$\Phi$ geod. flow on $\Sigma$

$\pi(T) := \# \{ \gamma$ closed primitive geodesics $/ \ \ell(\gamma) \leq T \}$

Aim $\exists\, h > 0$ top. entropy of $\Phi$ s.t.

$$\pi(T) \sim \frac{e^{hT}}{hT} \qquad \left( \Rightarrow h := \lim_{T \to +\infty} \frac{1}{T} \log \pi(T) \right)$$

$$\Uparrow$$

$$\exists\, c \in (0,h) \text{ s.t. } \quad \pi(T) = \int_1^{e^{hT}} \frac{1}{\log s}\, ds + O(e^{cT})$$

$$\longleftrightarrow$$

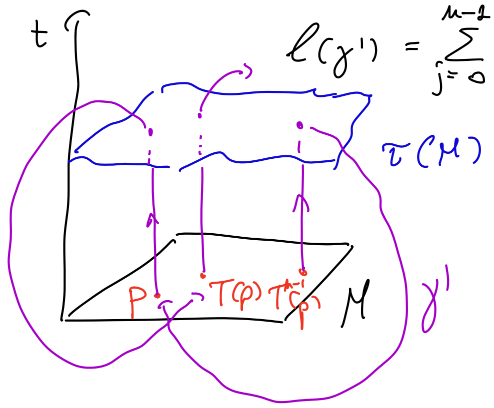
$$\alpha \in \mathbb{C}, \qquad Z^{\Phi}(\alpha) := \prod_{\gamma \in ppo} \left(1 - e^{-\alpha h \ell(\gamma)}\right)^{-1} \quad \text{zeta function}$$

Let's assume that $\exists\ T: M \to M$ base map, $\exists\ \tau : M \to \mathbb{R}^+$ positive s.t. $(\Phi, m)$ is isomorphic to $V_t : M_\tau \to M_\tau$, with $m = \mu \times \ell$

$$Z^V(\alpha) := \prod_{\gamma' \in ppo(V)} \left(1 - e^{-\alpha h \ell(\gamma')}\right)^{-1} =$$

$\gamma'$ per. orbit for $V$ of length $\ell(\gamma')$ $\longleftrightarrow$ $p$ periodic point for $T$ of period $n$

$$\ell(\gamma') = \sum_{j=0}^{n-1} \tau(T^j(p))$$




$$= \exp \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{p \in Fix(T^n)} e^{-\alpha h \left( \sum_{j=0}^{n-1} \tau(T^j(p)) \right)} =$$

$$\left[ \tau \equiv 1 \quad = \exp \sum_{n=1}^{+\infty} \frac{1}{n} e^{-\alpha h n} \# Fix(T^n) = Z^T\left(e^{-\alpha h}\right) \right]$$

$$= \exp \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{p \in \mathrm{Fix}(T^n)} \left[ \prod_{j=0}^{n-1} e^{-\alpha h \tau(T^j(p))} \right] \text{ "=" } Z^T_\tau$$

$$\longleftrightarrow$$

- Dynamical zeta functions for the subshifts of finite type.

$\mathcal{A}$ finite alphabet, $M$ primitive transition matrix

$$\Omega_M = \left\{ \omega = (\omega_0 \omega_1 \dots) \in \mathcal{A}^{\mathbb{N}_0} \,/\, \omega_i \in \mathcal{A} \ \forall i \geq 0,\ m_{\omega_i \omega_{i+1}} = 1 \ \forall i \geq 0 \right\}$$

shift map $\sigma : \Omega_M \to \Omega_M$, $(\sigma(\omega))_i = \omega_{i+1}$.

Let $\vartheta \in (0,1)$, $d_\vartheta(\omega, \tilde\omega) := \vartheta^{N(\omega,\tilde\omega)}$ where

$$N(\omega, \tilde\omega) := \inf \{ i \in \mathbb{N}_0 \,/\, \omega_i \neq \tilde\omega_i \}.$$

$(\Omega_M, d_\vartheta)$ is compact.

$$\left. \begin{aligned} B_\vartheta := \{ u : \Omega_M \to \mathbb{C} \,/\, \|u\|_\vartheta = \|u\|_\infty + |u|_\vartheta < +\infty \text{ where} \\ |u|_\vartheta := \sup_{\omega \neq \tilde\omega} \frac{|u(\omega) - u(\tilde\omega)|}{d_\vartheta(\omega, \tilde\omega)} \end{aligned} \right\} \text{ Lipschitz functions}$$

