Exponential decay of correlations for the geodesic flow on Σ
(closed negatively curved Riemannian surface)

$$\Phi = \{\varphi_t\} \;, \quad \varphi_t : T^1\Sigma \to T^1\Sigma \;, \quad \mu \text{ Liouville measure.}$$

$$C_\Phi(f,g,t) = \int_{T^1\Sigma} f \; U_t g \, d\mu \;, \quad \int_{T^1\Sigma} f \, d\mu = 0 \;, \quad U_t : g \mapsto g \circ \varphi_t$$

<u>Thm</u> $\exists\, C>0,\ \vartheta \in (0,1)$ s.t. $\forall f, g \in C^1(T^1\Sigma)$

$$|C_\Phi(f,g,t)| \le C \, \|f\|_1 \, \|g\|_1 \, \vartheta^t \qquad \forall t>0.$$

$$\int_{T^1\Sigma} f \; U_t g \, d\mu = \int_{T^1\Sigma} (f \circ \varphi_{-t}) \, g \, d\mu$$

$$\mathcal{L}_t : f \longmapsto f \circ \varphi_{-t}$$

<u>Def</u> Let $\lambda \in (0,1)$, $\mu > 1$ be the constants in the Anosov property of Φ.
Let $\sigma \in (0,+\infty)$ s.t. $e^\sigma \in (1, \min\{\lambda^{-1}, \mu\})$. Then:

$(z,v) \in T^1\Sigma$, $v \in T_z\Sigma$, $|v| = 1$

$$d_s((z,v),(z',v')) = \int_0^{+\infty} e^{\sigma t} \, d(\varphi_t(z,v), \varphi_t(z',v')) \, dt$$

$$d_u((z,v),(z',v')) = \int_{-\infty}^{0} e^{-\sigma t} \, d(\varphi_t(z,v), \varphi_t(z',v')) \, dt$$

$$d_{s,u} < +\infty \iff d(\varphi_t(z,v), \varphi_t(z',v')) \xrightarrow[t \to \pm\infty]{} 0$$

<u>Def</u> Fix $\delta > 0$, $\beta > 0$, for $h \in C^1(T^1\Sigma)$

$$H_{s,\beta}(h) := \sup_{d_s((z,v),(z',v'))<\delta} \frac{|h(z,v) - h(z',v')|}{d_s((z,v),(z',v'))^\beta}$$

$$|h|_{s,\beta} := \|h\|_\infty + H_{s,\beta}(h) \;, \qquad C_s^\beta = \overline{C^1}^{\,|\cdot|_{s,\beta}}$$

<u>Def</u> Let $f \in C^1(T^1\Sigma)$, we define

$$|f|_w := \sup_{h \in C_s^1,\ |h|_{s,1} \le 1} \int_{T^1\Sigma} f \cdot h \, d\mu$$

$\beta \in (0,1)$ $\quad \|f\|_{\mathcal{B}} := \sup_{h \in C^{\beta}_{s},\, |h|_{s,\beta} \le 1} \int_{T^1\Sigma} f \cdot h \, d\mu \quad + \quad H_{u,\beta}(f)$

$$\mathcal{B}_w := \overline{C^1}^{\,|\cdot|_w}, \qquad \mathcal{B} := \overline{C^1}^{\,\|\cdot\|_{\mathcal{B}}}$$

(*) Theorem (Liverani 2004) $\{\mathcal{L}_t\}$ is a strongly continuous group of operators on $\mathcal{B}$.
Moreover, $\exists\, C>0$, $\vartheta \in (0,1)$ s.t. $\forall f \in C^1(T^1\Sigma)$, $\int_{T^1\Sigma} f \, d\mu = 0$

$$\|\mathcal{L}_t f\|_{\mathcal{B}} \le C \|f\|_{C^1} \vartheta^t, \quad \forall t > 0.$$

Proof of Exp Decay

$$|C_V(f,g,t)| = \left| \int_{T^1\Sigma} (\mathcal{L}_t f) \cdot g \, d\mu \right| \le \|\mathcal{L}_t f\|_{\mathcal{B}} \, |g|_{s,\beta} \le C \|f\|_{C^1} \|g\|_{C^1} \vartheta^t$$

$\square$

Proof of (*) $\qquad L := \lim_{t \to 0} \frac{\mathcal{L}_t - \mathrm{Id}}{t}$ has a domain $D(L)$

Lemma 1 $\quad \|\mathcal{L}_t f\|_{\mathcal{B}} \le \|f\|_{\mathcal{B}}$

Then $\operatorname{spec}(L|_{\mathcal{B}}) \subset \{\operatorname{Re}(z) \le 0\}$. So

$$\{\operatorname{Re}(z) > 0\} \ni z \longmapsto R(z) := (z - L)^{-1} = \int_0^{+\infty} e^{-zt} \mathcal{L}_t \, dt$$

resolvent of L

Lemma 2 $\quad$ Let $f \in D(L) \cap D(L^2) \cap C^0$. Then:

- $\mathcal{L}_t f = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(\rho + i\eta)t} R(\rho + i\eta) f \, d\eta$ where $z = \rho + i\eta$, $\rho > 0$.

