Thm If $\Gamma < SL(2,\mathbb{R})$ is a lattice (discrete subgroup s.t. $\Sigma = \Gamma \backslash \mathbb{H}^2$ is of finite volume) then the action of non-elliptic elements $g \in SL(2,\mathbb{R})$ on $T^1(\Gamma \backslash \mathbb{H}^2)$ is ergodic.

Proof

$$SL(2,\mathbb{R}) \ni h \mapsto p(h) = \langle f \circ h, f \rangle_{L^2} \quad , \quad f \in L^2(T^1\Sigma).$$

p is continuous and if f is τ_b-invariant ($\tau_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$) then
$$p(\tau_b^m h \tau_b^n) = p(h).$$

Let $\{\varepsilon_k\} \searrow 0$ s.t. $b\varepsilon_k \in \mathbb{R} \setminus \mathbb{Q}$ and it's possible to find sequences $\{n_k\}, \{m_k\} \subset \mathbb{Z}$ s.t.

$$| b\varepsilon_k n_k - 1 | < \varepsilon_k$$

$$\left| m_k b \varepsilon_k + \frac{n_k b \varepsilon_k}{n_k b \varepsilon_k + 1} \right| < \varepsilon_k .$$

$$h_k = \exp(\varepsilon_k H_-) = \begin{pmatrix} 1 & 0 \\ \varepsilon_k & 1 \end{pmatrix} \xrightarrow[k \to +\infty]{} Id$$

$$\tau_b^{n_k} h_k \tau_b^{m_k} = \begin{pmatrix} 1 + n_k b \varepsilon_k & (1 + n_k b\varepsilon_k) m_k b + n_k b \\ \varepsilon_k & 1 + m_k b \varepsilon_k \end{pmatrix} \xrightarrow[k \to +\infty]{} \begin{pmatrix} 2 & r \\ 0 & \frac{1}{2} \end{pmatrix} =: \bar{h}$$

$\bar{h}$ is a hyperbolic element, then $\bar{h} \sim \lambda_a$ for some $a \in \mathbb{R} \setminus \{\pm 1, 0\}$.

$$\left. \begin{array}{l} p(\tau_b^{n_k} h_k \tau_b^{m_k}) = p(h_k) \xrightarrow[k \to +\infty]{} p(Id) = \|f\|_{L^2}^2 \\ \quad \downarrow k \to +\infty \\ p(\bar{h}) \end{array} \right] \Rightarrow p(\bar{h}) = \|f\|_{L^2}^2$$

$\Rightarrow$ f is $\bar{h}$-invariant. $\Rightarrow$ f is constant μ-a.e. $\square$

Remarks f is λ_a-invariant and $p(\lambda_a^n h \lambda_a^m) = p(h) \quad \forall n, m \in \mathbb{Z}$

and

$$\lambda_a^n \tau_b \lambda_a^{-n} \xrightarrow[n \to -\infty]{} Id \quad , \quad a > 1$$

$$\begin{pmatrix} a^n & 0 \\ 0 & a^{-n} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-n} & 0 \\ 0 & a^n \end{pmatrix} = \begin{pmatrix} 1 & ba^{2n} \\ 0 & 1 \end{pmatrix} = \tau_{ba^{2n}}$$

Prop. Let $\Phi = \{\varphi_t\}$ be the geodesic flow on $\mathbb{H}^2$, and $\{u_s^\pm\}$ the positive (negative) horocycle flow.

In $PSL(2,\mathbb{R})$ φ_t is given by the right action of $\lambda_{e^{t/2}}$

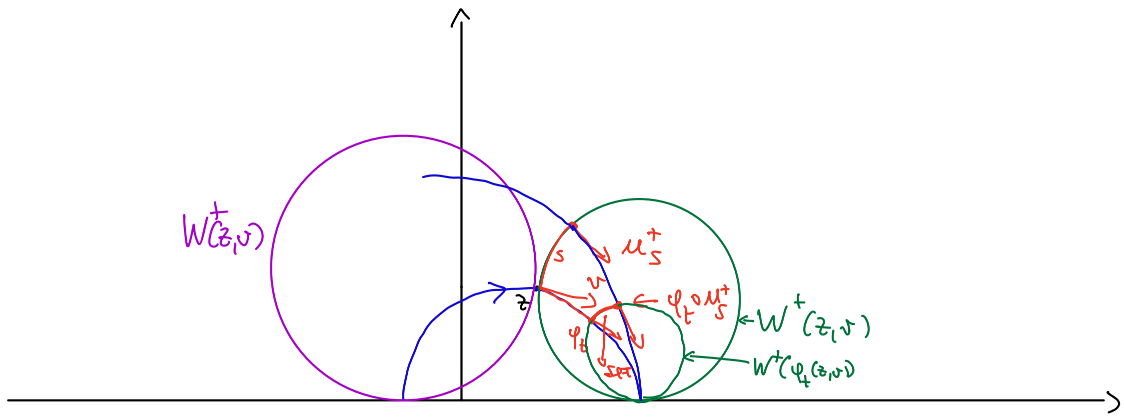
$u_s^\pm$ " " " " " " τ_s

$\exp(sH)$

Then $\forall\, t, s \in \mathbb{R}$

$$\varphi_t \circ u_s^\pm = u^\pm_{se^{-t}} \circ \varphi_t$$

$$\left[\quad \varphi_{-t} \circ u_s^+ \circ \varphi_t = u^+_{se^t} \xrightarrow[t \to -\infty]{} Id \quad \right]$$



