$T: S^1 \to S^1$, $T \in C^2$, $|T'| \geq \lambda > 1$, $m =$ reference measure (Lebesgue)

$\mathcal{L}: L^1 \to L^1$, $U: L^\infty \to L^\infty$, $g \mapsto g \circ T$

$$\int_{S^1} f\,(Ug)\,dx = \int_{S^1} (\mathcal{L}f)\, g\, dm \qquad \forall f \in L^1,\ g \in L^\infty.$$

$$(\mathcal{L}f)(x) = \sum_{T(y)=x} \frac{1}{|T'(y)|} f(y)$$

$$N := \max_{x \in S^1} \#\{ y \,/\, T(y) = x \} \leq \|T'\|_\infty < +\infty$$

<u>Remark</u> $\mathcal{L}: L^1 \to L^1$, linear, positive, $\|\mathcal{L}\|_1 = 1$, $\|\mathcal{L}^j f\|_1 \leq \|f\|_1$ $\forall j \in \mathbb{N}$

$$(\mathcal{L}f)'(x) = \operatorname{sgn}(T') \sum_{T(y)=x} \left(\frac{f'(y)}{|T'(y)|^2} - \frac{T''(y)}{|T'(y)|^3} f(y) \right)$$

$$\Rightarrow \quad \|(\mathcal{L}f)'\|_1 \leq \left\| \mathcal{L}\left(\frac{f'}{|T'|}\right)\right\|_1 + \left\| \mathcal{L}\left(\frac{T''}{(T')^2} f\right)\right\|_1$$

$$\leq \frac{1}{\lambda} \|f'\|_1 + \frac{\|T''\|_\infty}{\lambda^2} \|f\|_1$$

$$\|(\mathcal{L}^j f)'\|_1 \leq \frac{1}{\lambda} \|(\mathcal{L}^{j-1} f)'\|_1 + \frac{\|T''\|_\infty}{\lambda^2} \|\mathcal{L}^{j-1} f\|_1$$

$$\leq \frac{1}{\lambda}\left(\frac{1}{\lambda} \|(\mathcal{L}^{j-2} f)'\|_1 + \frac{\|T''\|_\infty}{\lambda^2} \underbrace{\|\mathcal{L}^{j-2} f\|_1}_{\leq \|f\|_1} \right) + \frac{\|T''\|_\infty}{\lambda^2} \|f\|_1$$

$$= \frac{1}{\lambda^2} \|(\mathcal{L}^{j-2} f)'\|_1 + \frac{\|T''\|_\infty}{\lambda^2} \left(1 + \frac{1}{\lambda}\right) \|f\|_1 \leq \ldots$$

$$\ldots \leq \frac{1}{\lambda^j} \|f'\|_1 + \frac{\|T''\|_\infty}{\lambda^2} \left(\sum_{i=0}^{+\infty} \lambda^{-i} \right) \|f\|_1$$

<u>Step 1</u> $\exists\, h \in W^{1,2}$ s.t. $\mathcal{L}h = h$ and $d\mu = h\,dx$ is T-invariant and $h > 0$.

$$g_n := \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}^j 1, \quad 1 \in W^{1,2}. \quad , \text{ need that } \|g_n\|_{W^{1,2}} \leq \text{cost } \forall n.$$

Given $f \in W^{1,2}$, $\quad \|\mathcal{L}^j f\|_{W^{1,2}}$.

$$\forall j, \quad \|\mathcal{L}^j f\|_2^2 = \int_{S^1} (\mathcal{L}^j f)(\mathcal{L}^j f)\, dx = \int_{S^1} f \cdot (U^j \mathcal{L}^j f)\, dx \leq$$

$$\leq \|f\|_2 \left(\int_{S^1} (\mathcal{L}^j f)^2 \circ T^j \, dx \right)^{1/2} = \|f\|_2 \left(\int_{S^1} (\mathcal{L}^j 1)(\mathcal{L}^j f)^2 dx \right)^{1/2}$$

$$\leq \|f\|_2 \, \|\mathcal{L}^j 1\|_\infty \, \|\mathcal{L}^j f\|_2 \leq \|f\|_2 \, \|\mathcal{L}^j 1\|_{W^{1,1}} \, \|\mathcal{L}^j f\|_2$$

$$W^{1,1}(S^1) \subset C^0(S^1)$$

$$= (\underbrace{\|\mathcal{L}^j 1\|_1}_{\leq 1} + \underbrace{\|(\mathcal{L}^j 1)'\|_1}_{\leq \frac{\|T''\|_\infty}{\lambda(\lambda-1)}}) \|f\|_2 \, \|\mathcal{L}^j f\|_2 \leq C_1 \|f\|_2 \, \|\mathcal{L}^j f\|_2$$

