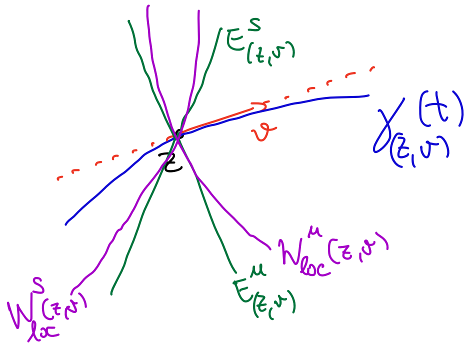
- Σ smooth closed connected Riemannian surface with negative curvature orientable
- $\Phi = \{\varphi_t\}$ the geodesic flow on Σ is Anosov.



Theorem (Stable and Unstable Manifold Theorem)

Let M smooth connected closed manifold and $\Phi = \{\varphi_t\}$ a C^r, $r \geq 1$, Anosov flow on M, with constants $\lambda \in (0,1)$, $\mu > 1$.

Then $\forall p \in M$ there exist $W^s_{loc}(p)$ and $W^u_{loc}(p)$ which are C^r embedded disks depending continuously on p in the C^1 topology, called local strong stable and unstable manifolds, s.t.:

(i) $T_p W^s_{loc}(p) = E^s_p$, $T_p W^u_{loc}(p) = E^u_S$;

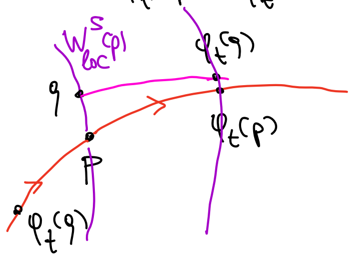
(ii) $\varphi_t(W^s_{loc}(p)) \subseteq W^s_{loc}(\varphi_t(p)) \quad \forall t > 0$

$\varphi_t(W^u_{loc}(p)) \subseteq W^u_{loc}(\varphi_t(p)) \quad \forall t < 0$

(iii) $\forall \delta > 0 \quad \exists C_\delta > 0$ s.t.

$$d(\varphi_t(p), \varphi_t(q)) \leq C_\delta (\lambda+\delta)^t \, d(p,q) \quad \forall q \in W^s_{loc}(p) \quad \forall t > 0$$

$$d(\varphi_t(p), \varphi_t(q)) \leq C_\delta (\mu-\delta)^t \, d(p,q) \quad \forall q \in W^u_{loc}(p) \quad \forall t < 0$$



(iv) There exist a continuous family of neighbourhoods U_p in M s.t.

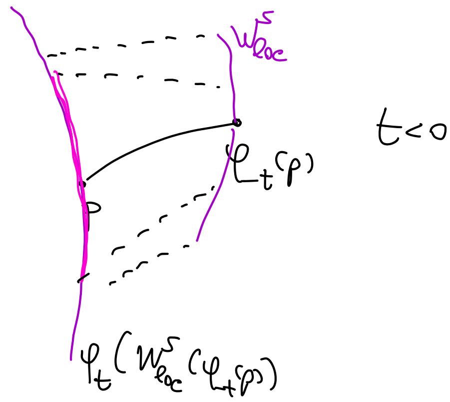
$$W^s_{loc}(p) = \{ q \in U_p \,/\, \varphi_t(q) \in U_{\varphi_t(p)} \ \forall t > 0, \ d(\varphi_t(p), \varphi_t(q)) \xrightarrow[t \to +\infty]{} 0 \}$$

$$W^u_{loc}(p) = \{ q \in U_p \,/\, \varphi_t(q) \in U_{\varphi_t(p)} \ \forall t < 0, \ d(\varphi_t(p), \varphi_t(q)) \xrightarrow[t \to -\infty]{} 0 \}$$

