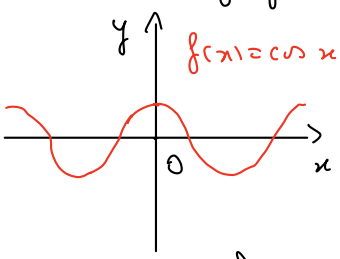
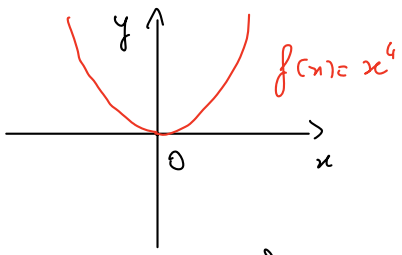
Funzione inversa

Def $f: D \subseteq \mathbb{R} \to \mathbb{R}$. Si dice che f è <u>iniettiva su D</u> se dati $x, x' \in D$, avere $f(x) = f(x')$ implica $x = x'$.

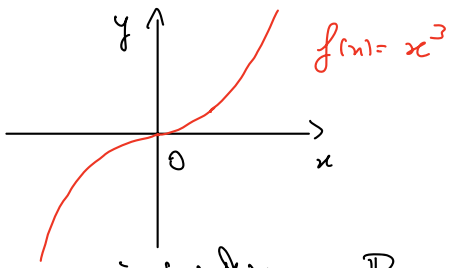
Oss Se $y \in f(D)$ allora la retta orizzontale passante per y interseca il grafico di f esattamente in un punto.



non è iniettiva su $\mathbb{R}$

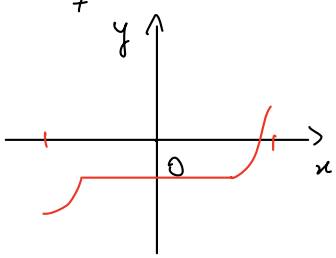


non è iniettiva su $\mathbb{R}$

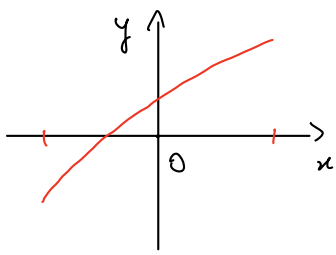


è iniettiva su $\mathbb{R}$

Def $f: D \subseteq \mathbb{R} \to \mathbb{R}$ si dice CRESCENTE su D se per ogni $x, x' \in D$ con $x \leq x'$, si ha $f(x) \leq f(x')$.



crescente non iniettiva

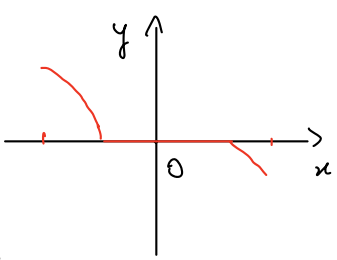


strettamente crescente iniettiva

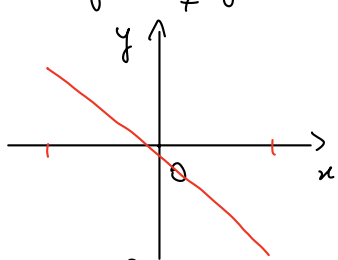
$f: D \subseteq \mathbb{R} \to \mathbb{R}$ si dice STRETTAMENTE CRESCENTE su D se per ogni $x, x' \in D$ con $x \lneq x'$, si ha $f(x) \lneq f(x')$.

$f: D \subseteq \mathbb{R} \to \mathbb{R}$ si dice DECRESCENTE su D se per ogni $x, x' \in D$ con $x \leq x'$, si ha $f(x) \geq f(x')$.

$f: D \subseteq \mathbb{R} \to \mathbb{R}$ si dice STRETTAMENTE DECRESCENTE su D se per ogni $x, x' \in D$ con $x \lneq x'$, si ha $f(x) \gneq f(x')$.



decrescente, non iniettiva



strettamente decrescente, iniettiva

Def Data $f: D \subseteq \mathbb{R} \to \mathbb{R}$ iniettiva, si definisce FUNZIONE INVERSA di f su D la funzione f^{-1} (oppure $(f|_D)^{-1}$) definita su $f(D)$

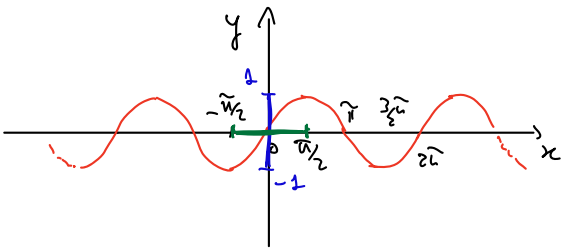
$$(f|_D)^{-1} : f(D) \subseteq \mathbb{R} \to \mathbb{R}$$

in modo che $f^{-1}(y) = x$, dove $x \in D$ è l'unico che verifica $f(x) = y$.

