Flows under a function

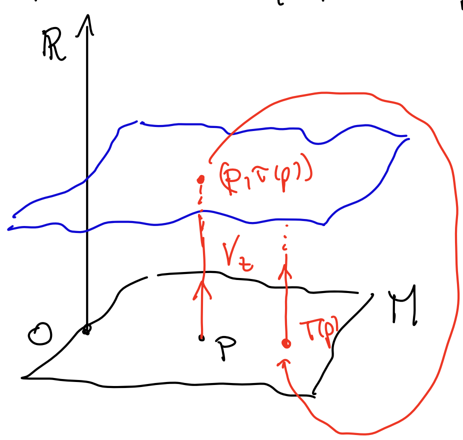
- M manifold, $T : M \to M$ base map, ν measure on M
  $\tau : M \to [0,+\infty)$, roof function, $\tau \in L^1(M,\nu)$

$$M_\tau = \{ (p,s) \in M \times \mathbb{R} \;/\; 0 \le s \le \tau(p) \} \big/ \sim$$

with $(p, \tau(p)) \sim (T(p), 0)$

- $V = \{V_t\}$, flow under the function τ

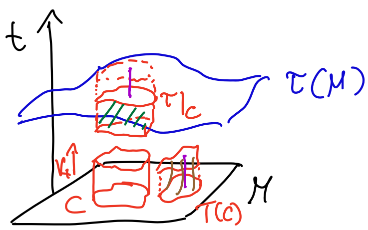
$$M_\tau \ni (p,s) \longmapsto V_t(p,s) = (p, s+t)$$



Thm Let Φ measure-preserving flow on a Lebesgue space with essentially no fixed points, then Φ is isomorphic to a flow under a function.

Prop Let (M,ν) be a measure space and $T : M \to M$ measure-preserving. Let $\tau : M \to [0,+\infty)$ be measurable and $\tau \in L^1(M,\nu)$. Then $V = \{V_t\}$ on M_τ preserves the probability measure $\mu_\tau := \dfrac{\nu \times \ell}{\int_M \tau \, d\nu}$, where ℓ is the Lebesgue measure on $\mathbb{R}$.

proof



$A = C \times (s_1, s_2) \subset M \times \mathbb{R}$

$\nu(C) > 0, \quad \mu_\tau(A) > 0$

$$\mu_\tau(V_t(A)) = \mu_\tau(A) \quad \forall\, t \in \mathbb{R}.$$

$$\mu_\tau(V_t(A)) = \mu_\tau(\{(p,s) \,/\, p \in C,\ t \le s \le \tau(p)\}) +$$

$A = C \times (0, s_2)$

$$\mu_\tau\{(T(p),s) \,/\, p \in C,\ 0 \le s \le s_2 + t - \tau(p)\})$$

$$= \int_C (\tau(p) - t)\, d\nu + \int_{T(C)} (s_2 + t - \tau(T^{-1}(p)))\, d\nu = \int_C s_2\, d\nu = \mu_\tau(A)$$

$$\int_C (s_2 + t - \tau(p))\, d\nu$$

$\square$

Prop $T: M \to M$ is ergodic wrt ν prob. meas.

$\Updownarrow$

$V = \{V_t\}$ is ergodic wrt $\mu_\tau = \dfrac{\nu \times \ell}{\int_M \tau\, d\nu}$

proof If T is ergodic, let $f: M_\tau \to \mathbb{R}$ be V-invariant ($f \circ V_t = f$ for all t and μ_τ-a.e.). Then one can define

$$g: M \to \mathbb{R}, \quad g(p) = \frac{1}{\tau(p)} \int_0^{\tau(p)} f(p,s)\, ds, \quad \nu\text{-a.e.}$$

with $g(p) = f(p,0)$, $g(T(p)) = f(T(p),0)$.

But $f(T(p),0) = f \circ V_t(p,0)$ for $t = \tau(p)$, then $f(T(p),0) = f(p,0)$.

hence g is T-invariant $\Rightarrow$ g is ν-a.e. constant.

$\Rightarrow$ f is μ_τ-a.e. constant.

