \to +\infty} e^{-x} =$

$= +\infty + 0 = +\infty$