Def (Anosov flow)

M smooth connected manifold, $\Phi = \{\varphi_t\}$ smooth flow on M without fixed points.

A compact Φ-invariant $\Lambda \subset M$ is <u>hyperbolic</u> if there is a Φ-invariant splitting $TM|_\Lambda = E^c \oplus E^u \oplus E^s$ and $\exists\ C \geq 1,\ \lambda \in (0,1)$, $\mu > 1$ s.t. $\forall p \in \Lambda$ we have:

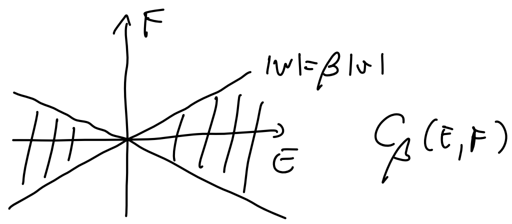
- $E^c_p = \mathrm{Span}\left(\frac{d}{dt}\varphi_t|_{t=0}(p)\right)$
- $\forall\ w \in E^s,\ |(d_p\varphi_t)(w)| \leq C\lambda^t |w| \quad \forall\ t > 0$;
- $\forall\ w \in E^u,\ |(d_p\varphi_t)(w)| \leq C\left(\frac{1}{\mu}\right)^{|t|} |w| \quad \forall\ t < 0$.

Φ is Anosov if M is closed and hyperbolic.

Alekseev cone field criterion

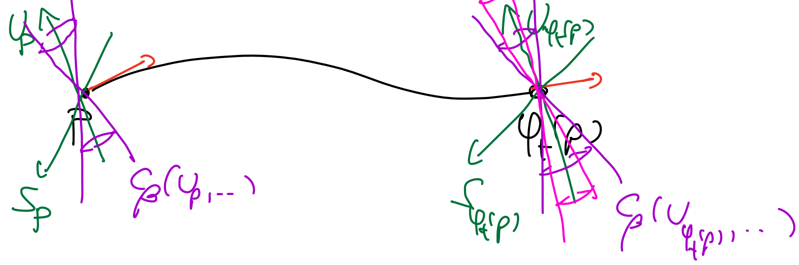
Def Given linear spaces E, F and $\beta \in (0,1)$, the β-cone $C_\beta(E,F)$ is defined as

$$C_\beta(E,F) = \{ v + w \ /\ v \in E,\ w \in F,\ |w| < \beta |v| \}$$



Prop M, $\Phi = \{\varphi_t\}$ as before, a compact Φ-invariant $\Lambda \subset M$ is hyperbolic if and only if $\exists\ C \geq 1,\ \lambda, \beta \in (0,1)$ and a decomposition $T_pM = E^c_p \oplus S_p \oplus U_p \quad \forall\ p \in \Lambda$ s.t.

- $E^c_p = \mathrm{Span}\left(\frac{d}{dt}\Big|_{t=0} \varphi_t(p)\right)$
- $d_p\varphi_t\left(\overline{C_\beta(U_p, E^c_p \oplus S_p)}\right) \subset C_\beta(U_{\varphi_t(p)}, E^c_{\varphi_t(p)} \oplus S_{\varphi_t(p)}) \quad \forall\ t > 0$



- $d_p\varphi_t\left(\overline{\mathcal{C}_\beta(\mathcal{S}_p, E^c_p \oplus U_p)}\right) \subset \mathcal{C}_\beta(\mathcal{S}_{\varphi_t(p)}, E^c_{\varphi_t(p)} \oplus U_{\varphi_t(p)}) \quad \forall t<0$

- $\forall w \in \mathcal{C}_\beta(U_p, E^c_p \oplus \mathcal{S}_p)$, $\quad |d_p\varphi_t(w)| \le C\lambda^{-t}|w| \quad \forall t<0$

- $\forall w \in \mathcal{C}_\beta(\mathcal{S}_p, E^c_p \oplus U_p)$, $\quad |d_p\varphi_t(w)| \le C\lambda^{t}|w| \quad \forall t>0$

$\longleftrightarrow$

Σ with curvature $K \le 0$, $\quad \Phi = \{\varphi_t\}$ geodesic flow., $\varphi_t : T^1\Sigma \circlearrowleft$.