$$C_1 := 1 + \frac{\|T''\|_\infty}{\lambda(\lambda-1)}$$

$$\Longrightarrow \boxed{\|\mathcal{L}^j f\|_2 \leq C_1 \|f\|_2 \quad \forall f \in L^2, \ \forall j \in \mathbb{N}}$$

$$\forall j, \quad (\mathcal{L}^j f)(x) = \sum_{T^j(y)=x} \frac{1}{|(T^j)'(y)|} f(y)$$

$$\Big[(\mathcal{L}^2 f)(x) = \mathcal{L}(\mathcal{L} f)(x) = \mathcal{L}\Big(\sum_{T(y)=x} \frac{1}{|T'(y)|} f(y) \Big)(x)$$

$$= \sum_{T(y)=x} \sum_{T(z)=y} \frac{1}{|T'(y)|} \frac{1}{|T'(z)|} f(z) = \sum_{T^2(z)=x} \frac{1}{|(T^2)'(z)|} f(z) \Big]$$

<u>Rem</u> $\quad (\mathcal{L}_\varphi f)(x) = \sum_{i \in I} e^{\varphi(\psi_i(x))} f(\psi_i(x)) \qquad \psi_i := (T|_{I_i})^{-1}$

$$\varphi(t) = -\log|T'|$$

$$(\mathcal{L}^2_\varphi f)(x) = \sum_{i \in I} \mathcal{L}\left(e^{\varphi(\psi_i(x))} f(\psi_i(x))\right) = \sum_{i \in I} \sum_{j \in I} e^{\varphi(\psi_j(x))} e^{\varphi(\underbrace{\psi_i(\psi_j(x))}_{T(\psi_i(\psi_j(x)))})} f(\psi_i(\psi_j(x)))$$

$$= \sum_{i,j \in I} e^{\varphi(\psi_i(\psi_j(x))) + \varphi(\psi_j(x))} f(\psi_i(\psi_j(x)))$$

$$= \sum_{T^2(z)=x} e^{\varphi(z) + \varphi(T(z))} f(z)$$

$\|(\mathcal{L}^j f)'\|_2$

$$(\mathcal{L}^j f)'(x) = \operatorname{sgn}(T') \cdot \sum_{T^j(y)=x} \left[\frac{f'(y)}{|(T^j)'(y)|^2} - \frac{(T^j)''(y)}{|(T^j)'(y)|^3} f(y) \right]$$

$$\|(\mathcal{L}^j f)'\|_2 \le \left\| \mathcal{L}^j\left(\frac{f'}{|(T^j)'|}\right)\right\|_2 + \left\| \mathcal{L}^j\left(\frac{(T^j)''\cdot f}{|(T^j)'|^2}\right)\right\|_2$$

$$\le C_1 \left\|\frac{f'}{|(T^j)'|}\right\|_2 + C_1 \left\|\frac{(T^j)''\cdot f}{|(T^j)'|^2}\right\|_2$$

$$\le C_1 \lambda^{-j} \|f'\|_2 + C_1 \left\|\frac{(T^j)''}{|(T^j)'|^2}\right\|_\infty \|f\|_2$$

$$\left\|\frac{(T^j)''}{|(T^j)'|^2}\right\|_\infty \le \frac{1}{\lambda - 1} \cdot \left\|\frac{T''}{T'}\right\|_\infty =: C_2$$

$$\boxed{\forall f \in W^{1,2}, \quad \|(\mathcal{L}^j f)'\|_2 \le C_1 \lambda^{-j} \|f'\|_2 + C_1 \cdot C_2 \cdot \|f\|_2, \quad \forall j \in \mathbb{N}}$$

$$g_n = \frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}^j 1, \qquad \|g_n\|_{W^{1,2}} \le \frac{1}{n}\sum_{j=0}^{n-1} \|\mathcal{L}^j 1\|_{W^{1,2}} \le C_1(1+C_2) \ \forall n.$$

$$\Uparrow$$

$$\|\mathcal{L}^j 1\|_{W^{1,2}} = \|\mathcal{L}^j 1\|_2 + \|(\mathcal{L}^j 1)'\|_2 \le C_1 + C_1 \cdot C_2 = C_1(1+C_2) \quad \forall j$$