Def Strong stable and unstable manifolds

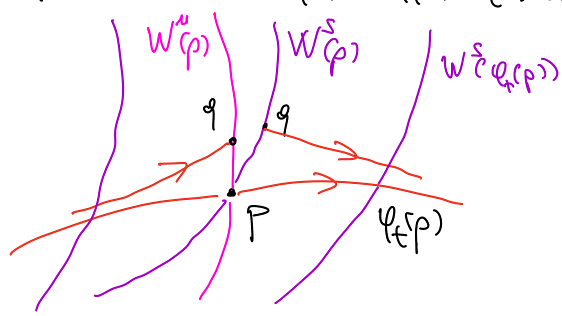
$$W^s(p) := \bigcup_{t<0} \varphi_t\left(W^s_{loc}(\varphi_{-t}(p))\right)$$

$$= \left\{ q \in M \;/\; d(\varphi_t(p), \varphi_t(q)) \xrightarrow[t\to+\infty]{} 0 \right\}$$



$$W^u(p) := \bigcup_{t>0} \varphi_t\left(W^u_{loc}(\varphi_{-t}(p))\right)$$

$$= \left\{ q \in M \;/\; d(\varphi_t(p), \varphi_t(q)) \xrightarrow[t\to-\infty]{} 0 \right\}$$

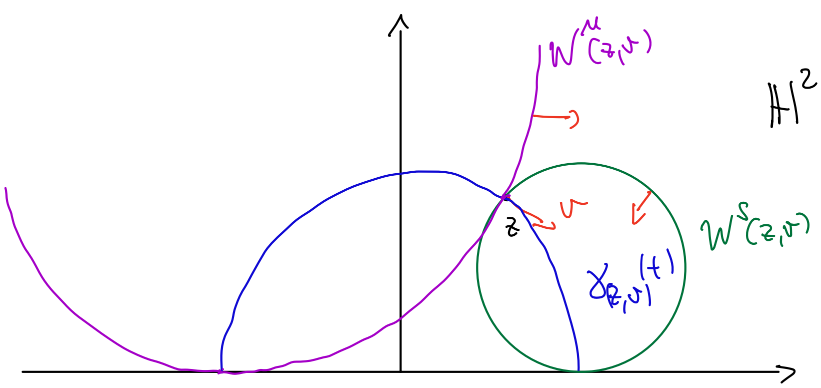


Weak stable and unstable manifolds

$$W^{cs}(p) := \bigcup_{t\in\mathbb{R}} \varphi_t(W^s(p))$$

$$W^{cu}(p) := \bigcup_{t\in\mathbb{R}} \varphi_t(W^u(p))$$

Ex



Theorem (Hopf, Hirsch-Pugh '75) The geodesic flow on Σ admits C^1 foliation in strong stable and unstable manifolds.

Ergodicity

Σ Riemannian surface, m volume form on Σ $(dx_1 \wedge dx_2)$
$T^1\Sigma$, μ Liouville measure, $\mu = m \times \ell$, ℓ = Lebesgue measure on S^1.

Def - Let (M,μ) be a probability space, $T: M \to M$ be a measurable map. Then T is <u>measure preserving</u> (or μ is T-invariant) if $T_*\mu = \mu$ or $\mu(T^{-1}(A)) = \mu(A)$ $\forall A \subset M$ measurable.

- Let (M,μ) be a probability space, $\Phi = \{\varphi_t\}$ a measurable flow is <u>measure preserving</u> if φ_t is measure preserving $\forall t \in \mathbb{R}$.

Theorem (Birkhoff) Let $\Phi = \{\varphi_t\}$ be a measurable flow on (M,μ) prob. space, and assume that Φ is measure preserving. Let $f \in L^1(M,\mu)$, then

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T f(\varphi_t(p))\,dt = \lim_{T\to-\infty} \frac{1}{|T|}\int_T^0 f(\varphi_t(p))\,dt =: f_\Phi(p)$$

exist for μ-a.e. $p \in M$. And $\int_M f_\Phi(p)\,d\mu = \int_M f(p)\,d\mu$.

Def A measurable flow $\Phi = \{\varphi_t\}$ on (M,μ) prob. space is <u>ergodic</u> (or μ is ergodic for Φ) if $A \subset M$ measurable and Φ-invariant (i.e. $\varphi_t(A) = A$ $\forall t \in \mathbb{R}$) then $\mu(A) = 0$ or $\mu(A^c) = 0$.

