$\Sigma = \Gamma \backslash \mathbb{H}^2$, $\Gamma = PSL(2,\mathbb{Z})$, $\pi : \mathbb{H}^2 \to \Sigma$

modular surface $\quad d\pi : T^1\mathbb{H}^2 \to T^1\Sigma$

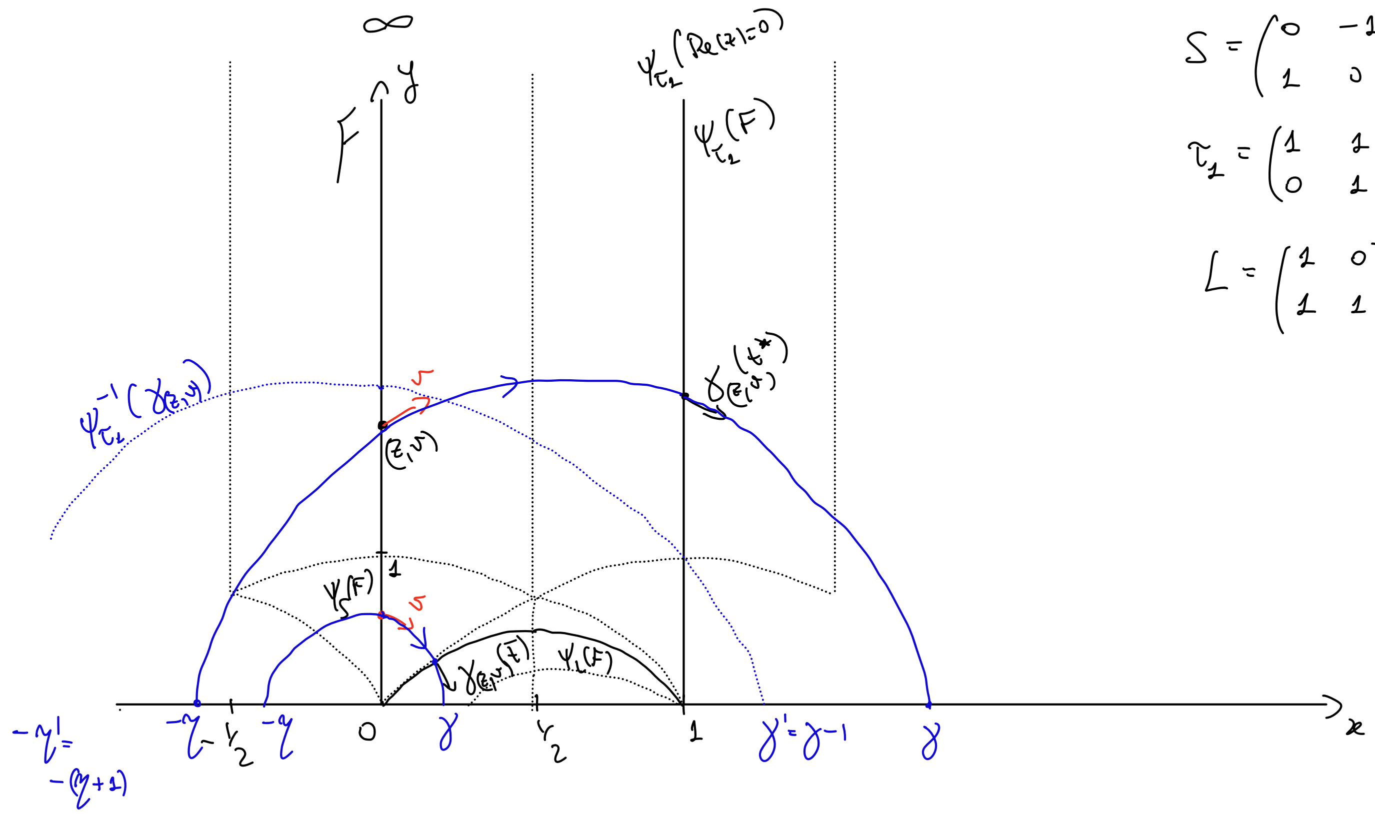
$$\begin{array}{ccc} T^1\mathbb{H}^2 & \xrightarrow{\varphi_t} & T^1\mathbb{H}^2 \\ d\pi \downarrow & & \downarrow d\pi \\ T^1\Sigma & \xrightarrow[\varphi_t]{} & T^1\Sigma \end{array}$$