We use that $W^{1,2}(S^1) \Subset L^2(S^1)$, then $\exists\, n_k \nearrow +\infty$ and $h \in L^2$ s.t. $g_{n_k} \xrightarrow[k\to+\infty]{} h$ in L^2, L^1.

$$\mathcal{L} h = \mathcal{L}\left(\lim_{k\to+\infty} g_{n_k}\right) = \lim_{k\to+\infty} \mathcal{L} g_{n_k} = \lim_{k\to+\infty} \left(\frac{1}{n_k}\sum_{j=0}^{n_k-1} \mathcal{L}^{j+1} 1\right) =$$

$$= \lim_{k\to+\infty} \left[g_{n_k} + \frac{1}{n_k}\left(\mathcal{L}^{n_k} 1 - 1\right)\right] = \lim_{k\to+\infty} g_{n_k} = h. \quad \checkmark$$

$$\|g'_{n_k}\|_2 \le C_1(1+C_2) \ \forall n \Rightarrow \exists\, k_i \nearrow +\infty, \ \exists\, h' \in L^2 \text{ s.t.}$$

$$g'_{n_{k_i}} \xrightarrow{L^2} h', \quad i \to +\infty$$

$$\forall \psi \in C^\infty(S^1), \quad \int_{S^1} h \cdot \psi' \, dx = \lim_{i \to +\infty} \int g_{m_{k_i}} \psi' \, dx = \lim_{i \to +\infty} \int g'_{m_{k_i}} \psi \, dx = \int_{S^1} h' \psi \, dx$$

$\Rightarrow$ h' is the weak derivative of h $\Rightarrow$ $h \in W^{1,2}$. ✓

$d\mu = h(x)\, dx$

$$\mu(S^1) = \int_{S^1} h(x)\, dx = \lim_{k \to +\infty} \int_{S^1} g_{m_k} \, dx = \lim_{k \to +\infty} \frac{1}{m_k} \sum_{j=0}^{m_k - 1} \int_{S^1} \mathcal{L}^j 1 \, dx =$$

$$= \lim_{k \to +\infty} \frac{1}{m_k} \sum_{j=0}^{m_k-1} \int_{S^1} 1 \, dx = 1. \quad ✓$$

$h \geq 0$, because $\mathcal{L}$ is positive $\Rightarrow g_{m_k} > 0$. ✓

$$\forall g \in C^0, \quad \int_{S^1} g \circ T \, d\mu = \int_{S^1} (Ug) \cdot h \, dx = \int_{S^1} g \cdot (\mathcal{L} h)\, dx = \int_{S^1} g\, h \, dx = \int_{S^1} g \, d\mu.$$

$\Rightarrow$ μ is T-invariant ✓

$h(x) > 0 \ \forall x \in S^1$. If $\exists x_0$ s.t. $h(x_0) = 0$, then

$$0 = h(x_0) = (\mathcal{L} h)(x_0) = \sum_{T(y) = x_0} \frac{1}{|T'(y)|} h(y) \Rightarrow h(y) = 0 \ \forall y \in T^{-1}(x_0)$$

$$\Rightarrow h(y) = 0 \quad \forall y \in \bigcup_{j=1}^{+\infty} T^{-j}(x_0)$$

Since $\{ y \ / \ \exists j \text{ s.t. } T^j(y) = x_0 \}$ is dense $\forall x_0 \in S^1 \xRightarrow{h \in C^0} h \equiv 0$. ↯.

$\Big(\forall \varepsilon > 0 \ \forall z \in S^1 \ \exists j$ s.t. $|T^j(z - \varepsilon, z + \varepsilon)| \geq 1$. $\Rightarrow \exists y \in (z - \varepsilon, z + \varepsilon)$ s.t. $T^j(y) = x_0 \Big)$

✓

<u>Step 2</u> T has exponential decay of correlations on $W^{1,2}$ and L^2

$\Leftrightarrow$ $\exists C > 0$, $\vartheta \in (0,1)$ s.t. $\forall f \in W^{1,2}$, $g \in L^2$

$$|C_T(f, g, n)| = \left| \int_{S^1} f \cdot (U^n g)\, d\mu - \left(\int_{S^1} f \, d\mu\right)\left(\int_{S^1} g \, d\mu\right) \right| \leq C \vartheta^n \|f\|_{W^{1,2}} \|g\|_2$$

Theorem (Hennion) Let $(B_W, |\cdot|_W)$ and $(B, \|\cdot\|_B)$ two Banach spaces s.t. B is compactly embedded B_W.

Let $\mathcal{L}: B_W \to B_W$ linear continuous operator, s.t. $\mathcal{L}(B) \subset B$.

