Theorem (Hennion) $(\mathcal{B}_w, |\cdot|_w)$, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ Banach spaces

$\mathcal{B} \subset \mathcal{B}_w$ compactly

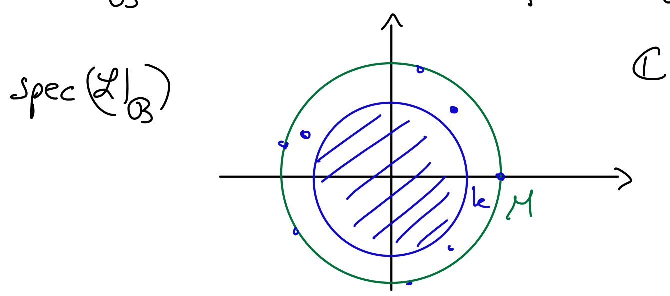
$\mathcal{L} : \mathcal{B}_w \to \mathcal{B}_w$, $\quad \mathcal{L}(\mathcal{B}) \subseteq \mathcal{B}$.

If $\exists\, C_H > 0$, $M > k > 0$, s.t.

$$|\mathcal{L}^j f|_w \le C_H M^j |f|_w \qquad \forall f \in \mathcal{B}_w$$

$$\|\mathcal{L}^j f\|_{\mathcal{B}} \le C_H k^j \|f\|_{\mathcal{B}} + C_H M^j |f|_w \qquad \forall f \in \mathcal{B}$$

then $\quad \operatorname{spec}(\mathcal{L}|_{\mathcal{B}}) \subseteq B(0, M)$, $\quad \operatorname{esspec}(\mathcal{L}|_{\mathcal{B}}) \subseteq B(0, k)$.



Toral automorphisms $\qquad \mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$, $\quad A \in SL(2, \mathbb{Z})$

$$T_A : \mathbb{T}^2 \to \mathbb{T}^2, \qquad T_A(x, y) = A \binom{x}{y} \pmod{\mathbb{Z}^2}$$

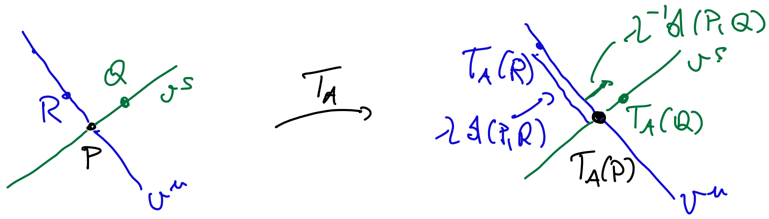
Ex Arnold cat map $\quad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

A is hyperbolic if all eigenvalues are in $\mathbb{C} \setminus \{|\xi| = 1\}$.

A symmetric

$\Rightarrow$ Eigenvalues $= \{\lambda, \lambda^{-1}\}$, $|\lambda| > 1$,

Eigenvectors $= \{v^u, v^s\}$, $\quad A v^u = \lambda v^u$, $\quad A v^s = \lambda^{-1} v^s$



Theorem If A is hyperbolic and symmetric, then T_A preserves the Lebesgue measure m on $\mathbb{T}^2$, it is ergodic and mixing wrt m, and it has exponential decay of correlations on C^∞ observables.

<u>proof</u> • Let $A = 1 \Rightarrow m$ is T_A-invariant.

• mixing $\Rightarrow$ ergodicity

• $C_{T_A}(f,g,j) \xrightarrow[j\to+\infty]{} 0 \qquad \forall f,g \in L^2(\mathbb{T}^2)$

$$\int_{\mathbb{T}^2} f \cdot \overline{(g \circ T_A^j)}\, dm - \left(\int_{\mathbb{T}^2} f\, dm\right)\left(\int_{\mathbb{T}^2} g\, dm\right)$$

$$f(x,y) = e^{2\pi i (mx+ny)}, \quad g(x,y) = e^{2\pi i (m'x+n'y)}$$

for $m, n, m', n' \in \mathbb{Z}$

$$(g\circ T_A^j)(x,y) = e^{2\pi i (<(m',n'), T_A^j(x,y)>)} = e^{2\pi i (<(m',n'), A^j(x,y)>)}$$
$$= e^{2\pi i (<A^j(m',n'), (x,y)>)}$$

$$C_{T_A}(f,g,j) = \int_{\mathbb{T}^2} e^{2\pi i (<(m,n) - A^j(m',n'), (x,y)>)} \, dm$$

$$(m',n') = c_u v^u + c_s v^s \Rightarrow A^j(m',n') = c_u \lambda^j v^u + c_s \lambda^{-j} v^s$$