Def (Ruelle transfer operators)

Given $u \in B_\vartheta$, $\mathcal{L}_u : B_\vartheta \to B_\vartheta$

$$f \mapsto (\mathcal{L}_u f)(\omega) := \sum_{\sigma(\tilde\omega) = \omega} e^{u(\tilde\omega)} f(\tilde\omega)$$

$u$ potential

$$(u(\tilde\omega) = -\log |T'(\tilde\omega)|)$$

Theorem (Ruelle-Perron-Frobenius Theorem)

(a) $u$ is real, $u : \Omega_M \to \mathbb{R}$, $u \in B_\vartheta$. Then:

- $\mathcal{L}_u$ has a simple maximal real eigenvalue $\beta(u)$ with a real positive eigenfunction $f_\beta \in B_\vartheta$;

- $\mathcal{L}_u$ has a spectral gap;
- $\mathcal{L}_u^*$, dual of $\mathcal{L}_u$, has a unique eigenmeasure $m$ s.t.

$\mathcal{L}_u^* m = \beta(u)\, m$, and $f_\beta\, m$ is $\sigma$-invariant.

$$\Big[ \langle m, f_\beta (v\circ\sigma) \rangle = \frac{1}{\beta(u)} \langle \mathcal{L}_u^* m, f_\beta (v\circ\sigma) \rangle =$$

$$= \frac{1}{\beta(u)} \langle m, \mathcal{L}_u ( f_\beta (v\circ\sigma)) \rangle = \langle m, f_\beta\, v \rangle$$

$$\mathcal{L}_u ( f_\beta (v\circ\sigma))(\omega) = \sum_{\sigma(\tilde\omega)=\omega} e^{u(\tilde\omega)} f_\beta(\tilde\omega)\,(v\circ\sigma)(\tilde\omega) =$$

$$= v(\omega)\cdot (\mathcal{L}_u f_\beta)(\omega) = \beta(u)\, v(\omega)\, f_\beta(\omega) \Big]$$

**Def** $u, v \in B_\theta$ are called cohomologous, $v \sim u$, if $\exists\, w \in C^0(\Omega_M)$ s.t. $v = u + w\circ\sigma - w$

**Lemma** If $u$ is real, $u \in B_\theta$, with $\mathcal{L}_u$ with eigenvalue $\beta(u)$, and eigenf. $f_\beta$ then

$$v = u - \log\beta(u) - (\log f_\beta)\circ\sigma + \log f_\beta \sim u - \log\beta$$

and $\mathcal{L}_v 1 = 1$ (in this case $v$ is called "normalised").

$$(\mathcal{L}_v 1)(\omega) = \sum_{\sigma(\tilde\omega)=\omega} e^{u(\tilde\omega)} \frac{1}{\beta(u)} \frac{f_\beta(\tilde\omega)}{f_\beta(\sigma(\tilde\omega))} = \frac{1}{\beta(u)\cdot f_\beta(\omega)} (\mathcal{L}_u f_\beta)(\omega) = 1.$$

(b) $u$ is real and normalised, $u \in B_\theta$, then:

- everything as in (a) with $\beta(u) = 1$, $f_\beta \equiv 1$, $m$ is $\sigma$-invariant

(c) $u$ is not real, $\mathrm{Re}(u)$ is normalised, $u \in B_\theta$, then:

- $\mathcal{L}_u$ has spectral radius bounded by $1$;
- If $\mathcal{L}_u$ has a maximal eigenvalue, $\beta(u) \in \{|z| = 1\}$, it is simple and unique on $\{|z|=1\}$, $\mathcal{L}_u$ has spectral gap;
- If $\mathcal{L}_u$ has no eigenvalue on $\{|z|=1\}$ then the spectral radius of $\mathcal{L}_u$ is $< 1$.

(d) Let $D \subset B_\theta$ denote the set of potentials $u$ for which $\mathcal{L}_u$ has a maximal eigenvalue $\beta(u)$. Then:

- real functions are in $D$, and $D$ is open;
- let $P(u) := \log \beta(u)$ [if $\beta(u)$ is real, $P(u) = \log \beta(u) + 2\pi i \cdot 0$]
  then $P$ is an analytic function
- if $u \sim v + c + 2\pi i \tilde{v}$ with $\tilde{v}$ integer-valued, then

$$P(u) = P(v) + c$$

- $P$ is called the <u>pressure</u>.