<u>Def</u> A vector field J along a geodesic γ is a <u>Jacobi field</u> if

$$D_{\dot\gamma} D_{\dot\gamma} J = R(\dot\gamma, J)\dot\gamma$$

<u>Prop</u>
- Given $u, v \in T_{\gamma(0)}\Sigma$, there exists a Jacobi field J with $J(0)=u$, $D_{\dot\gamma}J(0)=v$.
- If $J(0)$ and $D_{\dot\gamma}J(0)$ are orthogonal to $\dot\gamma(0)$, then the associated Jacobi field J and $D_{\dot\gamma}J$ are orthogonal to $\dot\gamma$ $\forall t$.

<u>proof</u>

$$\dot\gamma(\langle D_{\dot\gamma}J, \dot\gamma\rangle) = \langle D_{\dot\gamma}D_{\dot\gamma}J, \dot\gamma\rangle + \langle D_{\dot\gamma}J, \overset{0}{\overset{\shortparallel}{D_{\dot\gamma}\dot\gamma}}\rangle =$$

$$= \langle R(\dot\gamma, J)\dot\gamma, \dot\gamma\rangle = \langle \underbrace{R(\dot\gamma,\dot\gamma)\dot\gamma}_{=0}, J\rangle = 0$$

$$\dot\gamma(\langle J, \dot\gamma\rangle) = \langle D_{\dot\gamma}J, \dot\gamma\rangle = 0$$

$\square$

<u>Prop</u> Given $\gamma : [0,1] \to \Sigma$ geodesic, let $h_\gamma : (-\varepsilon, \varepsilon) \times [0,1] \to \Sigma$ smooth be a geodesic variation of γ (that is, $[0,1] \ni t \mapsto h_\gamma(s,t)$ is a geodesic $\forall s \in (-\varepsilon, \varepsilon)$, and $h_\gamma(0,t) = \gamma(t)$), then $\frac{\partial h_\gamma}{\partial s}(0,t)$ is a Jacobi field.

proof $\quad \dot h_\gamma = \frac{\partial}{\partial t} h_\gamma$

$$R\left(\dot h_\gamma, \frac{\partial h_\gamma}{\partial s}\right)\dot h_\gamma = D_{\dot h_\gamma} D_{\frac{\partial h_\gamma}{\partial s}} \dot h_\gamma - D_{\frac{\partial h_\gamma}{\partial s}}\big(\overbrace{D_{\dot h_\gamma}\dot h_\gamma}^{0}\big) - D_{\underbrace{[\dot h_\gamma, \frac{\partial h_\gamma}{\partial s}]}_{0}} \dot h_\gamma =$$

$$= D_{\dot h_\gamma}\left(D_{\frac{\partial h_\gamma}{\partial s}}\dot h_\gamma\right) = D_{\dot h_\gamma}\Big(D_{\dot h_\gamma}\frac{\partial h_\gamma}{\partial s} + \underbrace{[\dot h_\gamma, \tfrac{\partial h_\gamma}{\partial s}]}_{0}\Big) =$$

$$= D_{\dot h_\gamma} D_{\dot h_\gamma} \frac{\partial h_\gamma}{\partial s}$$

$\square$

Examples $\quad$ - $h_\gamma(s,t) = \gamma(t+s)$ is a geodesic variation

$$\frac{\partial h_\gamma}{\partial s}(0,t) = \dot\gamma(t), \quad J(0) = \dot\gamma(0), \quad D_{\dot\gamma} J(0) = 0$$

- $h_\gamma(s,t) = \gamma((1+s)t)$ " " since

$$\dot h_\gamma(s,t) = (1+s)\dot\gamma((1+s)t)$$

$$\frac{\partial h_\gamma}{\partial s}(0,t) = t\dot\gamma(t), \quad J(0) = 0, \quad D_{\dot\gamma} J(0) = \dot\gamma(0)$$