Corollary The geodesic and horocycle flows on $\Sigma = \Gamma \backslash \mathbb{H}^2$ with $\Gamma < SL(2,\mathbb{R})$ a lattice, are ergodic with respect to the Liouville measure $\mu = m \times \ell$.

$\longleftrightarrow$

Non-constant curvature case.

Thm (Hopf) Let Σ be a smooth closed connected orientable Riemannian surface with negative curvature. Then, the geodesic flow $\Phi = \{\varphi_t\}$ on Σ is ergodic wrt the Liouville measure $\mu = m \times \ell$.

proof The Liouville measure μ is finite and Φ-invariant. $\mu \sim \alpha \wedge d\alpha$

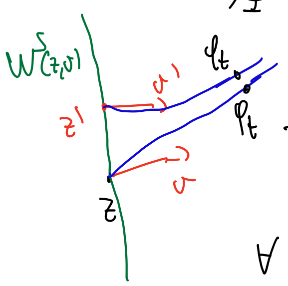
If $g : T^1\Sigma \to \mathbb{C}$, $g \in L^1(T^1\Sigma, \mu)$, $\exists\, g_\Phi$ s.t.

$$g_\Phi(z,v) = \lim_{T\to\pm\infty} \frac{1}{|T|}\int_0^{|T|} g(\varphi_t(z,v))\,dt \quad \text{for } \mu\text{-a.e. } (z,v).$$

Let $f \in L^1(T^1\Sigma,\mu)$ and assume that f is Φ-invariant. Wlog $f \geq 0$,
$\forall C>0$, let $f_C := \min\{f, C\}$. $f_C \in L^1$, bounded then
$\exists \{g_n\} \subset C^0(T^1\Sigma)$ s.t. $\|f_C - g_n\|_{L^1} \xrightarrow[n\to+\infty]{} 0$.

Then let $(z,v) \in T^1\Sigma$ and $W^s(z,v)$ be the strong stable manifold.
Then, $\forall n$ $\exists N_n^s \subset W^s(z,v)$ s.t. $\forall (z',v') \in W^s(z,v) \setminus N_n^s$
$\mu(N_n^s) = 0$

$$\exists (g_n)_\Phi(z',v') = \lim_{T\to+\infty} \frac{1}{T}\int_0^T g_n(\varphi_t(z',v'))$$



Then $(g_n)_\Phi(z',v') = (g_n)_\Phi(z,v)$

$\forall \varepsilon>0$ $\exists T_\varepsilon$ s.t. $d(\varphi_t(z,v), \varphi_t(z',v')) < \varepsilon$ $\forall t > T_\varepsilon$

Then $\forall \eta>0$ $\exists T_\varepsilon$ s.t. $|g_n(\varphi_t(z,v)) - g_n(\varphi_t(z',v'))| < \eta$ $\forall t > T_\varepsilon$

$$\Rightarrow \quad |(g_n)_\Phi(z,v) - (g_n)_\Phi(z',v')| < \eta.$$

Then $$\|f - g_n\|_{L^1} \geq \int_{T^1\Sigma} |f(z,v) - g_n(z,v)|\,d\mu = \int_{T^1\Sigma} |f(\varphi_t(z,v)) - g_n(\varphi_t(z,v))|\,d\mu$$

μ is Φ-invariant

f is Φ-invariant

$$= \int_{T^1\Sigma} |f(z,v) - g_n(\varphi_t(z,v))|\,d\mu$$

$$\Rightarrow \int_{T^1\Sigma} \Big| \underbrace{f(z,v)}_{=\frac{1}{T}\int_0^T f(z,v)\,dt} - \frac{1}{T}\int_0^T g_n(\varphi_t(z,v))\,dt \Big|\,d\mu \leq \frac{1}{T}\int_0^T \left(\int_{T^1\Sigma} |f(z,v) - g_n(\varphi_t(z,v))|\,d\mu \right) dt$$