<u>Def</u> The dynamical zeta function associated to $u \in B_\theta$, is

$w \in \mathbb{C}$, $$Z_u^\sigma(w) := \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \sum_{\omega \in \mathrm{Fix}(\sigma^n)} \prod_{j=0}^{n-1} e^{u(\sigma^j(\omega))}$$

$$Z_0^\sigma = Z.$$

$$\left[ Z^V(\alpha) = \exp \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{p \in \mathrm{Fix}(T^n)} \prod_{j=0}^{n-1} e^{-\alpha h \tau(T^j(p))} = Z^T_{-\alpha h \tau}(1) \right]$$

converges in $\{\mathrm{Re}(\alpha) > 1\}$

<u>Prop</u> If $u$ is real, $u \in B_\theta$, then $Z_u^\sigma$ has radius of convergence $e^{-P(u)} = \beta(u)^{-1}$. In particular, $P(0) = h(\sigma)$.

<u>proof</u> Let $u(\omega) = u(\omega_0, \omega_1)$.

$$Z_u^\sigma(w) = \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \sum_{\omega \in \mathrm{Fix}(\sigma^n)} \prod_{j=0}^{n-1} e^{u(\sigma^j(\omega))} =$$

$$= \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \sum_{i_1, i_2, \dots, i_n \in A} e^{u(i_1,i_2)} e^{u(i_2,i_3)} \cdots e^{u(i_{n-1},i_n)} \, m_{i_1 i_2} \cdots m_{i_{n-1} i_n} m_{i_n i_1}$$

$$= \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \operatorname{trace}(M_u^n), \quad M_u = (m_{hk} e^{u(h,k)})$$

$$= \frac{1}{\det(1 - w M_u)} \quad \text{converges for } \{|w| < \lambda_{M_u}^{-1}\}.$$

$$(\mathcal{L}_u f)(\omega) = \sum_{\sigma(\tilde\omega)=\omega} e^{u(\tilde\omega)} f(\tilde\omega) = \sum_{i\in\mathcal{A}} m_{i\omega_0} e^{u(i,\omega_0)} f(i\omega)$$

$$F_\vartheta := \{ f\in B_\vartheta \,/\, f(\omega)=f(\omega_0)\}, \quad F_\vartheta \cong \mathbb{C}^{\#\mathcal{A}}$$

$$f\in F_\vartheta, \quad (\mathcal{L}_u f)(\omega) = \sum_{i\in\mathcal{A}} m_{i\omega_0} e^{u(i,\omega_0)} f(i) =$$

$$= \left[ (f(1)\ f(2) \dots )\, M_u \right]_{\omega_0}$$

$$\Rightarrow \beta(u) = \lambda_{M_u} = e^{P(u)} \qquad \square$$

Q. $Z^\sigma_{-\alpha h\tau}$ converges at 1 ?

Yes, if $P(-\alpha h\tau)<0 \Rightarrow 1\in\{|w|<e^{-P(-\alpha h\tau)}\}$.

<u>Theorem</u> Let $u\in B_\vartheta$, s.t. $\mathrm{Re}(u)$ is normalised, then $P(\mathrm{Re}(u))=0$. Then:

- if the spectral radius of $\mathcal{L}_u$ is $<1$, then $\exists\, \varepsilon>0$ s.t. if $v\in B_\vartheta$ with $\|u-v\|_\vartheta<\varepsilon$, then $Z^\sigma_v$ converges at $w=1$;

- if $\mathcal{L}_u$ has a maximal eigenvalue $\beta(u)\in\{|\zeta|=1\}$, then $\exists\,\varepsilon>0$ s.t. if $v\in B_\vartheta$, with $\|u-v\|_\vartheta<\varepsilon$, the function

$$\bar Z^\sigma_v(w) := \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \sum_{\omega\in \mathrm{Fix}(\sigma^n)} \left[ \prod_{j=0}^{n-1} e^{v(\sigma^j(\omega))} - e^{nP(v)} \right]$$

converges at $w=1$, and

$$v \longmapsto Z^\sigma_v(1) := \frac{\bar Z^\sigma_v(1)}{1-e^{P(v)}}$$

gives an extension of $Z^\sigma_u(1)$ to $\{\|u-v\|_\vartheta<\varepsilon\}$, which is non-zero and analytic if 1 is not an eigenvalue of $\mathcal{L}_u$ (if $P(u)\neq 0$).