Rem If $\varphi_{\bar t}$ is ergodic for a specific $\bar t$ then $\Phi = \{\varphi_t\}$ is ergodic

Prop Let $\Phi = \{\varphi_t\}$ be a measurable flow on (M,μ) prob., then the following are equivalent:

(i) Φ is ergodic;

(ii) Any Φ-invariant $f \in L^1(M,\mu)$ [i.e. $f \circ \varphi_t = f$ μ-a.e. $\forall t \in \mathbb{R}$] is constant μ-a.e.;

(iii) If μ is Φ-invariant, then if $\nu \ll \mu$ is Φ-invariant it must be that $\nu = \mu$;

(iv) The time maps φ_t are ergodic for all t but countably many;

(v) For $f \in L^1(M,\mu)$, $f_{\Phi}(p) = \int_M f \, d\mu$ for μ-a.e. $p \in M$.

<u>Theorem</u> The geodesic flow $\Phi = \{\varphi_t\}$ on Σ is a <u>contact flow</u>. That is $\exists$ 1-form α on $T^1\Sigma$ s.t. $\alpha \wedge d\alpha$ is non-degenerate, the vector field $X := \frac{d}{dt}\Big|_{t=0} \varphi_t$ satisfies $\alpha(X) = 1$, $d\alpha(X, \xi) = 0$ $\forall \xi \in T(T^1\Sigma)$.

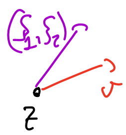
Then Φ preserves the Liouville measure μ.

<u>proof</u> Cfr. Paternain book, 1999.

$\pi : T\Sigma \to \Sigma$, $\pi(z,v) = z$.

$$\alpha_{(z,v)}(\xi) := \langle d_{(z,v)}\pi(\xi), v \rangle, \qquad \forall \xi \in T_{(z,v)}(T\Sigma).$$

[<u>Rem</u>



$$\xi = (\underbrace{\xi_1, \xi_2}_{z}, \underbrace{\xi_3, \xi_4}_{v})$$

$$\alpha = v_1 \, dx_1 + v_2 \, dx_2 \qquad , \qquad d\alpha = dv_1 \wedge dx_1 + dv_2 \wedge dx_2 \quad]$$

Then $d\alpha$ is a symplectic 2-form on $T(T\Sigma)$ and, if $H(z,v) = \frac{1}{2}|v|^2$, then X satisfies

$$-d\alpha(X, \xi) = dH(\xi) \qquad \forall \xi \in T(T\Sigma).$$

Then $\forall (z,v) \in T^1\Sigma$,

$$\alpha_{(z,v)}(X) = \langle d_{(z,v)}\pi(X), v \rangle = \langle v, v \rangle = 1.$$

$$d\alpha(X, \xi) = -dH(\xi) = 0 \qquad \forall \xi \in T_{(z,v)}(T^1\Sigma)$$

$\Downarrow$

H is constant on $T^1\Sigma$

Then Φ preserves α, $d\alpha$ and $\alpha \wedge d\alpha$.

$$L_X \alpha = \underbrace{i_X(d\alpha)}_{\overset{\shortparallel}{0}} + \underbrace{d(i_X\alpha)}_{\overset{\shortparallel}{0}} \qquad \square$$

$\underline{\text{Ex}}$ The geodesic flow on $(\mathbb{T}^2, \text{flat})$ and on $(S^2, \text{constant } K = +1)$ is not ergodic.

For $\mathbb{T}^2$, let $A \subset T^1\mathbb{T}^2$ be

$$A = \mathbb{T}^2 \times [\vartheta_1, \vartheta_2] \quad \text{with } [\vartheta_1, \vartheta_2] \subsetneq S^1.$$

Then $\mu(A) = m(\mathbb{T}^2) \cdot \ell([\vartheta_1, \vartheta_2]) = \frac{|\vartheta_2 - \vartheta_1|}{2\pi} > 0$, $\mu(A^c) > 0$,

and $\forall (z, v) \in A$, $\varphi_t(z, v) \in A \quad \forall t \in \mathbb{R}$.

$$\varphi_t(z,v) = (z + tv \pmod{\mathbb{Z}^2}, v).$$

$\underline{\text{Thm}}$ The geodesic flow on a quotient of $\mathbb{H}^2$ with finite volume is ergodic.

It follows from

$\underline{\text{Thm}}$ Let $\Gamma < SL(2,\mathbb{R})$ be a $\underline{\text{lattice}}$ (that is Γ is discrete and $\Sigma = \Gamma / \mathbb{H}^2$ has finite volume). Then, the right-action of any non-elliptic $g \in SL(2,\mathbb{R})$ on Σ is measure preserving and ergodic wrt the Liouville measure μ on $T^1\Sigma$.