$$\leq \|f - g_n\|_{L^1}$$

$$\Rightarrow \lim_{n\to+\infty} \|f - (g_n)_\Phi\|_{L^1} = 0.$$

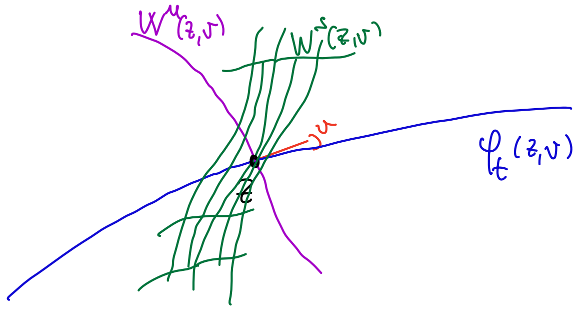
Then $\exists\, N^s := \bigcup_n N_n^s$, $\mu(N^s)=0$, s.t. $\forall\, (z',v') \in W^s(z,v) \setminus N^s$

$$f(z',v') = \lim_{n\to+\infty} (g_n)_{\Phi}(z',v') = \lim_{n\to+\infty} (g_n)_{\Phi}(z,v) = f(z,v)$$

Then

f Φ-invariant $\Rightarrow$ $\exists\, N^s, N^u$, $\mu(N^s)=\mu(N^u)=0$ s.t.

$$f(z',v') = f(z,v) \quad \forall\, (z',v') \in \left(W^s(z,v)\setminus N^s\right) \cup \left(W^u(z,v)\setminus N^u\right)$$



$$W^{cs}(z,v) = \bigcup_{t\in\mathbb{R}} \varphi_t(W^s(z,v))$$

Let $(z,v) \in T^1\Sigma$, U be an open set s.t. $U \cap W^{cs}(z,v) = \bigcup_{t\in(-T,T)} W^s_{loc}(\varphi_t(z,v))$

Then $\exists$ a measurable family of positive function $h_t : W^s_{loc}(\varphi_t(z,v)) \to \mathbb{R}$ s.t.

$\forall\, A \subset U$ measurable

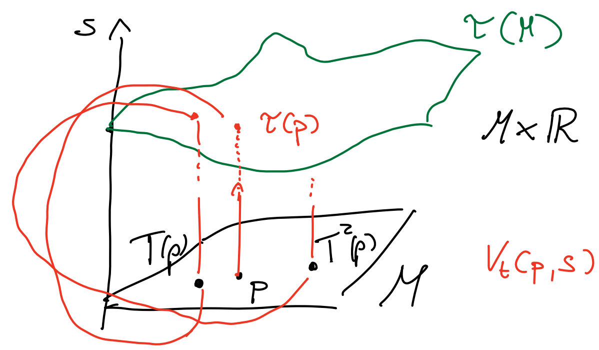
$$\underset{\mu|_{W^{cs}(z,v)}}{\underset{\shortparallel}{\mu_{cs}}}(A \cap W^{cs}(z,v)) = \int_{-T}^{T} \left(\int_{W^s_{loc}(\varphi_t(z,v))} h_t(z',v')\, \chi_A(z',v')\, d\underset{\mu|_{W^s_{loc}}}{\underset{\shortparallel}{\mu_s}}(z',v') \right) dt$$

Then f Φ-invariant $\Rightarrow$ f is constant μ-a.e. $\square$

Flows under a function

$T : M \to M$, M manifold

$\tau : M \to [0,+\infty)$

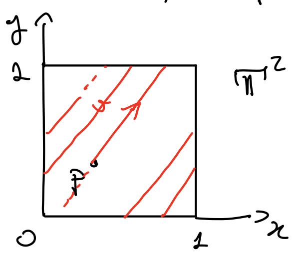


Def Given a manifold M, a map T, and a function $\tau: M \to [0,+\infty)$, let
$$M_\tau := \{(p,s) \,/\, p\in M,\ 0\le s\le \tau(p)\}/_\sim$$
with $(p,\tau(p)) \sim (T(p),0)$. The function τ is called the roof function, the map T is called the base map.
Then we define the semi-flow $V_t : M_\tau \to M_\tau$ given by
$$(p,s) \mapsto V_t(p,s) = (p, s+t) \quad \forall\, t\ge 0.$$
If T is invertible then $\{V_t\}$ is a flow.
$\{V_t\}$ is called the flow on M under the function τ. (special flows, suspension flows)

Warning Suspension flows is sometimes used for flows under a constant function.

Ex Poincaré section and map.

$\{\varphi_t\}$ on $\mathbb{T}^2$, $p \mapsto p + tv$, for $v\in\mathbb{R}^2$ fixed.



If $v=(v_1,v_2)$ has $v_2\ne 0$ then $M=\{y=0\}$ is a Poincaré section

Then the Poincaré map $T: M\to M$, $(x,0)\mapsto \varphi_{t(x,0)}(x,0)$ where
$t(x,0) = \min\{t>0 \,/\, \varphi_t(x,0)\in M\} = \frac{1}{v_2}$
$$S^1 = M \ni (x,0) \mapsto T(x,0) = \left(\left\{x + \frac{v_1}{v_2}\right\}, 0\right) \in M$$