In particolare, $f^{-1}(f(x)) = x$ e $f(f^{-1}(y)) = y$.

Funzioni Trigonometriche inverse

- $\sin : \mathbb{R} \to \mathbb{R}$, $D = \mathbb{R}$ dominio naturale., $\operatorname{Im} \sin = [-1, 1]$



$\sin$ non è iniettiva su $\mathbb{R}$

$\sin\left([-\frac{\pi}{2}, \frac{\pi}{2}]\right) = [-1, 1]$, $\sin$ è iniettiva su $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Quindi posso definire $\left(\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}\right)^{-1} : [-1, 1] \subseteq \mathbb{R} \to \mathbb{R}$

e si chiama funzione arcoseno, $\operatorname{arcsin} = \left(\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}\right)^{-1}$.

L'immagine di arcsin è $\operatorname{arcsin}([-1, 1]) = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

ES $\operatorname{arcsin}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$, $\sin\left(\operatorname{arcsin}\left(-\frac{1}{2}\right)\right) = -\frac{1}{2}$.

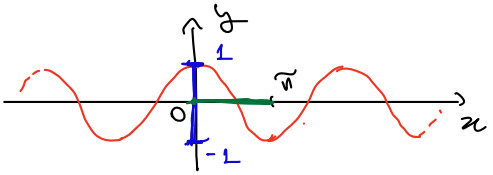
OSS $\operatorname{arcsin}\left(\sin\left(\frac{3}{2}\pi\right)\right) = \operatorname{arcsin}(-1) = -\frac{\pi}{2}$ $\left(= \operatorname{arcsin}\left(\sin\left(-\frac{\pi}{2}\right)\right)\right)$.

$\frac{3}{2}\pi \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(x) = -1$, $\operatorname{arcsin}(\sin(x)) = \operatorname{arcsin}(-1) = -\frac{\pi}{2}$

$x = -\frac{\pi}{2} + 2k\pi$, $k \in \mathbb{Z}$.

- $\cos : \mathbb{R} \to \mathbb{R}$, $D = \mathbb{R}$ dominio naturale, $\operatorname{Im} \cos = [-1, 1]$

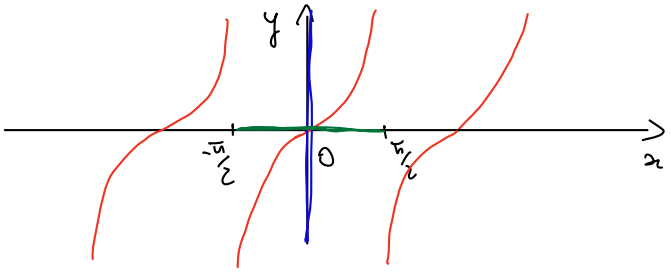


cos non è iniettiva su $\mathbb{R}$

$\cos([0,\pi]) = [-1,1]$, $\cos$ è iniettiva su $[0,\pi]$, quindi esiste $\left(\cos|_{[0,\pi]}\right)^{-1} : [-1,1] \subseteq \mathbb{R} \to \mathbb{R}$, detta funzione arcocoseno, $\arccos$, con immagine $\arccos([-1,1]) = [0,\pi]$.

Es $\arccos(\cos(\pi/2)) = \pi/2$, $\arccos(\cos(\frac{3}{2}\pi)) = \frac{\pi}{2}$ ($\frac{3}{2}\pi \notin [0,\pi]$)

- $\tan : \mathbb{R} \to \mathbb{R}$, $D = \mathbb{R}$ dominio naturale, $\operatorname{Im}\tan = \mathbb{R}$



$\tan$ non è iniettiva su $\mathbb{R}$.

