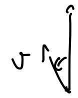
$$(z,v) \in T^1\mathbb{H}^2 \cong PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/_{\pm I}$$

$$z = x+iy\,, \quad v = iy(\cos\vartheta + i\sin\vartheta)$$



$$SL(2,\mathbb{R}) \ni g_{(z,v)} = \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}\begin{pmatrix} \cos\vartheta/2 & \sin\vartheta/2 \\ -\sin\vartheta/2 & \cos\vartheta/2 \end{pmatrix}$$

$$(z,v) = d\,\varphi_{g_{(z,v)}}(i,i)$$

<u>Prop</u> The geodesic flow on $\mathbb{H}^2$ is conjugate to the right action of $\{\lambda_a\}_{a>0}$ on $SL(2,\mathbb{R})$.

$$g_{(z,v)} \leftrightarrow (z,v) \quad \text{then} \quad \varphi_t(z,v) \leftrightarrow g_{\varphi_t(z,v)} = g_{(z,v)}\underbrace{\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}}_{=\lambda_{e^{t/2}}}$$

<u>proof</u> $(i,i) \leftrightarrow g_{(i,i)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\varphi_t(i,i) = (\underbrace{ie^t}_{=0+ie^t}, ie^t)\,, \quad g_{\varphi_t(i,i)} = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$$

$$(z,v) \leftrightarrow g_{(z,v)} \iff (z,v) = d\,\varphi_{g_{(z,v)}}(i,i)$$

$$\varphi_t(z,v) \qquad d\,\varphi_{g_{(z,v)}}(ie^t, ie^t) = \varphi_t(z,v)$$

$$\Updownarrow$$

$$g_{\varphi_t(z,v)} = g_{(z,v)}\, g_{(ie^t, ie^t)} = g_{(z,v)}\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \qquad \square$$

<u>Remark</u> $\gamma_{(z,v)}(t)$, $\quad \gamma^{\pm}_{(z,v)} := \lim_{t\to\pm\infty} \gamma_{(z,v)}(t)$

$$g_{(z,v)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})\,, \quad \gamma^{\pm}_{(z,v)} = \lim_{t\to\pm\infty}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} i =$$

$$= \lim_{t\to\pm\infty} \begin{pmatrix} a e^{t/2} & b e^{-t/2} \\ c e^{t/2} & d e^{-t/2}\end{pmatrix} i = \lim_{t\to\pm\infty} \frac{a e^{t/2} i + b e^{-t/2}}{c e^{t/2} i + d e^{-t/2}}$$

$$\gamma^+_{(z,v)} = \frac{a}{c}, \quad \gamma^-_{(z,v)} = \frac{b}{d}, \quad \left(\frac{1}{0} = \infty\right)$$

$$T_{(z,v)}(T^1 \mathbb{H}^2) \cong sl(2,\mathbb{R}) := \{ a \in \mathcal{M}(2\times 2, \mathbb{R}) \,/\, \operatorname{tr} a = 0\} \quad \forall (z,v)$$

$\dim sl(2,\mathbb{R}) = 3$, basis $X = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$, $H_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $H_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

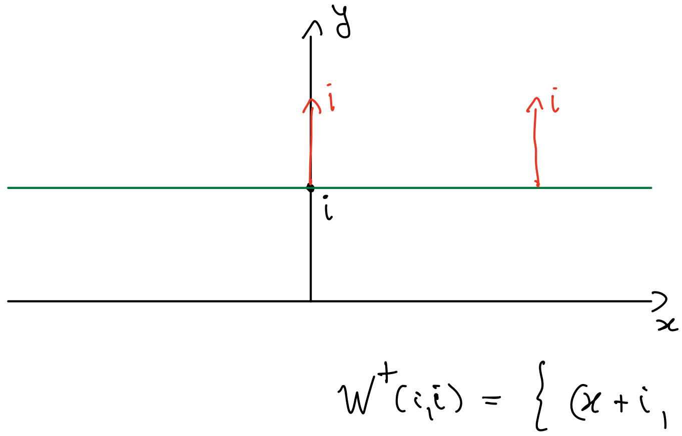
A perturbation of (z,v) is of the form (z',v') s.t. $g_{(z',v')} = g_{(z,v)} \exp(\delta a)$
$a \in \{X, H_+, H_-\}$.