If $\exists\, \tilde{C} > 0$, $M > 0$, $\rho \in (0, M)$ s.t.

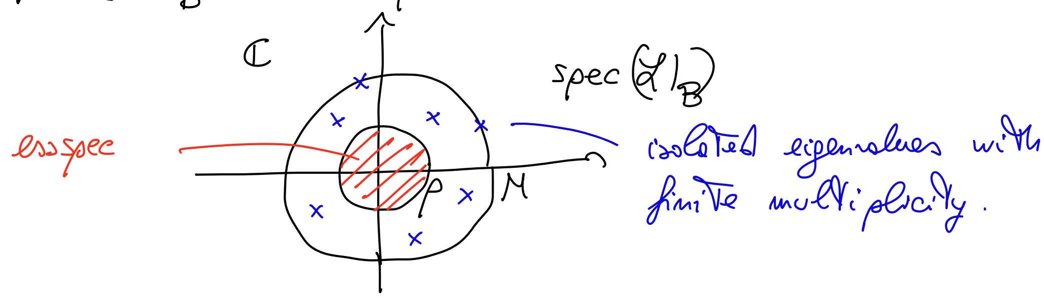
$$|\mathcal{L}^j v|_W \leq \tilde{C} \cdot M^j |v|_W \quad \forall v \in B_W,$$

$$\|\mathcal{L}^j v\|_B \leq \tilde{C} \rho^j \|v\|_B + \tilde{C} M^j |v|_W \quad \forall v \in B,$$

then

$$\operatorname{spec}(\mathcal{L}|_B) \subseteq B(0, M) = \{ z \in \mathbb{C} \,/\, |z| \leq M \}$$

$$\operatorname{ess\,spec}(\mathcal{L}|_B) \subseteq B(0, \rho)$$



We apply Hennion's Thm to $(L^2(S^1), \|\cdot\|_2) = (B_W, |\cdot|_W)$,

$(W^{1,2}(S^1), \|\cdot\|_{W^{1,2}}) = (B, \|\cdot\|_B)$, $\mathcal{L}$ is the transfer operator

- $$\|\mathcal{L}^j f\|_2 \leq C_1 \|f\|_2 \qquad \forall j, \quad \forall f \in W^{1,2}$$

- $$\|\mathcal{L}^j f\|_{W^{1,2}} = \|\mathcal{L}^j f\|_2 + \|(\mathcal{L}^j f)'\|_2 \leq$$

$$\leq C_1 \|f\|_2 + C_1 \lambda^{-j} \|f'\|_2 + C_1 \cdot C_2 \|f\|_2$$

$$\leq C_1 (\lambda^{-1})^j \|f\|_{W^{1,2}} + C_1 (1 + C_2) \|f\|_2, \quad \forall j \ \forall f \in W^{1,2}$$

$\tilde{C} = \max\{C_1, C_1(1+C_2)\}$, $M = 1$, $\rho = \frac{1}{\lambda} \in (0,1)$.

$\Rightarrow$ $\operatorname{spec}(\mathcal{L}|_{W^{1,2}}) \subseteq B(0,1)$, $\operatorname{ess\,spec}(\mathcal{L}|_{W^{1,2}}) \subseteq B(0, \lambda^{-1})$

Then by the spectral theorem,

$$\forall f \in W^{1,2}, \qquad \mathcal{L}^j f = K^j f + R^j f$$

where K, R are projections on $\operatorname{spec} \mathcal{L} \cap \{\lambda^{-1} < |\xi|\}$ and $\{|\xi| \leq \lambda^{-1}\}$

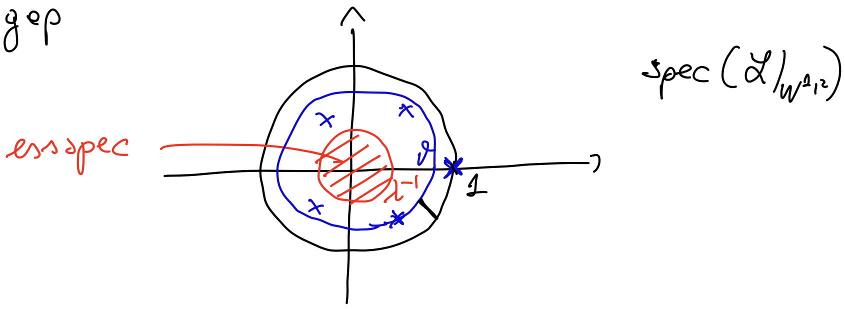
$KR = RK = 0$, $\quad \|R^j f\|_{W^{1,2}} \leq \lambda^{-j} \|f\|_{W^{1,2}}$.

