Mixing and decay of correlations

Def $\Phi = \{\varphi_t\}_{t\in\mathbb{R}}$ on (M,μ), μ is Φ-invariant probability measure, we say that Φ is mixing wrt μ if $\forall A, B \subset M$ measurable,

$$\lim_{t\to+\infty} \mu(A \cap \varphi_{-t}(B)) = \mu(A)\mu(B)$$

$$\mu(A\cap\varphi_{-t}(B)) = \int_M \chi_A(p)\,(\chi_B\circ\varphi_t)(p)\,d\mu \xrightarrow[t\to+\infty]{} \left(\int_M \chi_A\,d\mu\right)\left(\int_M \chi_B\,d\mu\right)$$

Def We say that Φ has decay of correlations if $\forall f, g \in L^2(M,\mu)$

$$C_\Phi(f,g,t) := \int_M f\cdot(g\circ\varphi_t)\,d\mu - \left(\int_M f\,d\mu\right)\left(\int_M g\,d\mu\right) \xrightarrow[t\to+\infty]{} 0$$

Prop Φ has decay of correlation iff it is mixing.

$$\longleftrightarrow$$

$\Sigma = \Gamma \backslash \mathbb{H}^2$, Γ is a lattice ($\Gamma < SL(2,\mathbb{R})$ discrete and s.t. Σ has finite volume). $\Phi = \{\varphi_t\}$ be the geodesic flow on Σ, μ Liouville measure on $T^1\Sigma$, $\mu = m\times l$. We proved that μ is Φ-invariant and ergodic.

Def (M,μ) and $\Phi = \{\varphi_t\}$ measurable and measure preserving, we call Koopman operator $U_t : L^p(M,\mu) \to L^p(M,\mu)$, $p\geq 1$, $\forall t\in\mathbb{R}$.

$$g \longmapsto g\circ\varphi_t$$

Rem If $p\in[1,\infty)$, then U_t is an isometry on L^p.

$$C_\Phi(f,g,t) = \int f\,(U_t g)\,d\mu - \left(\int f\,d\mu\right)\left(\int g\,d\mu\right).$$

Thm If $\Gamma < SL(2,\mathbb{R})$ is a lattice then $\Phi = \{\varphi_t\}$ the geodesic flow on $\Sigma = \Gamma\backslash\mathbb{H}^2$ is mixing wrt μ.

proof It's enough to prove that $\forall g \in L^2(T^1\Sigma, \mu)$, $U_t g \xrightarrow[t\to+\infty]{L^2} \int_{T^1\Sigma} g \, d\mu$.

U_t is an isometry on $L^2 = L^2(T^1\Sigma, \mu)$, then

$\|(U_t g)\|_2 = \|g\|_2 \implies \exists\, t_k \nearrow +\infty$ s.t. $\exists\, g_\infty \in L^2$, $U_{t_k} g \xrightarrow[k\to+\infty]{} g_\infty$.

- $\|(U_{t_k} g)\circ u_s^+ - U_{t_k} g\|_2 = \|U_{t_k}(g \circ u^+_{se^{-t_k}}) - U_{t_k} g\|_2 =$

$$(U_{t_k} g)\circ u_s^+ = g \circ (\varphi_{t_k} \circ u_s^+) = g \circ (u_{se^{-t_k}} \circ \varphi_{t_k})$$

$= \|g \circ u^+_{se^{-t_k}} - g\|_2 \xrightarrow[k\to+\infty]{} 0.$

$$u^+_{se^{-t_k}} \longleftrightarrow \begin{pmatrix} 1 & se^{-t_k} \\ 0 & 1 \end{pmatrix} \xrightarrow[k\to+\infty]{} \mathrm{Id}$$

- $(U_{t_k} g)\circ u_s^+ \xrightarrow[k\to+\infty]{} g_\infty \circ u_s^+$ since

$\forall f \in L^2$ $\quad \int_{T^1\Sigma} f \cdot (U_{t_k} g)\circ u_s^+ \, d\mu = \int_{T^1\Sigma} (f \circ u^+_{-s})(U_{t_k} g)\, d\mu \xrightarrow[k\to+\infty]{} \int_{T^1\Sigma} (f \circ u^+_{-s}) g_\infty \, d\mu$

$$= \int_{T^1\Sigma} f \cdot g_\infty \circ u_s^+ \, d\mu$$

$$\begin{cases} \|(U_{t_k} g)\circ u_s^+ - U_{t_k} g\|_2 \to 0 \\ (U_{t_k} g)\circ u_s^+ - U_{t_k} g \xrightarrow{L^2} g_\infty \circ u_s^+ - g_\infty \end{cases} \implies \|g_\infty \circ u_s^+ - g_\infty\|_2 = 0$$

$\implies g_\infty$ is u_s^+-invariant $\implies g_\infty$ is constant μ-a.e. $\implies g_\infty = \int_{T^1\Sigma} g \, d\mu$

$\implies \forall g \in L^2$, $\quad U_t g \xrightarrow[t\to+\infty]{} \int_{T^1\Sigma} g \, d\mu$. $\quad\square$

Theorem (Anosov) $\quad$ Let Φ be a measure preserving Anosov flow on (M, μ), then

either $\quad$ the strong stable and unstable foliations are ergodic, and Φ is mixing;

or $\quad$ Φ on M is isomorphic to a flow under a constant roof function with an Anosov base map.