Rem $\quad$ Orthogonal Jacobi fields form a 2D space

Prop $\quad$ Let J be a Jacobi field along a geodesic γ with J and $D_{\dot\gamma}J$ orthogonal to $\dot\gamma$. Then let N be the orthonormal direction to $\dot\gamma$, and define

$$\rho(t) := \frac{\dot\gamma(\langle J, N\rangle)}{\langle J, N\rangle}$$

Then $\rho(t)$ satisfies the Riccati differential equation

$$\dot\rho(t) = -K(\gamma(t)) - \rho^2(t)$$

proof $\quad \langle N, \dot\gamma\rangle = 0, \quad D_{\dot\gamma} N = 0$

$$\dot\gamma(\langle J, N\rangle) = \langle D_{\dot\gamma} J, N\rangle$$

$$\dot\gamma(\langle D_{\dot\gamma} J, N\rangle) = \langle D_{\dot\gamma} D_{\dot\gamma} J, N\rangle = \langle R(\dot\gamma, J)\dot\gamma, N\rangle$$

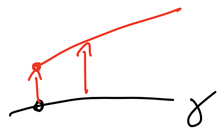
Then $$K(\gamma(t)) = \frac{\langle \dot\gamma, R(\dot\gamma,J)J\rangle}{\underbrace{\langle\dot\gamma,\dot\gamma\rangle}_{=1}\langle J,J\rangle - \underbrace{\langle\dot\gamma,J\rangle^2}_{=0}} = -\frac{\langle J, R(\dot\gamma,J)\dot\gamma\rangle}{\langle J,J\rangle} \underset{J=\pm|J|N}{=} -\frac{\langle N, R(\dot\gamma,J)\dot\gamma\rangle}{\langle J,N\rangle}$$

It follows that

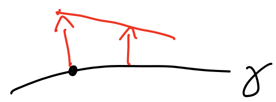
$$\dot\rho = \frac{\dot\gamma(\dot\gamma\langle J,N\rangle)}{\langle J,N\rangle} - \underbrace{\frac{[\dot\gamma\langle J,N\rangle]^2}{\langle J,N\rangle^2}}_{\rho^2} = \underbrace{\frac{\langle R(\dot\gamma,J)\dot\gamma,N\rangle}{\langle J,N\rangle}}_{-K(\gamma(t))} - \rho^2 \qquad \square$$

Rem $\quad \rho(t) = \frac{\dot\gamma(\langle J,N\rangle)}{\langle J,N\rangle}$

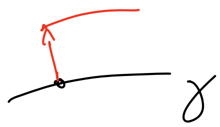
$\rho > 0$



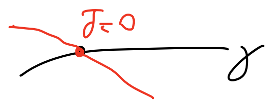
$\rho < 0$



$\rho = 0$

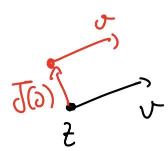


$\langle J,N\rangle = 0$



Theorem (Eberlein '73) Let Σ be a closed connected smooth Riemannian surface with non-positive curvature ($K \le 0$). If every geodesic in Σ contains a point where $K<0$, then the geodesic flow on Σ is Anosov.

proof



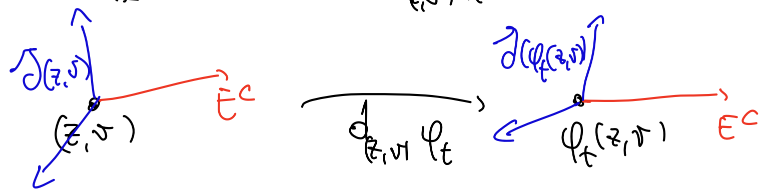
$D_{\dot\gamma}J(0)=0$

$J(0)=0$

$(z,v)\in T^1\Sigma$, $\qquad T_{(z,v)}(T^1\Sigma) \cong \mathcal{J}(z,v)\oplus E^c$

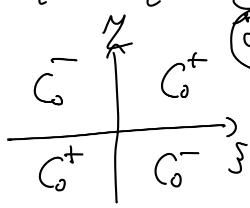
$E^c = \mathrm{Span}\left(\frac{d}{dt}\Big|_{t=0}\varphi_t(z,v)\right)$, $\quad \mathcal{J}(z,v) =$ Orthogonal Jacobi fields.

