

Sistemi lineari di m eq. in n variabili, $m, n \in \mathbb{N}$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}, \quad Ax = b$$

$A = (a_{ij}) \in \mathbb{M}(m \times n)$ matrice dei coefficienti

$x_1, \dots, x_n$ variabili

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m \quad \text{termini noti}$$

$$Ax = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix}$$

$$\mathbb{R}^n \ni x \longmapsto Ax \in \mathbb{R}^m, \quad A \in \mathbb{M}(m \times n)$$

$Ax = b$. Dato $b \in \mathbb{R}^m$, cerco $x \in \mathbb{R}^n$ t.c. $Ax = b$

$Ax = b$ ammette almeno una soluzione se b appartiene all'immagine

$$\text{Immagine di } A = \text{Im}(A) := \{ y \in \mathbb{R}^m / \exists x \in \mathbb{R}^n, Ax = y \}.$$

ES $\begin{cases} 2x - y + z = 3 \\ x - z = 1 \end{cases} \quad A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$A \in \mathbb{M}(2 \times 3), \quad b \in \mathbb{R}^2, \quad \mathbb{R}^3 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^2$$

Il sistema ammette almeno una soluzione se $\exists \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ t.c.

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = b$$

$$C = \left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 0 & -1 & 1 \end{array} \right)$$

$$C_2 \rightarrow C_2 - \frac{1}{2} C_1 \quad C = \left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \end{array} \right)$$

$$C_1 \rightarrow \frac{1}{2} C_1, \quad C_2 \rightarrow 2C_2, \quad C = \left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -3 & -1 \end{array} \right)$$

$$C_1 \rightarrow C_1 + \frac{1}{2} C_2, \quad C = \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & -1 \end{array} \right)$$

Il sistema è equivalente a $\begin{cases} x - z = 1 \\ y - 3z = -1 \end{cases} \Leftrightarrow \begin{cases} x - z = 1 \\ y - 3z = -1 \\ z = t \end{cases}$

$$\Leftrightarrow \begin{cases} x = 1 + t \\ y = -1 + 3t \\ z = t \end{cases} \quad \text{Soluzioni } W = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \text{Span} \left(\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \right)$$

In particolare, $A \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 + (-1) \cdot (-1) + 1 \cdot 0 \\ 1 \cdot 1 + 0 \cdot (-1) + (-1) \cdot 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = b$

$$A \begin{pmatrix} 1+t \\ -1+3t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \quad \forall t \in \mathbb{R}$$



Def Dati due spazi vettoriali V e W , si chiama **APPLICAZIONE LINEARE** da V a W una funzione $L: V \rightarrow W$ tale che
(i) $L(\underline{0}) = \underline{0}$;

- (i) $\forall v_1, v_2 \in V$, $L(v_1 + v_2) = L(v_1) + L(v_2)$;
 (ii) $\forall v \in V$, $\forall \lambda \in \mathbb{R}$, $L(\lambda v) = \lambda L(v)$.

Prop $A \in \mathcal{M}(m \times n)$, $A: \mathbb{R}^n \rightarrow \mathbb{R}^m$ è un'applicazione lineare.
 $\left[A \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0, \quad A \left[\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right] = A \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right]$

$Ax = b$ ammette almeno una soluzione se $b \in \text{Im}(A)$

$$A \in \mathcal{M}(m \times n)$$

Prop • $\text{Im}(A) = \{ y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n \text{ che verifica } Ax = y \}$ è un sottospazio vettoriale di $\mathbb{R}^m$.

• $\text{Im}(A) = \text{Span} (A^1, \dots, A^n)$, A^i = colonne i -esime di A .

ES $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \in \mathcal{M}(2 \times 3)$, $A^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $A^2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A: \mathbb{R}^3 \rightarrow \mathbb{R}^2$, $\text{Im}(A)$ è un sottosp. vett. di $\mathbb{R}^2$

$$\text{Im}(A) = \text{Span} \left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ lin. indep. $\Rightarrow \text{Im}(A)$ ha dimensione ≥ 2

$$\Rightarrow \text{Im}(A) = \mathbb{R}^2.$$

Quindi $Ax = b$ ha almeno una soluzione, per ogni $b \in \mathbb{R}^2$.

Def Si chiama RANGO di A la dimensione di $\text{Im}(A)$.

$A \in \mathcal{M}(m \times n)$, $A: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\text{Im}(A) \subseteq \mathbb{R}^m$ sottosp. vett.