Then $\forall m,n,m',n' \in \mathbb{Z}$, $\exists\, k>0$ s.t. $j>k \Rightarrow A^j(m',n') \neq (m,n)$

$\Rightarrow \quad C_{T_A}(f,g,j) = 0 \quad \forall j > k.$

• $U_{T_A} : L^\infty(\mathbb{T}^2) \to L^\infty(\mathbb{T}^2), \quad g \mapsto U_T g = g \circ T_A.$

$f \in L^1,\ g \in L^\infty$

$$\int_{\mathbb{T}^2} f \cdot U_{T_A} g\, dm = \int_{\mathbb{T}^2} \underbrace{(f\circ T_A^{-1})}_{=\mathcal{L}f}\, g\, dm$$

$$\mathcal{L} : L^1 \to L^1, \quad f \mapsto \mathcal{L}f = f \circ T_A^{-1}$$

$f \in C^\infty, \quad \partial_u f := <v^u, \nabla f>, \quad \partial_s f := <v^s, \nabla f>$

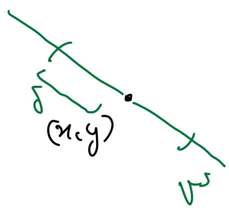
$\psi \in C_0^\infty([-\delta,\delta])$, $\delta>0$ fixed ,

$$\|\psi\|_{C^q} = \max_{0\le q'\le q} \|\psi^{(q')}\|_\infty$$

q'-th derivative

For $p,q\in\mathbb{N}_0$,

$$\|f\|_{p,q} := \sup_{(x,y)\in\mathbb{T}^2} \sup_{0\le p'\le p} \left(\sup_{\substack{\psi\in C_0^\infty(-\delta,\delta)\\ \|\psi\|_{C^q}\le 1}} \int_{-\delta}^{\delta} \partial_u^{p'} f((x,y)+t v^s)\,\psi(t)\,dt \right)$$



$$B^{p,q} := \overline{C^\infty}^{\|\cdot\|_{p,q}}$$

<u>Lemma</u> $\forall\, p,q\in\mathbb{N}_0$ we have:

(i) If $p>0$ then $B^{p,q}$ is continuously embedded in $(C^q)^*$;

(ii) $\partial_u : B^{p+1,q}\to B^{p,q}$ is bounded, and its kernel is the set of constant functions.

(iii) If $p>0$ then $B^{p,q}\subset B^{p-1,q+1}$ compactly.

For us it's enough to consider $p=1,\ q=0$.

$$B^{1,0}\subset B^{0,1}$$

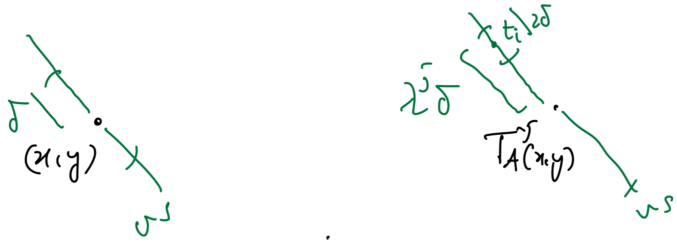
Aim $\exists\, C_H>0$, $1>k>0$ s.t.

$$\|\mathcal{L}^j f\|_{0,1} \le C_H \|f\|_{0,1}$$

$$\|\mathcal{L}^j f\|_{1,0} \le C_H k^j \|f\|_{1,0} + C_H \|f\|_{0,1}$$

$\|\mathcal{L}^j f\|_{0,1}$ $\displaystyle\int_{-\delta}^{\delta} (\mathcal{L}^j f)((x,y)+t v^s)\,\psi(t)\,dt =$

$$= \int_{-\delta}^{\delta} f\left(T_A^{-j}((x,y)+t v^s)\right)\psi(t)\,dt =$$



$$= \lambda^{-j} \int_{-\delta\lambda^j}^{\delta\lambda^j} f\left(T_A^{-j}(x,y) + t v^s\right) \psi(\lambda^{-j} t)\, dt =$$

$$= \sum_i \lambda^{-j} \int_{t_i-\delta}^{t_i+\delta} f\left(T_A^{-j}(x,y) + \tau v^s\right) \psi(\lambda^{-j}\tau)\, \underbrace{\psi_i(\tau)}_{C^\infty \text{ partition of unity}}\, d\tau$$

$$\forall i \qquad \left| \int_{t_i-\delta}^{t_i+\delta} \cdots \right| \leq C_1 \|f\|_{0,1} \|\psi\|_{C^1}$$

$$\|\mathcal{L}^j f\|_{0,1} \leq C_1 \|f\|_{0,1}$$

$$\|\mathcal{L}^j f\|_{1,0} \qquad \int_{-\delta}^{\delta} \partial_u^{p'} (\mathcal{L}^j f)\left((x,y) + t v^s\right) \psi(t)\, dt$$