$\mathcal{C} = \{ (z,v) \in T^1\mathbb{H}^2 \,/\, Re(z)=0,\ \mathcal{I}(v) \in (-\pi,0) \}$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\psi_S(z) = -\frac{1}{z}$

$\tau_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\psi_{\tau_1}(z) = z+1$

$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ $\psi_L(z) = \frac{z}{z+1}$



Given $(z,v) \in \mathcal{C}$, $Re(z)=0$, $\mathcal{I}(v) \in (-\pi,0)$, then $\gamma_{(z,v)}(t)$ for $t>0$ intersects a copy of $\mathcal{C}$ on $\{Re(z)=1\}$ iff $\gamma^+_{(z,v)} = \lim_{t\to+\infty} \gamma_{(z,v)}(t) > 1$.

Let $P : \mathcal{C} \to \mathcal{C}$ Poincaré map, if $\gamma^+_{(z,v)} > 1$ then

$$P(z,v) = d\psi_{\tau_1}^{-1}(\gamma_{(z,v)}(t^*), \gamma'_{(z,v)}(t^*))$$

if $\gamma^+_{(z,v)} < 1$ then $P(z,v) = d\psi_L^{-1}(\gamma_{(z,v)}(\bar{t}), \gamma'_{(z,v)}(\bar{t}))$.

We change coordinates, $\mathcal{G} : \mathcal{C} \to \mathcal{M} := \{ (\gamma,\eta) \,/\, \gamma,\eta \in (0,+\infty) \}$

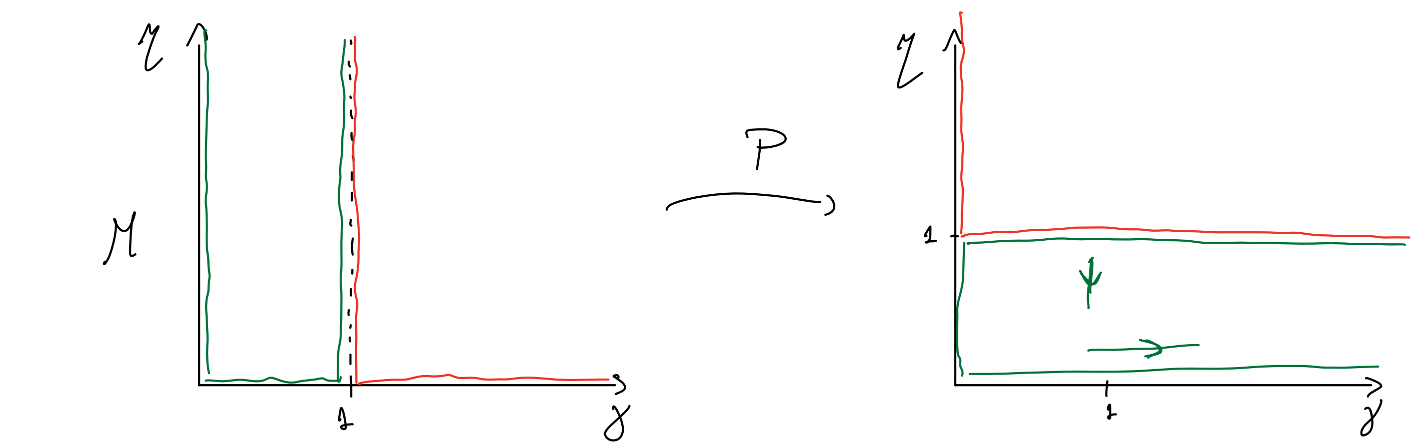
$\gamma := \gamma^{+}_{(z,v)}$, $\eta := -\gamma^{-}_{(z,v)}$