$\tan(-\frac{\pi}{2}, \frac{\pi}{2}) = \mathbb{R}$, $\tan$ è iniettiva su $(-\frac{\pi}{2}, \frac{\pi}{2})$, quindi esiste

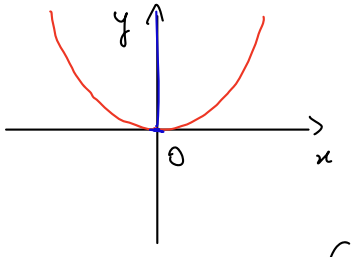
$\operatorname{arctan} := \left(\tan|_{(-\frac{\pi}{2},\frac{\pi}{2})}\right)^{-1} : \mathbb{R} \to \mathbb{R}$, con immagine $\operatorname{arctan}(\mathbb{R}) = (-\frac{\pi}{2}, \frac{\pi}{2})$.

la funzione arcotangente.

$\longleftarrow$

Oss

$f(x) = x^2$, $D = \mathbb{R}$ dominio naturale, però, $\operatorname{Im} f = [0,+\infty)$.



f non è iniettiva su $\mathbb{R}$

$x^2 = y \iff x = \pm\sqrt{y}$.

$\left(f|_{[0,+\infty)}\right)^{-1} : [0,+\infty) \to \mathbb{R}$

$\operatorname{Im} = [0,+\infty)$

$\left(f|_{[0,+\infty)}\right)^{-1}(y) = \sqrt{y}$

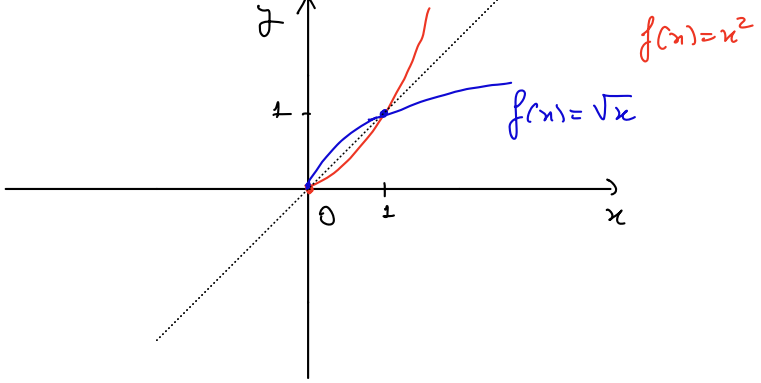
$f\left(\left(f|_{[0,+\infty)}\right)^{-1}\right)(y) = y$

$\left(f|_{[0,+\infty)}\right)^{-1}(f(x)) = x$

$\left(f|_{(-\infty,0]}\right)^{-1} : [0,+\infty) \to \mathbb{R}$

$\operatorname{Im} = (-\infty,0]$

$\left(f|_{(-\infty,0]}\right)^{-1}(y) = -\sqrt{y}$



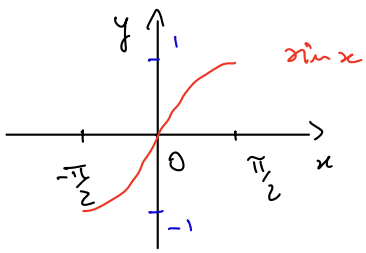
$$\text{graf } f = \{(x,y)\in\mathbb{R}^2 : x\in D=[0,+\infty),\ y=x^2\}$$

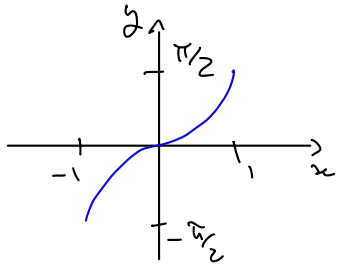
$$(x,x^2) \qquad \{(1,1),\ (2,4),\ (3,9),\ \dots\}$$

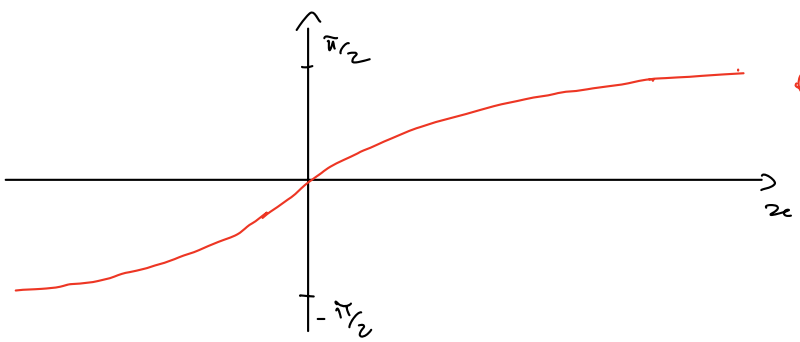
$$\text{graf}\left(f|_{[0,+\infty)}\right)^{-1} = \{(x,y)\in\mathbb{R}^2 : x\in[0,+\infty),\ y=\sqrt{x}\}$$