$$p' = 0, 1.$$

$$p' = 0, \qquad \left| \quad " \quad \right| \leq C_1 \|f\|_{0,1}$$

$$p' = 1,$$

$$\partial_u(\mathcal{L}^j f) = \langle v^u, \nabla(\mathcal{L}^j f)\rangle = \langle v^u, \nabla(f \circ T_A^{-j})\rangle$$
$$= \langle A^{-j} v^u, (\nabla f)\circ T_A^{-j}\rangle = \lambda^{-j} \langle v^u, \mathcal{L}^j(\nabla f)\rangle$$
$$= \lambda^{-j} \mathcal{L}^j(\partial_u f)$$

$$\left| \int_{-\delta}^{\delta} \partial_u(\mathcal{L}^j f)\left((x,y)+t v^s\right) \psi(t)\, dt \right| = \left| \int_{-\delta}^{\delta} \lambda^{-j} \mathcal{L}^j(\partial_u f)\left((x,y) + t v^s\right) \psi(t)\, dt \right|$$

$$\leq C_1 \lambda^{-j} \|\partial_u f\|_{0,1} \|\psi\|_{C^1}. \leq C_1 \lambda^{-j} \|f\|_{1,0} \|\psi\|_{C^1}$$

$$\|\mathcal{L}^j f\|_{1,0} \leq C_1 \lambda^{-j} \|f\|_{1,0} + C_1 \|f\|_{0,1}.$$

$\Rightarrow$ Hennion's Thm work on $B^{1,0} \subset B^{0,1}$. with $k = \lambda^{-1} \in (0,1)$.

- $\operatorname{spec}(\mathcal{L}|_{\mathcal{B}^{1,0}}) \cap \{|\zeta| > \lambda^{-1}\} = \{1\}$

Let $\ell \in \{|\zeta| > \lambda^{-1}\}$, $f \in \mathcal{B}^{1,0}$ s.t. $\mathcal{L}f = \ell f$, $f \neq 0$.

$\partial_u(\ell f) = \ell \partial_u f = \partial_u(\mathcal{L}f) = \lambda^{-1}\mathcal{L}(\partial_u f)$

$\Rightarrow \partial_u f \in \mathcal{B}^{0,0}$ is an eigenfunct with eigenv. $\ell\lambda$

But $\operatorname{spec}(\mathcal{L}|_{\mathcal{B}^{0,0}}) \subset B(0,1) \Rightarrow |\ell\lambda| \leq 1$

$\Rightarrow \lambda^{-1} < |\ell| \leq \lambda^{-1}$ contradiction

$\Rightarrow \partial_u f = 0 \Rightarrow f = \text{cst} \Rightarrow \ell = 1$.

- $$C_{T_A}(f,g,j) = \int_{\mathbb{T}^2} f\, U^j_{T_A} g\, dm - \left(\int_{\mathbb{T}^2} f\, dm\right)\left(\int_{\mathbb{T}^2} g\, dm\right)$$

If $f, g \in C^\infty(\mathbb{T}^2)$, then

$$\int_{\mathbb{T}^2} f\, U^j_{T_A} g\, dm = \int_{\mathbb{T}^2} (\mathcal{L}^j f)\, g\, dm = \int_{\mathbb{T}^2} \left(\int_{\mathbb{T}^2} f\, dm\right) g\, dm + \int_{\mathbb{T}^2} (R^j f)\, g\, dm$$

$$\mathcal{L}^j|_{\mathcal{B}^{1,0}} = \Pi_1 + R^j$$

$$\Pi_1(f) = \int_{\mathbb{T}^2} f\, dm, \quad \|R^j f\|_{1,0} \leq \lambda^{-j}\|f\|_{1,0}$$

$$|C_{T_A}(f,g,j)| = \left|\int_{\mathbb{T}^2} (R^j f)\, g\, dm\right| \leq C\lambda^{-j}\|f\|_{1,0}\|g\|_\infty \qquad \square$$

<u>Rem</u> Wlog $\int_X f\, d\mu = 0$ wrt T-inv. meas μ. ($\Rightarrow \int Ug\, d\mu = \int g\, d\mu$)

Let f_0 have zero-mean, $f = f_0 + c$, $c \in \mathbb{R}$

$$C_T(f,g,j) = \int_X f\, U^j_T g\, d\mu - \left(\int_X f\, d\mu\right)\left(\int_X g\, d\mu\right) =$$

$$= \int_X f_0\, U^j_T g\, d\mu + c\int_X U^j_T g\, d\mu - c\int_X g\, d\mu = \int_X f_0\, U^j_T g\, d\mu$$