If V is ergodic on M_τ, let $g: M \to \mathbb{R}$ be T-invariant.

Then one can define $f(p,s) = g(p)$ $\forall\, p \in M$, $s \in [0, \tau(p))$.

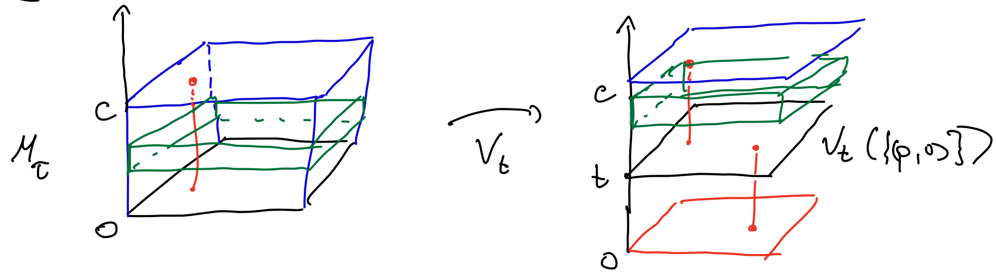
Then f is V-invariant $\Rightarrow$ f is μ_τ-a.e. constant

$\Rightarrow$ g is ν-a.e. constant. $\square$

Remarks

- V ergodic $\Leftrightarrow$ $V_t: M_\tau \to M_\tau$ is ergodic for all t up to a countable set.

Ex Let $\tau : M \to \mathbb{R}$, $\tau \equiv c > 0$, (M_τ, V).

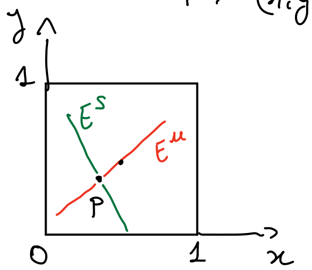


V_{cn} is not ergodic $\forall n \in \mathbb{Z}$.

- $M = \mathcal{A}^{\mathbb{Z}}$ with $\mathcal{A} = \{1, \dots, N\}$, $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ shift
- $M = \mathbb{T}^2$, $T : M \to M$ hyperbolic automorphism

Ex T = Arnold cat map.

$$T : (x,y) \longmapsto \overbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}^{A} \begin{pmatrix} x \\ y \end{pmatrix} \pmod{\mathbb{Z}^2}.$$



$E^s = \mathrm{Span}(v_s)$

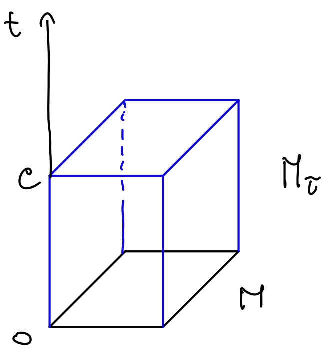
$E^u = \mathrm{Span}(v_u)$

$A v_s = \lambda_- v_s$, $A v_u = \lambda_+ v_u$

$0 < \lambda_- < 1 < \lambda_+$

$\tau : M \to (0, +\infty)$

$\tau \equiv c > 0$



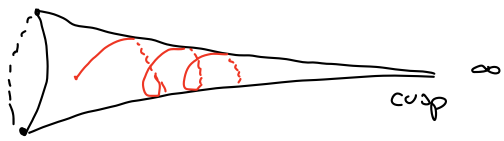
V on M_τ is Anosov, ergodic wrt $\mu_\tau := \dfrac{m \times \ell}{c}$