$\underline{\text{proof}}$ $T^1\Sigma \cong \Gamma / PSL(2,\mathbb{R}) \ni h \longmapsto hg$ for a fixed $g \in SL(2,\mathbb{R})$.

Let $g = \lambda_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $a > 1$. Let $f \in L^2(T^1\Sigma, \mu)$ which is λ_a-invariant. Then we prove that f is constant μ-a.e..

We use that $SL(2,\mathbb{R}) \ni h \mapsto f \circ h \in L^2$ is unitary $(\|f \circ h\|_{L^2} = \|f\|_{L^2})$ and continuous. Then

$$SL(2,\mathbb{R}) \ni h \longmapsto p(h) := \langle f \circ h, f \rangle_{L^2}.$$

p has the following properties:

- it is continuous;
- $p(\lambda_a^m h \lambda_a^n) = \langle f \circ (\lambda_a^m h \lambda_a^n), f \rangle_{L^2} =$

$$= \int_M f(\lambda_a^m h \lambda_a^n) f \, d\mu = \int_M f(\lambda_a^m h) f(\lambda_a^{-n}) \, d\mu = p(h)$$

$f \circ \lambda_a^m = f \qquad \forall n, m \in \mathbb{Z}$

Let $h = \tau_b = \begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}$, then $\lambda_a^n \tau_b \lambda_a^{-n} = \begin{pmatrix}a^n & 0\\ 0 & a^{-n}\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}a^{-n} & 0\\ 0 & a^{n}\end{pmatrix} \xrightarrow[n\to-\infty]{} \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$\Rightarrow \quad p(\tau_b) = p(\lambda_a^n \tau_b \lambda_a^{-n}) \xrightarrow[n\to-\infty]{} p(Id) = \|f\|^2_{L^2}$.

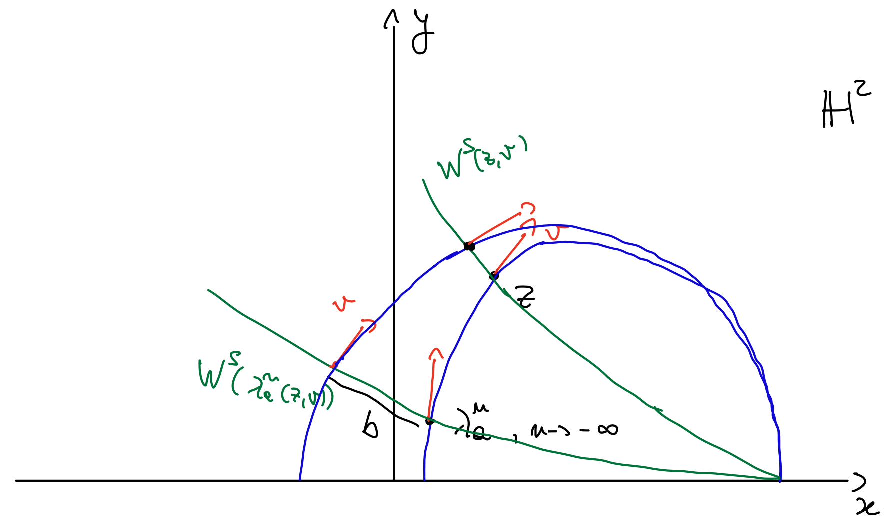
$$\langle f\circ\tau_b, f\rangle_{L^2} = \|f\|^2_{L^2} \quad \forall\, \tau_b. \quad \left(|\langle f\circ\tau_b, f\rangle| \le \|f\|^2_{L^2} \right)$$

$\Rightarrow \quad f\circ\tau_b = f \quad \mu\text{-a.e.}$

Let $h = \exp(\gamma H_-) = \begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix}$, then $f\circ h = f$ μ-a.e.

Last step, $SL(2,\mathbb{R}) = \langle \{\tau_b\}_{b\in\mathbb{R}}, \{\exp(\gamma H_-)\}_{\gamma\in\mathbb{R}} \rangle$, then any f λ_a-invariant is $SL(2,\mathbb{R})$-invariant, hence f is constant μ-a.e.

$\square$ hyperbolic elements.



$$\boxed{\lambda_a^n \tau_b \lambda_a^{-n}} \xrightarrow[n\to-\infty]{} Id$$