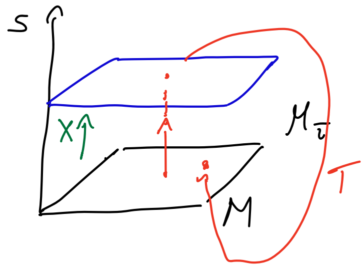
Theorem If Φ is a contact Anosov flow on Σ surface (connected compact orientable Riemannian) then Φ is mixing wrt μ on $T^1\Sigma$.

proof If $X = \frac{d}{dt}\Big|_{t=0} \varphi_t$ is the vector field generating the flow, $\exists$ α 1-form

$$\alpha(X) = 1, \quad i_X d\alpha = 0 \quad (d\alpha(X,\xi) = 0 \ \forall \xi), \quad d\alpha \text{ is non-degenerate.}$$

$$\alpha \wedge d\alpha \text{ "=" } \mu .$$

Let's assume that Φ is a flow under a constant roof function, then



$$\ker \alpha = E^s \oplus E^u \Rightarrow \alpha = a(s)\, ds$$

$$\Rightarrow d\alpha = 0. \ ↯$$

$\square$

Exponential decay of correlations

1. Smooth expanding maps in 1D.

$$T : S^1 \to S^1, \quad T \in C^2$$

$$\exists\ \lambda > 1 \quad \text{s.t.} \quad |T'(x)| \geq \lambda > 1 \quad \forall x \in S^1.$$

Theorem $\exists!$ μ probability measure on S^1 which is T-invariant, ergodic, mixing, $\mu \sim m$ Lebesgue measure that is $d\mu = h(x)\,dx$ with $h > 0$. Moreover, $h \in W^{1,2}(S^1)$ and T has exponential decay of correlations on $W^{1,2}(S^1)$ and $L^2(S^1)$, this means $\exists\ C > 0$, $\vartheta \in (0,1)$ s.t. $\forall f \in W^{1,2}$, $\forall g \in L^2$

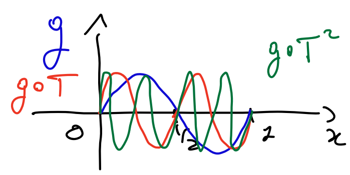
$$|C_T(f,g,n)| = \left| \int_{S^1} f \cdot (g \circ T^n)\, d\mu - \left(\int_{S^1} f\, d\mu\right)\left(\int_{S^1} g\, d\mu\right) \right| \leq$$

$$\leq C\, \vartheta^n \cdot \|f\|_{W^{1,2}} \cdot \|g\|_2 .$$

$$\int_{S^1} f \cdot (g \circ T^n)\, d\mu = \int_{S^1} f\, \underbrace{(g \circ T^n)}_{U_T^n g}\, h\, dx \overset{\text{reference measure } m}{=} \int_{S^1} g \cdot \mathcal{L}^n(fh)\, dx$$

Ex $T(x) = 2x \pmod 1$ $\qquad g(x) = \sin(2\pi x)$

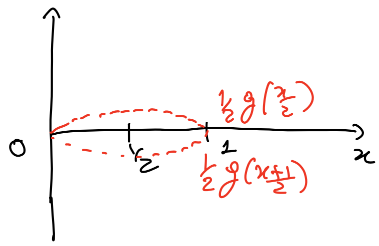
$S^1 = \mathbb{R}/\mathbb{Z}$



$$\int_{S^1} f\cdot(g\circ T)\,dx = \int_{S^1} (\mathcal{L}f)\, g\, dx$$

$$(\mathcal{L}f)(x) = \frac{1}{2} f\left(\frac{x}{2}\right) + \frac{1}{2} f\left(\frac{x+1}{2}\right)$$

$g(x) = \sin(2\pi x)$



$\mathcal{L}g = 0$

Def $\mathcal{L}$ is the transfer operator, $\mathcal{L}: L^1(M,m) \to L^1(M,m)$, m = reference measure is defined as the dual of $U: L^\infty(M,m) \to L^\infty(M,m)$, $g \mapsto g\circ T$

$$\int_M f\cdot(Ug)\,dm = \int_M (\mathcal{L}f)\, g\, dm$$

Prop

- $\forall f \in L^1(M,m)$, $\int_M \mathcal{L}f\,dm = \int_M f\,dm$
- $\mathcal{L}$ is a linear positive operator, with $\|\mathcal{L}f\|_1 \le \|f\|_1$ $\forall f \in L^1$.
- $\|\mathcal{L}\| = 1$.
- $\operatorname{spec}(\mathcal{L}|_{L^1}) \subseteq B(0,1) = \{\zeta \in \mathbb{C} / |\zeta| \le 1\}$.

Proof of Theorem $T: S^1 \to S^1$, $|T'| \ge \lambda > 1$, hence $N_x := \#\{y \in S^1 / T(y) = x\} \le \|T'\|_\infty < +\infty$

m = reference measure = Lebesgue measure

$$(\mathcal{L}f)(x) = \sum_{T(y)=x} \frac{1}{|T'(y)|} f(y), \quad \|\mathcal{L}1\|_\infty = \Big\|\sum_{T(y)=x} \frac{1}{|T'(y)|}\Big\|_\infty \le \frac{N}{\lambda}$$

Step 1 $\exists\, h \in W^{1,2}$ s.t. $\mathcal{L}h = h$.

$g_n := \frac{1}{n}\sum_{j=0}^{n-1} \mathcal{L}^j 1$. We prove that $\{g_n\} \subset W^{1,2}$, $\|g_n\|_{W^{1,2}} \le C < +\infty$.