$$(x,\sqrt{x}) \qquad \{(1,1),\ (4,2),\ (9,3),\ \dots\}$$

Es





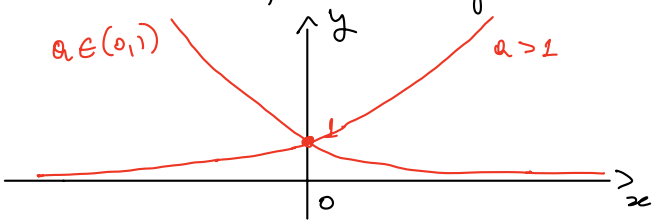


Funzioni esponenziali e logaritmiche.

- $f(x) = a^x$, $a\in(0,+\infty)$, $a\neq 1$., $f(x)>0 \quad \forall x\in\mathbb{R}$

  Caso $f(x)=e^x$, e numero di Nepero, $e\sim 2.7\dots\in\mathbb{R}\setminus\mathbb{Q}$.

  $D=\mathbb{R}$ dominio naturale, $\text{Im } f = (0,+\infty)$, $\inf f = 0$, $\sup f = +\infty$



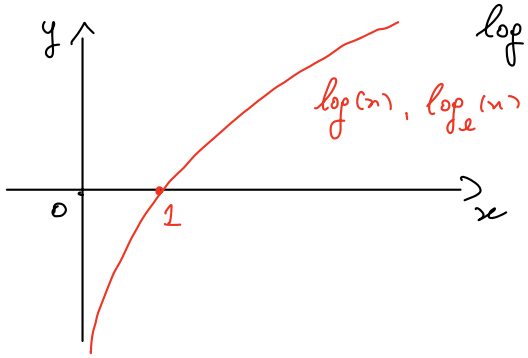
f è iniettiva su $\mathbb{R}$

oss $(a^x)^{x'} = a^{x \cdot x'}$

- Esiste $f^{-1} : (0, +\infty) \subseteq \mathbb{R} \to \mathbb{R}$, funzione inversa di e^x. e si chiama $\ln$, $\log_e$, logaritmo naturale.

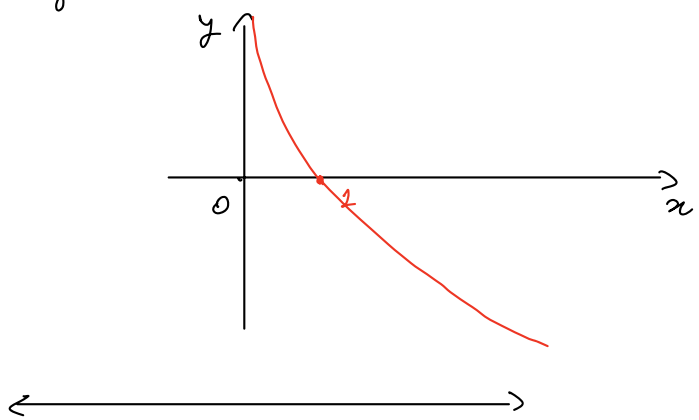
$\log_e(x)$, $\log(x)$ è tale che $e^{\log x} = x \quad \forall x \in (0, +\infty)$

$\log(e^x) = x \quad \forall x \in \mathbb{R}$



- $\log_{10}(x)$ è la funzione inversa di 10^x, quindi $10^{\log_{10}(x)} = x \quad \forall x \in (0, +\infty)$

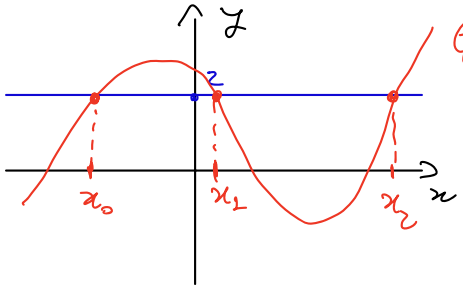
- $\log_a(x)$ è la funzione inversa di a^x.



Problema Risolvere $e^x + e^{-x} = 2$

I MODO $f(x) = e^x + e^{-x}$

Il problema diventa: trovare $x \in D$ tale che $f(x) = 2$



$D = \mathbb{R}$

II MODO Risolvere $e^x = 2 - e^{-x}$

$f(x) = e^x$, $g(x) = 2 - e^{-x}$

Il problema diventa: trovare $x \in D$ t.c. $f(x) = g(x)$