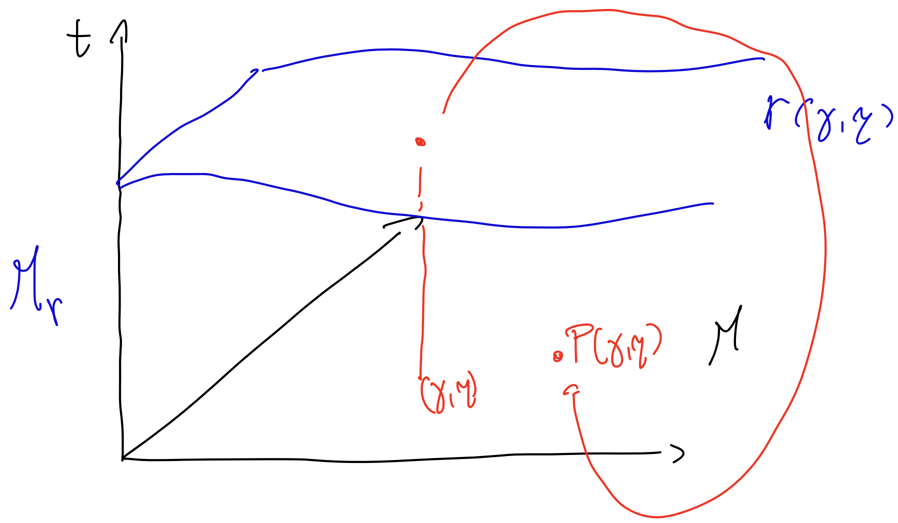
$$P(\gamma,\eta) = \begin{cases} (\gamma-1,\ \eta+1), & \text{if } \gamma > 1 ; \\ \left(\frac{\gamma}{1-\gamma},\ \frac{\eta}{1+\eta}\right), & \text{if } \gamma \in (0,1) . \end{cases}$$

$\gamma' = \psi_L^{-1}(\gamma)$, $\eta' = -\psi_L^{-1}(-\eta)$ $\qquad \psi_L^{-1}(z) = \psi_{L^{-1}}(z) = \frac{z}{1-z}$

$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $L^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$$P : M \to M , \quad P(\gamma,\eta) = \begin{cases} (\gamma-1,\ \eta+1), & \text{if } \gamma > 1 ; \\ \left(\frac{\gamma}{1-\gamma},\ \frac{\eta}{1+\eta}\right), & \text{if } \gamma \in (0,1) . \end{cases}$$





Let $(z,v) \in \mathbb{C}$, $\mathrm{Re}(z)=0$, $\mathrm{Im}(v) \in (-\pi,0)$, $(z,v) \in T^1\mathbb{H}^2 \cong PSL(2,\mathbb{R})$

$(z,v) \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g_{(z,v)}$ $\qquad$ $\gamma_{(z,v)}(t) \leftrightarrow g_{\gamma_{(z,v)}(t)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$

$$(z',v') \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} = \begin{pmatrix} a e^{t/2} & b e^{-t/2} \\ c e^{t/2} & d e^{-t/2} \end{pmatrix} \qquad \gamma = \frac{a}{c}, \ \eta = -\frac{b}{d}$$

$$z' = \frac{a e^{t/2} i + b e^{-t/2}}{c e^{t/2} i + d e^{-t/2}} \quad , \quad Re(z') = 1$$

$t^* \in (0,+\infty)$, the first return time of $\gamma_{(z,v)}(t)$ to $\mathcal{C}$, if $\gamma > 1$,

is given by $t^* = \frac{1}{2} \log\left(\frac{ac+d^2}{ac-c^2}\right)$

$$\Rightarrow \quad r(\gamma,\eta) = \frac{1}{2} \log\left(\frac{1+\frac{1}{\eta}}{1-\frac{1}{\gamma}}\right) \quad \text{if } \gamma > 1$$

$\bar{t}$ satisfies $\quad |z' - \frac{1}{2}| = \frac{1}{2}$, the first return time of $\gamma_{(z,v)}(t)$ to $\mathcal{C}$, if $\gamma < 1$, and is given by

$$\bar{t} = \frac{1}{2} \log\left(\frac{b^2-bd}{ac-a^2}\right)$$

$$\Rightarrow \quad r(\gamma,\eta) = \frac{1}{2} \log\left(\frac{1+\eta}{1-\gamma}\right) \quad \text{if } \gamma < 1.$$

- Find the invariant measure for $P: \mathcal{M} \to \mathcal{M}$.

