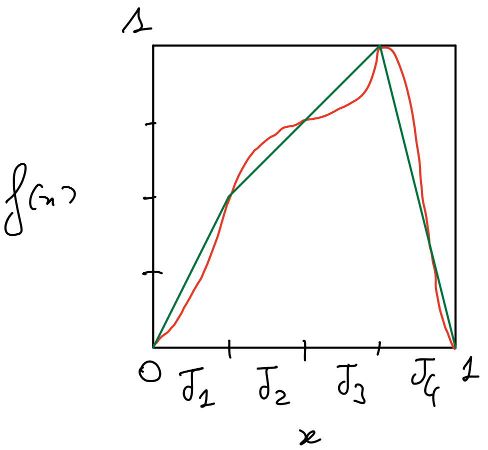
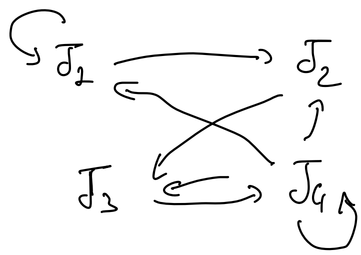
Zeta functions counting periodic orbits



$\mathcal{J} = \{ J_1, J_2, J_3, J_4 \}$ is a Markov partition for f if

$$f(J_i) \cap J_j \neq \emptyset \Rightarrow J_j \subseteq f(J_i)$$



$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$\mathcal{A} = \{ 1, 2, 3, 4 \}$ $\qquad \Omega = \mathcal{A}^{\mathbb{N}_0} = \{ \omega = (\omega_0 \, \omega_1 \dots) \, / \, \omega_i \in \mathcal{A} \;\; \forall i \in \mathbb{N}_0 \}$

$$\varphi : [0,1] \to \Omega \quad , \quad \varphi(x) = \omega(x) = (\omega_i)_{i \geq 0}$$

$$f^j(x) \in J_k \iff \omega_j = k$$

Transition matrix $M = (m_{ij}) \in \mathcal{M}(4 \times 4, \{0,1\})$

$$\Omega_M = \{ \omega \in \Omega \, / \, m_{\omega_i \omega_{i+1}} = 1 \;\; \forall i \geq 0 \}$$

$$\begin{tikzcd} {[0,1]} \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowright"] & {[0,1]} \arrow[d, "\varphi"] \\ \Omega_M \arrow[r, "\sigma"'] & \Omega_M \end{tikzcd}$$

The shift map $(\sigma(\omega))_i = \omega_{i+1} \quad \forall i \geq 0$

Def $\quad \mathcal{A}$ finite alphabet, $M \in \mathcal{M}(\# \mathcal{A} \times \# \mathcal{A}, \{0,1\})$, Ω_M

(Ω_M, σ) subshift of finite type

- Counting periodic orbits for a subshift of finite type.

$\omega \in \mathrm{Fix}(\sigma^n)$

Fixed $n \in \mathbb{N}$, $\omega \in \Omega_M$ is periodic of period n iff
$$\omega = (\overline{s_0 s_1 \dots s_{n-1}}) = (s_0 s_1 \dots s_{n-1} s_0 s_1 \dots)$$

$\boxed{n=1}$ $\quad \# \mathrm{Fix}(\sigma) = \text{trace } M$

$\boxed{n=2}$ $\quad \# \mathrm{Fix}(\sigma^2) = \text{trace } M^2$

$$(M \cdot M)_{ii} = \sum_{k \in A} m_{ik} m_{ki}$$

Lemma $\quad \# \mathrm{Fix}(\sigma^n) = \text{trace}(M^n)$.

Cor (of Perron-Frobenius Thm). If M is primitive ($\exists\, m \in \mathbb{N}$ s.t. M^m has positive entries) then $\exists$ unique maximal real eigenvalue λ_M of M, $\lambda_M > 1$.
Then Eigenvalues$(M) = \{\lambda_M, \lambda_2, \dots \lambda_{\# A}\}$ with $|\lambda_i| < \lambda_M$ $\forall i = 2, \dots, \# A$.

Then $\quad \text{trace}(M^n) = \lambda_M^n + \sum_{i \geq 2} \lambda_i^n \sim \lambda_M^n$

$$\Rightarrow \quad \boxed{h(\sigma) := \lim_{n \to +\infty} \frac{1}{n} \log \# \mathrm{Fix}(\sigma^n)} = \log \lambda_M$$

topological entropy

- Topological entropy in general

Thm (Artin-Mazur '65) There is a C^1-dense set F of diffeos of a smooth compact manifold for which
$$h(T) := \limsup_{n \to +\infty} \frac{1}{n} \log \# \mathrm{Fix}(T^n) \in (0, +\infty) \quad \forall T \in F.$$

Def Given a map $T : \Omega \to \Omega$, $\quad Z^T(w) := \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \# \mathrm{Fix}(T^n)$
$w \in \mathbb{C}$

is the Artin-Mazur zeta function of T.