$(p,s) \in M_\tau$, $T_{(p,s)} M_\tau = E^c \oplus E^u \oplus E^s$

$E^c = \mathrm{Span}(0,1)$

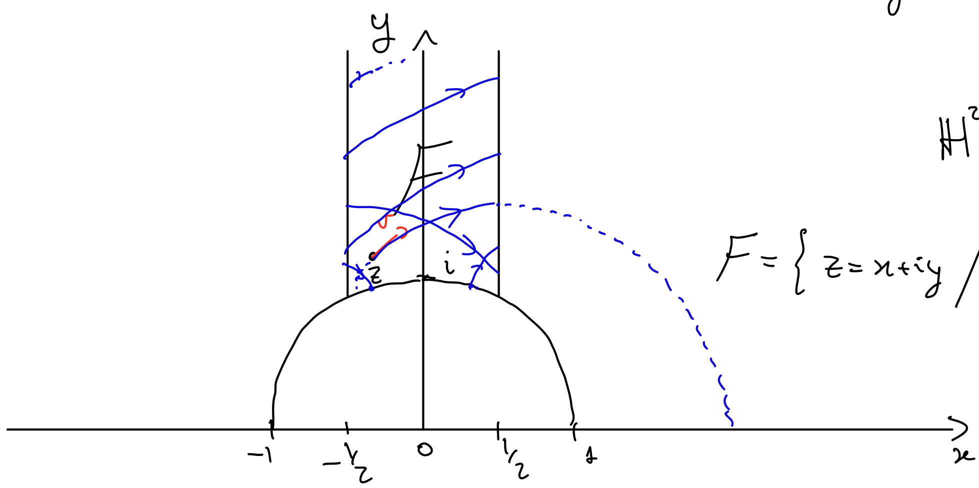
Example $\mathbb{H}^2$, $\Gamma = SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$ is a lattice

$\Gamma \backslash \mathbb{H}^2 = \Sigma$ is called the modular surface, is an orbifold which has a smooth Riemannian surface with two singular points with finite volume (connected, orientable).



$m(\Sigma) < +\infty$

$dm = \frac{1}{y^2}\, dx\, dy$ volume form on $\mathbb{H}^2$.



$\mathbb{H}^2$

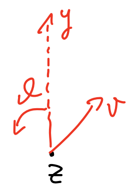
$F = \left\{ z = x+iy \,/\, |z|>1,\ -\frac{1}{2} < x < \frac{1}{2} \right\}$

$\pi : \mathbb{H}^2 \to \Sigma = \Gamma \backslash \mathbb{H}^2$.,

$z \in F$, $v \in T_z \mathbb{H}^2$, $|v| = 1$

$$\varphi_t(z,v) = \left(\gamma_{(z,v)}(t),\ \gamma'_{(z,v)}(t) \right) \in T^1 \mathbb{H}^2$$

$$\tilde{\varphi}_t : T^1\Sigma \to T^1\Sigma, \qquad \tilde{\varphi}_t(z,v) = d\pi\left(\varphi_t(z,v)\right)$$



$$\mathcal{C} := \left\{ (z,v) \in T^1\Sigma \,/\, z \in F,\ \mathrm{Re}(z) = 0,\ \theta(v) \neq 0, \pi \right\}$$

is a good Poincaré section for $\{\tilde{\varphi}_t\}$ geodesic flow on $T^1\Sigma$.

$I_+ = \{ z \in \mathbb{H}^2 \,/\, z = x+iy,\ x = 0,\ y > 1 \}$, $\quad I_- = \{ z = x+iy \,/\, x = 0,\ 0 < y < 1 \}$

$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z})$, $\quad \psi_S(I_+) = I_-$, $\quad \psi_S(I_-) = I_+$

$\psi_S(z) = -\frac{1}{z}$, $\quad z = iy$, $\psi_S(iy) = \frac{i}{y}$.

$$d\psi_S(z,v) = \left(\psi_S(z),\ d_z\psi_S(v)\right) = \left(\psi_S(z),\ \psi_S'(z)\,v\right)$$

$$\psi_S'(z) = \frac{1}{(cz+d)^2} = \frac{1}{z^2}$$

$$d\psi_S(iy,v) = \left(\frac{i}{y},\ -\frac{1}{y^2} v \right) = (z', v') \qquad \theta(v') = \theta(v) + \pi \pmod{2\pi}$$

$$\mathcal{C} := d\pi\left(\left\{(z,v)\in T^1\mathbb{H}^2 \,/\, z\in I_+\cup I_-\cup\{i\},\ \vartheta(v)\in(-\pi,0)\right\}\right)$$