- $R(z) f = \frac{1}{z} f - \frac{1}{z^2} L(f) + \frac{1}{z^2} R(z) L^2(f) \quad \forall z = \rho + i\eta, \rho > 0$

Lemma 3 $\quad \forall z$ with $\rho = \operatorname{Re}(z) > 0$, $\forall \beta' \in (0,\beta)$, $\exists C_1 > 0$ s.t. $\forall f \in C^1(T^1\Sigma)$

$$\|R(z) f\|_{\mathcal{B}} \le \rho^{-1} \|f\|_{\mathcal{B}}, \quad \|R(z)^j f\|_{\mathcal{B}} \le \frac{3}{(\rho + \sigma\beta')^j} \|f\|_{\mathcal{B}} + C_1 \rho^{-j} |f|_w$$

$\forall j \in \mathbb{N}$.

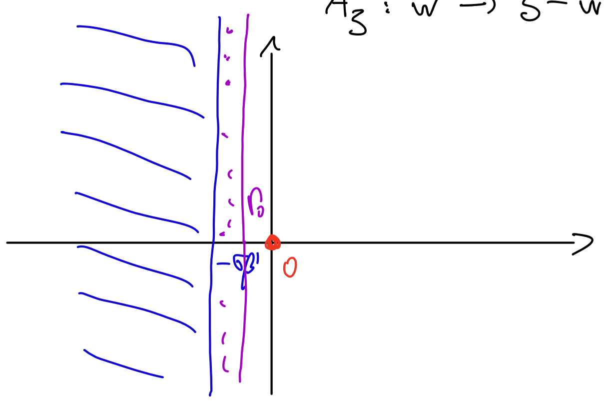
Then, $\text{ess spec}\left(R(\zeta)|_{\mathcal{B}}\right) \subset B\left(0, (\rho+\sigma\beta')^{-1}\right)$. Then,

$$L = \zeta - R(\zeta)^{-1} \quad \forall\, \zeta = \rho + i\eta,\ \rho > 0$$

$$\Rightarrow \quad \text{ess spec}\left(L|_{\mathcal{B}}\right) \subset \bigcap_{\zeta,\ \mathrm{Re}(\zeta)>0} A_\zeta\left(B\left(0, (\rho+\sigma\beta')^{-1}\right)\right) = \left\{\ell \in \mathbb{C} \,/\, \mathrm{Re}(\ell) \leq -\sigma\beta'\right\}$$

$$A_\zeta : w \to \zeta - w^{-1}$$

$\text{spec}(L|_{\mathcal{B}})$



$\mathbb{C}$

<u>Lemma 4</u> $\quad \exists\, r_0 > 0$ s.t. $\text{spec}(L) \cap \left\{\ell \in \mathbb{C} \,/\, \mathrm{Re}(\ell) > -r_0\right\} = \{0\}$

<u>Lemma 5</u> $\quad \exists\, r \in (0, r_0)$ for which $\exists\, C_2 > 0$ s.t. $\forall\, \rho \in (-r, 0)$

$$\| R(\rho + i\eta) \|_{\mathcal{B}} \leq C_2 \left(1 + \sqrt{|\eta|}\right)$$

$f \in D(L) \cap D(L^2) \cap C^0$, $\quad \int_{T^1\Sigma} f \, d\mu = 0$

$$\mathcal{L}_t f = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(-r+i\eta)t} R(-r+i\eta) f \, d\eta = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(-r+i\eta)t} \left(R(-r+i\eta) - \frac{1}{-r+i\eta}\right) f \, d\eta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{(-r+i\eta)t} \left(-\frac{1}{(-r+i\eta)^2} L(f) + \frac{1}{(-r+i\eta)^2} R(-r+i\eta) L^2(f) \right) d\eta$$

$$\| \mathcal{L}_t f \|_{\mathcal{B}} \leq \frac{e^{-rt}}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{r^2+\eta^2} \Big(\| L(f) \|_{\mathcal{B}} + \underbrace{\| R(-r+i\eta) \|_{\mathcal{B}}}_{\overset{\leq}{\text{L.5}}\ C_2(1+\sqrt{|\eta|})} \| L^2(f) \|_{\mathcal{B}} \Big) d\eta$$

$$\leq C_3 \left(\| L(f) \|_{\mathcal{B}} + \| L^2(f) \|_{\mathcal{B}} \right) e^{-rt}, \qquad \vartheta = e^{-r} \in (0,1)$$