• $\text{rank}(A) := \dim(\text{Im}(A)) \in \{0, 1, 2, \dots, m\}$

• $\text{rank}(A) \leq n$

ES $A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ -1 & 2 \end{pmatrix} \in \mathcal{M}(3 \times 2), \quad A: \mathbb{R}^2 \rightarrow \mathbb{R}^3$

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ 2x+3y \\ -x+2y \end{pmatrix} \in \mathbb{R}^3, \quad \text{Im}(A) \subseteq \mathbb{R}^3$$

$$\text{Im}(A) = \text{Span} \left(\underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}}_{A^1}, \underbrace{\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}}_{A^2} \right)$$

$$\text{rank}(A) \leq 3, \quad \text{rank}(A) \leq 2. \Rightarrow \text{rank}(A) \leq 2.$$

Se A^1 e A^2 sono lin. dip. $\Rightarrow \text{rank}(A)=1$, $\text{Im}(A)$ retta

" " " " indep. $\Rightarrow \text{rank}(A)=2$, $\text{Im}(A)$ piano

$\exists \lambda \in \mathbb{R}$ t.c. $A^1 = \lambda A^2$? NO, quindi A^1 e A^2 sono lin. indep.

Per $b \in \mathbb{R}^3$, $Ax=b$ ha almeno una soluzione? $\begin{cases} \text{SÌ, se } b \in \text{Im}(A) \\ \text{NO, se } b \notin \text{Im}(A) \end{cases}$

Prop Un sistema $Ax=b$ ha almeno una soluzione se e solo se $\text{rank}(A) = \text{rank}(C)$ dove $C = (A | b)$.

oss $A \in \mathcal{M}(m \times n), \quad C \in \mathcal{M}(m \times (n+1))$

$$A: \mathbb{R}^n \rightarrow \mathbb{R}^m, \quad C: \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m$$

$$\text{Im}(A) = \text{Span}(A^1, \dots, A^n), \quad \text{Im}(C) = \text{Span}(A^1, \dots, A^n, b)$$



ES
$$\begin{cases} x + 2y + z = 1 \\ 2x + ky + 3z = 0 \\ x + y + 2z = 0 \end{cases}, \quad k \in \mathbb{R} \text{ parametro}$$

Per quali valori di $k \in \mathbb{R}$ il sistema ha soluzioni?

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & k & 3 \\ 1 & 1 & 2 \end{pmatrix} \in \mathcal{M}(3 \times 3), C = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & k & 3 & 0 \\ 1 & 1 & 2 & 0 \end{array} \right) \in \mathcal{M}(3 \times 4)$$

$$\text{rank}(A) \leq 3, \quad \text{rank}(A) = \text{rank}(C) \leq 3$$

$A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, se $\text{rank}(A) = \underline{3}$, allora $Ax=b$ ha sol. $\forall b \in \mathbb{R}^3$

se $\text{rank}(A) \in \{1, 2\}$, studiare $\text{rank}(C)$

$$C = \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 2 & k & 3 & 0 \\ 1 & 1 & 2 & 0 \end{array} \right)$$

$$C_2 \rightarrow C_2 - 2C_1, \quad C = \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{k-4} & 1 & -2 \\ 1 & 1 & 2 & 0 \end{array} \right) \quad \boxed{k \neq 4}$$

$$C_3 \rightarrow C_3 - C_1, \quad C = \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{k-4} & 1 & -2 \\ 0 & -1 & 1 & -1 \end{array} \right)$$

$$C_3 \rightarrow C_3 + \frac{1}{k-4}C_2, \quad C = \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{k-4} & 1 & -2 \\ 0 & 0 & \boxed{\frac{k-3}{k-4}} & \frac{-k+2}{k-4} \end{array} \right) \quad \boxed{k \neq 3}$$

$$1 + \frac{1}{k-4} = \frac{k-3}{k-4} \quad -1 - \frac{2}{k-4} = \frac{-k+2}{k-4}$$

• Se $k \neq 3, 4$ allora il sistema ha soluzioni.

$$(\text{rank}(A) = \text{rank}(C) = 3)$$

$$\boxed{k=3} \quad C = \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{-1} & 1 & -2 \\ 0 & 0 & 0 & \boxed{1} \end{array} \right) \quad \begin{array}{l} \text{rank}(C) = 3 \\ \text{rank}(A) = 2 \end{array} \quad \begin{array}{l} \text{Non esistono} \\ \text{soluzioni} \end{array}$$

$$K=4$$