Z^T has radius of convergence $e^{-h(T)}$.

- Zeta function for subshift of finite type.

$$Z^\sigma(w) = \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \# \mathrm{Fix}(\sigma^n) = \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \operatorname{trace}(M^n) =$$

$$w^n \operatorname{trace}(M^n) = \operatorname{trace}((wM)^n)$$

$$= \exp \sum_{n=1}^{+\infty} \frac{1}{n} \operatorname{trace}((wM)^n) = \exp\left(\operatorname{trace}\left(\sum_{n=1}^{+\infty} \frac{(wM)^n}{n}\right)\right) =$$

$$\log(1-x) = -\sum_{n=1}^{+\infty} \frac{x^n}{n}, \quad |x|<1$$

$$= \exp\left(\operatorname{trace}\left(-\log(1-wM)\right)\right) = \det\left(\exp\left(-\log(1-wM)\right)\right)$$

Z^σ converges in $\{|w| < \lambda_M^{-1} = e^{-h(\sigma)}\}$, $\|wM\|<1$

$$= \frac{1}{\det(1-wM)} = Z^\sigma(w)$$ meromorphic on $\mathbb{C}$ with poles at λ_i^{-1} with finite order.

There is a simple pole at λ_M^{-1}.

- Zeta function for flows.

Let $T: \Omega \to \Omega$ be a map, $w = e^{-\alpha h(T)} = e^{-\alpha h}$, $\alpha \in \mathbb{C}$

$$Z^T(w) = \exp \sum_{n=1}^{+\infty} \frac{w^n}{n} \# \mathrm{Fix}(T^n)$$ $\{|w| < e^{-h}\}$

$\Updownarrow$

$\{\mathrm{Re}(\alpha) > 1\}$

$\downarrow$

$$Z^T(\alpha) = \exp \sum_{n=1}^{+\infty} \frac{e^{-\alpha h n}}{n} \# \mathrm{Fix}(T^n), \quad \# \mathrm{Fix}(T^n) = \sum_{\omega \in \mathrm{Fix}(T^n)} 1$$

$$= \exp \sum_{n=1}^{+\infty} \frac{e^{-\alpha h n}}{n} \sum_{\omega \in \mathrm{Fix}(T^n)} 1 =$$

$$= \exp \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{\omega \in \mathrm{Fix}(T^n)} (e^{-\alpha h})^n =$$

$\omega \in \mathrm{Fix}(T^n) \iff \exists\, \gamma$ primitive periodic orbit (ppo) of period $l(\gamma)$ s.t. $n = m\, l(\gamma)$ for $m \in \mathbb{N}$.

$\gamma \in$ ppo of period $l(\gamma)$ $\exists$ $l(\gamma)$ points in $\mathrm{Fix}(T^{l(\gamma)})$

$$= \exp \sum_{\gamma \in \mathrm{ppo}} \sum_{m=1}^{+\infty} \frac{1}{m\, l(\gamma)} (e^{-\alpha h})^{m\, l(\gamma)}\, l(\gamma) =$$

$$= \exp \sum_{\gamma \in \mathrm{ppo}} \sum_{m=1}^{+\infty} \frac{1}{m} e^{-\alpha m\, l(\gamma) h}$$

$$= \exp \sum_{\gamma \in \mathrm{ppo}} \left[- \log (1 - e^{-\alpha\, l(\gamma) h}) \right] =$$

$$= \prod_{\gamma \in \mathrm{ppo}} (1 - e^{-\alpha\, l(\gamma) h})^{-1} = Z^T(\alpha)$$

<u>Theorem</u> (Bowen '72) The geodesic flow Φ on a smooth closed connected orientable Riemannian surface with negative curvature Σ has countably many primitive periodic orbits, so $Z^{\Phi}(\alpha)$ is well defined. And $\exists$

$$h(\Phi) = \lim_{T \to +\infty} \frac{1}{T} \log \pi(T) < +\infty$$

where $\pi(T) := \# \{ \gamma \text{ ppo} \,/\, l(\gamma) \le T \}$. $h(\Phi)$ is the topological entropy and $Z^{\Phi}(\alpha)$ converges in $\mathrm{Re}(\alpha) > 1$.

$$Z^{\Phi}(\alpha) = \prod_{\gamma \in \mathrm{ppo}} (1 - e^{-\alpha\, l(\gamma)\, h(\Phi)})^{-1}.$$

<u>Theorem</u> (Parry-Pollicott '83, Pollicott-Sharp '88) For the geodesic flow Φ on Σ, we have

$$\pi(T) = \int_1^{e^{hT}} \frac{1}{\log s}\, ds + O(e^{cT})$$

$$e^{cT} = (e^{hT})^{\frac{c}{h}}, \quad \frac{c}{h} < 1$$

where $h = h(\Phi)$ and $c \in (0, h)$.

We will prove that $\pi(T) \sim \frac{e^{hT}}{hT} \sim \int_2^{e^{hT}} \frac{1}{\log s}\, ds$.