We have $\operatorname{spec}(\mathcal{L}|_{W^{1,2}}) \cap \{|\xi| = 1\} = \{1\}$ and 1 is simple.

$K = K_1 + \tilde{K}$, and $\exists\, \vartheta \in (0,1)$ s.t. $\|\tilde{K} f\|_{W^{1,2}} \leq \vartheta \|f\|_{W^{1,2}}$ $\forall f$,

$\vartheta := \max\{|\ell| \,/\, \ell \in \mathbb{C}, \ell$ is an eigenvalue of $\mathcal{L}|_{W^{1,2}}$ in $\{|\xi| > \lambda^{-1}\}$ and $\ell \neq 1\}$

spectral gap



$K_1 \tilde{K} = \tilde{K} K_1 = 0$

Then, $\quad \mathcal{L}^j f = K_1^j f + \tilde{R}^j f, \qquad \|\tilde{R}^j f\|_{W^{1,2}} \leq \vartheta^j \|f\|_{W^{1,2}}$.

$K_1 f = k_1(f) \cdot h, \qquad K_1^j = K_1, \qquad k_1 : W^{1,2} \to \mathbb{C}$

$$\forall f \in W^{1,2}, \quad \int_{S^1} f\, dx = \int_{S^1} \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}^j f\, dx = \frac{1}{n} \int_{S^1} \sum_{j=0}^{n-1} k_1(f)\, h\, dx + \frac{1}{n} \int_{S^1} \sum_{j=0}^{n-1} \tilde{R}^j f\, dx$$

$$\leq k_1(f) + \frac{1}{n} \sum_{j=0}^{n-1} \vartheta^j \cdot \|f\|_{W^{1,2}} = k_1(f) + o(1)$$

$$\Longrightarrow \quad k_1(f) = \int_{S^1} f\, dx$$

$f \in W^{1,2}$, $g \in L^2$

$$\int_{S^1} f \cdot (U^n g)\, d\mu = \int_{S^1} fh\, (U^n g)\, dx = \int_{S^1} (\mathcal{L}^n (fh))\, g\, dx \overset{fh \in W^{1,2}}{=}$$

$$= \int_{S^1} g \left(K_1^n (fh) + \tilde{R}^n (fh) \right) dx = \int_{S^1} g \left(h \cdot k_1(fh) \right) dx + \int_{S^1} g\, \tilde{R}^n (fh)\, dx$$

$$= \left(\int_{S^1} f\, d\mu \right) \left(\int_{S^1} g\, d\mu \right) + O\left(\|g\|_{L^2} \cdot \|\tilde{R}^n (fh)\|_{L^2} \right)$$

$$| C_T(f,g,n)| \le \|g\|_{L^2} \cdot \|\tilde{R}^n(fh)\|_{W^{1,2}} \le \|g\|_{L^2} \cdot \vartheta^n \|fh\|_{W^{1,2}} \le$$

$$\le c \cdot \|g\|_{L^2} \cdot \|f\|_{W^{1,2}} \|h\|_{W^{1,2}} \cdot \vartheta^n = C \|g\|_{L^2} \|f\|_{W^{1,2}} \vartheta^n .$$

<u>Step 3</u> μ is mixing and ergodic.

$$C_T(f,g,n) \xrightarrow[n\to+\infty]{} 0 \qquad \forall f,g \in L^2 .$$

$\forall \varepsilon > 0 \ \forall f \in L^2 \ \exists f_0 \in W^{1,2}$ s.t. $\|f\cdot h - f_0\|_2 < \varepsilon$.

$$C_T(f,g,n) = \int_{S^1} f\cdot h\,(U^n g)\,dx - \left(\int_{S^1} fh\,dx\right)\left(\int_{S^1} g\,d\mu\right) =$$

$$= \int_{S^1} (fh - f_0)(U^n g)\,dx + \int_{S^1} f_0 (U^n g)\,dx - \left(\int_{S^1} (fh - f_0)\,dx\right)\left(\int_{S^1} g\,d\mu\right)$$

$$- \left(\int_{S^1} f_0\,dx\right)\left(\int_{S^1} g\,d\mu\right) = \underbrace{C_T\left(\frac{f_0}{h}, g, n\right)}_{\substack{\downarrow\, n\to+\infty \\ 0}} + O(\varepsilon). \qquad \square$$