Let $f \in C^1(T^1\Sigma)$, $\int_{T^1\Sigma} f \, d\mu = 0$, $\quad \forall\, \varepsilon > 0 \ \overset{\exists\, C_4 > 0}{\exists}\ f_\varepsilon \in D(L^n) \cap C^1 \ \forall n$

s.t. $\quad \| f - f_\varepsilon \|_{\mathcal{B}} \leq \varepsilon^{\beta} \| f \|_{C^1} \cdot \sup_{t \in [0,1]} \| \varphi_{-t} \|_{C^1}$

$$\|L^2(f_\varepsilon)\|_{\mathcal{B}} \le \varepsilon^{-2} \|f\|_{C^1} C_4$$

Then $\forall t > 0$, $\forall \varepsilon > 0$

$$\|\mathcal{L}_t f\|_{\mathcal{B}} \le \|\mathcal{L}_t f_\varepsilon\|_{\mathcal{B}} + \|\mathcal{L}_t (f - f_\varepsilon)\|_{\mathcal{B}} \le$$

$$\le C_3 \left(\|L(f_\varepsilon)\|_{\mathcal{B}} + \|L^2(f_\varepsilon)\|_{\mathcal{B}} \right) e^{-rt} + \|f - f_\varepsilon\|_{\mathcal{B}}$$

$$\le C_5 \left(\varepsilon^{-2} e^{-rt} + \varepsilon^{\beta} \right) \|f\|_{C^1} = C_5 \|f\|_{C^1} \vartheta^t$$

Choosing $\varepsilon = e^{-r(\beta+2)^{-1} t}$, $\quad \varepsilon^{-2} e^{-rt} + \varepsilon^{\beta} = \Big(\underbrace{e^{-r\beta(\beta+2)^{-1}}}_{\vartheta} \Big)^t$ $\square$

$\longleftrightarrow$

Remark $\quad T : M \to M$ base map, $\quad \tau : M \to (0, +\infty)$ positive roof function

$$M_\tau = \{ (p, s) \,/\, p \in M, \; 0 \le s \le \tau(p) \} /_\sim$$

$$(p, \tau(p)) \sim (T(p), 0)$$

$$V_t : (p, s) \longmapsto (p, s+t).$$

$f, g : M_\tau \to \mathbb{C}$, $\quad C_V(f, g, t) = \int_{M_\tau} f \cdot U_t g \, d\mu$

$\int_{M_\tau} f \, d\mu = 0$

$\mu = \dfrac{m \times l}{\int_M \tau \, dm}$, $\quad m$ is a T-inv. prob. measure on M

$\tau \in L^1(M, m)$

l is Lebesgue

Idea (Pollicott, 85): Look at the analytic properties of $\hat{C}_V(\rho)$ and use Paley-Wiener Theorems.

$$\hat{C}_V(\rho) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{i\rho t} C_V(f, g, t) \, dt, \quad C_V(f, g, t) = 0 \;\; \forall t < 0.$$

$$C_V(f, g, t) = \int_{M_\tau} f(p, s) \, g(p, s+t) \, d\mu(p, s) =$$

$$g(p, s+t) = \sum_{n=0}^{+\infty} \int_0^{\tau(T^n(p))} g(T^n(p), v)\, \delta\Big(s+t-v-\underbrace{\sum_{j=0}^{n-1} \tau(T^j(p))}_{\tau^n(p)}\Big)\, dv$$

$$s+t = \sum_{j=0}^{n-1} \tau(T^j(p)) + v \quad , \quad v \in [0, \tau(T^n(p))).$$

$$= \frac{1}{\int_M \tau\, dm} \sum_{n=0}^{+\infty} \int_M \int_0^{\tau(p)} f(p,s) \left(\int_0^{\tau(T^n(p))} g(T^n(p), v)\, \delta(s+t-v-\tau^n(p))\, dv \right) ds\, dm$$

$$\hat{C}_v(\rho) = \frac{1}{2\pi i \int_M \tau\, dm} \int_M \sum_{n=0}^{+\infty} e^{i\rho\, \tau^n(p)} \underbrace{\left(\int_0^{\tau(p)} f(p,s)\, e^{-i\rho s}\, ds \right)}_{\overline{F}(p,\rho)} \underbrace{\left(\int_0^{\tau(T^n(p))} g(T^n(p), v)\, e^{i\rho v}\, dv \right)}_{G(T^n(p),\rho)} dm$$

$$t = v + \tau^n(p) - s$$

$$= \frac{1}{2\pi i \int_M \tau\, dm} \int_M \sum_{n=0}^{+\infty} e^{i\rho\, \tau^n(p)}\, \overline{F}(p,\rho)\, G(T^n(p), \rho)\, dm(p)$$

$$= \frac{1}{2\pi i \int_M \tau\, dm} \int_M \sum_{n=0}^{+\infty} G(p,\rho)\, \underline{\mathcal{L}^n \left(e^{i\rho\, \tau^n(p)}\, \overline{F}(p,\rho) \right)}\, dm(p)$$

$\underline{n=1}$ $\quad \mathcal{L}\left(e^{i\tau(p)} F(p) \right)$

$\underline{n=2}$ $\quad \mathcal{L}\left(\mathcal{L}\left(e^{i\tau(T(p))} e^{i\tau(p)} F(p) \right)\right)$