Let μ be a Borel measure on $\mathcal{M}$, μ is P-invariant if $\mu(P^{-1}(A)) = \mu(A) \quad \forall A$ Borel set in $\mathcal{M}$. Or, equivalently,

$$\int_{\mathcal{M}} g(P(\gamma,\eta))\, d\mu(\gamma,\eta) = \int_{\mathcal{M}} g(\gamma,\eta)\, d\mu(\gamma,\eta) \iff P_* \mu = \mu$$

Let's assume that $\mu \ll m$ on $\mathcal{M}$, $d\mu(\gamma,\eta) = h(\gamma,\eta)\, d\gamma\, d\eta$, and take $g \in C_0^\circ(\mathcal{M})$.

$$\int_M g(P(\gamma,\eta))\, h(\gamma,\eta)\, d\gamma\, d\eta = \int_M g(\gamma,\eta) \underbrace{h(P^{-1}(\gamma,\eta))\, |\det(dP^{-1}(\gamma,\eta))|}_{\mathcal{L}h}\, d\gamma\, d\eta =$$

$$P^{-1}(\gamma,\eta) = \begin{cases} (\gamma+1,\ \eta-1), & \text{if } \eta > 1 \\ \left(\frac{\gamma}{1+\gamma},\ \frac{\eta}{1-\eta}\right), & \text{if } \eta < 1 \end{cases}$$

$$dP^{-1}(\gamma,\eta) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } \eta > 1 \\ \begin{pmatrix} \frac{1}{(1+\gamma)^2} & 0 \\ 0 & \frac{1}{(1-\eta)^2} \end{pmatrix}, & \text{if } \eta < 1 \end{cases}$$

$$= \int_M g(\gamma,\eta)\, h(\gamma,\eta)\, d\gamma\, d\eta$$

$(\mathcal{L}h)(\gamma,\eta) = h(\gamma,\eta)$

transfer operator

$$(\mathcal{L}h)(\gamma,\eta) = h(\gamma+1,\eta-1)\, \chi_{\{\eta>1\}}(\gamma,\eta) + \frac{h\left(\frac{\gamma}{1+\gamma},\frac{\eta}{1-\eta}\right)}{(1+\gamma)^2(1-\eta)^2}\, \chi_{\{\eta<1\}}(\gamma,\eta)$$

$$h(\gamma,\eta) = \frac{1}{(\gamma+\eta)^2}$$

$$h(\gamma+1,\eta-1) = h(\gamma,\eta), \qquad \frac{h\left(\frac{\gamma}{1+\gamma},\frac{\eta}{1-\eta}\right)}{(1+\gamma)^2(1-\eta)^2} = \frac{1}{\left(\frac{\gamma}{1+\gamma}+\frac{\eta}{1-\eta}\right)^2}\,\frac{1}{(1+\gamma)^2(1-\eta)^2} =$$

$$= \frac{1}{(\gamma(1-\eta)+\eta(1+\gamma))^2} = \frac{1}{(\gamma+\eta)^2} = h(\gamma,\eta)$$

$d\mu = \frac{1}{(\gamma+\eta)^2}\, d\gamma\, d\eta$ is P-invariant, ergodic, $\mu(M) = +\infty$. $\left[\int_0^{+\infty} h(\gamma,\eta)\, d\eta = \frac{1}{\gamma}\right]$

Then the $V = \{V_t\}$ on M_r preserves the $d\mu_r(\gamma,\eta,s) = \frac{1}{(\gamma+\eta)^2}\, d\gamma\, d\eta\, ds$.

$$\mu_r(M_r) = \int_M \int_0^{r(\gamma,\eta)} 1\, d\mu_r = \int_M r(\gamma,\eta)\, h(\gamma,\eta)\, d\gamma\, d\eta < +\infty, \quad r \in L^1(M,\mu).$$