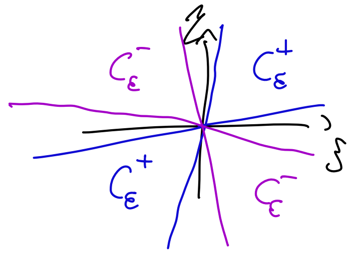
$J(z,v)$ is Φ-invariant, $\quad d_{(z,v)}\varphi_t(J(z,v)) = J(\varphi_t(z,v)) \quad \forall\, t\in\mathbb{R}.$



$$C_0^\pm := \{(\xi,\eta)\in\mathbb{R}^2 \,/\, \pm\xi\eta>0\}, \quad C_\varepsilon^\pm := \{(\xi,\eta)\in\mathbb{R}^2 \,/\, \varepsilon\eta \le \pm\xi \le \tfrac{\eta}{\varepsilon}\}$$

$J(z,v)$





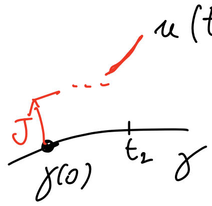
Following (Kourganoff 2018).

Step 1 $\quad \exists\, \kappa>0,\ t_0>0$ s.t. $\forall\, \gamma$ geodesic on Σ, $|\dot\gamma|=1$,

$$\int_0^{t_0} K(\gamma(t))\,dt \le -\kappa.$$

Step 2 $\quad \exists\, m>0,\ t_1>0$ s.t. $\forall\,\gamma$ geodesic on Σ, $|\dot\gamma|=1$, the solution to $\dot u = -K(\gamma(t)) - u^2$ with $u(0)=0$ satisfies

$$u(t_1)\ge m.$$



Let κ and t_0 be as in Step 1. First of all, $u(t)\ge 0$ $\forall\, t>0$ by comparison $\dot u \ge -u^2$. Then choose $t_1>t_0$, and let $a > \max\{1, t_1\kappa\}$, $\quad m = \frac{\kappa}{a}\left(1-\frac{t_1\kappa}{a}\right)>0$.

Let $t^* = \sup\{t\in[0,t_1] \,/\, u(t)\ge \frac{\kappa}{a}\}$, $\quad t^*=0$ if $u(t)<\frac{\kappa}{a}$ $\forall\, t\in[0,t_1]$.

Then, if $t^*=0$,

$$u(t_1) = \int_0^{t_1}\dot u(t)\,dt = \int_0^{t_1}[-K(\gamma(t)) - u^2]\,dt \ge \int_0^{t_1}\left[-\left(\tfrac{\kappa}{a}\right)^2\right]dt + \int_0^{t_0}[-K(\gamma(t))]\,dt$$

$$\ge \kappa - \frac{t_1\kappa^2}{a^2} > \frac{\kappa}{a} - \frac{t_1\kappa^2}{a^2} = m.$$

If $t^*>0$,

$$u(t_1) = u(t^*) + \int_{t^*}^{t_1}\dot u(t)\,dt \ge u(t^*) + \int_{t^*}^{t_1}(-u^2)\,dt \ge \frac{\kappa}{a} - \frac{t_1\kappa^2}{a^2} = m$$

Step 3 $\quad \forall (z,v) \in T^1\Sigma$,

$$d_{(z,v)}\varphi_{t_1}(C_0^+) \subset C_\varepsilon^+ \,, \quad d_{(z,v)}\varphi_{-t_1}(C_0^-) \subset C_\varepsilon^- .$$

Step 4 $\quad$ We can apply Alekseev cone field criterion.

Use that φ_t preserves the volume form of $T^1\Sigma$. $\square$