- $g_{(z',v')} = g_{(z,v)} \underset{\begin{pmatrix} e^{\delta/2} & 0 \\ 0 & e^{-\delta/2}\end{pmatrix}}{\underset{\shortparallel}{\exp(\delta X)}}$, for small δ.

  X = vector field of the geodesic flow.

- $g_{(z',v')} = g_{(z,v)} \exp(\delta H_+) = g_{(z,v)} \tau_\delta = g_{(z,v)} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$

$(z,v) = (i,i)$, $W^+(i,i) := \left\{ (z',v') \,/\, g_{(z',v')} = \underbrace{g_{(i,i)} \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}} \right\}_{\delta\in\mathbb{R}}$



$\tau_\delta(i) = \delta + i$

$$W^+(i,i) = \{ (x+i, i) \,/\, x\in\mathbb{R} \} \subset T^1 \mathbb{H}^2.$$

- $W^-(i,i) := \left\{ (z',v') \,/\, g_{(z',v')} = g_{(i,i)} \exp(\delta H_-) = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \right\}_{\delta\in\mathbb{R}}$

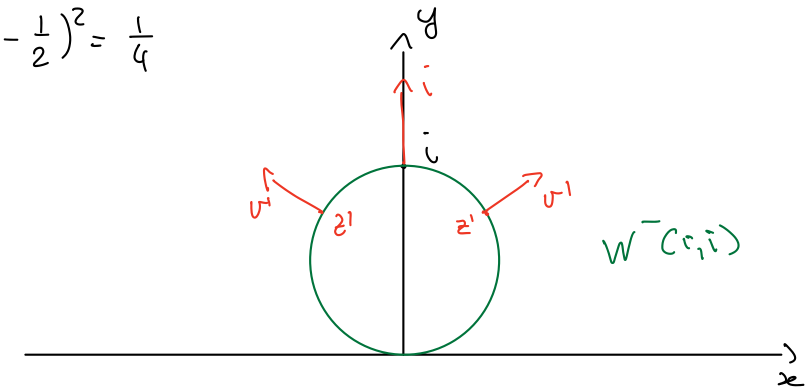
$H_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad \exp(\delta H_-) = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}$

$$(z',v') = \left(x'+iy',\; iy'(\cos\vartheta' + i\sin\vartheta')\right)$$

$$x'+iy' = \begin{pmatrix}1&0\\ \delta&1\end{pmatrix} i = \frac{i}{\delta i+1} = \frac{i(1-\delta i)}{1+\delta^2} = \underbrace{\frac{\delta}{1+\delta^2}}_{x'} + \underbrace{\frac{1}{1+\delta^2}}_{y'}\, i$$

$$\vartheta' = -2\arctan\frac{c}{d} = -2\arctan\delta = -2\arctan\frac{x'}{y'}.$$

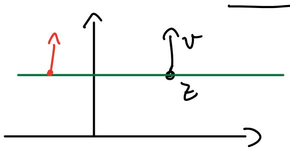
$$(x')^2 + \left(y'-\frac{1}{2}\right)^2 = \frac{1}{4}$$



- For $(z,v)\in T^1\mathbb{H}^2$, $\quad W^\pm(z,v) = d\psi_{g(z,v)}\left(W^\pm(i,i)\right)$

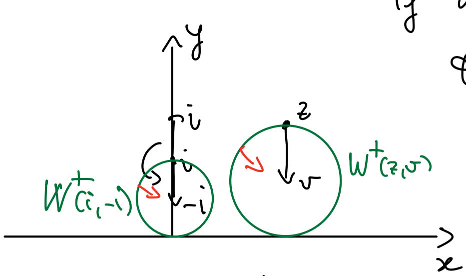
Def Given $\underset{(x,y,\vartheta)}{(z,v)}\in T^1\mathbb{H}^2$ and $\gamma_{(z,v)}(t)$ the geodesic for (z,v), we call:

- <u>Positive Horocycle</u> of (z,v) the set $W^+(z,v)$, which is



  - if $\vartheta=0$, $W^+(z,v)$ is the horizontal line through z with vertical vectors pointing upward;
  - if $\vartheta=\pi$, $W^+(z,v) = d\psi_{g(z,-v)}\circ d\psi_S\left(W^+(i,i)\right)$ is the circle through z tangent to $\{y=0\}$ at $x=\mathrm{Re}(z)=\gamma^+_{(z,v)}$, with vectors pointing inside.

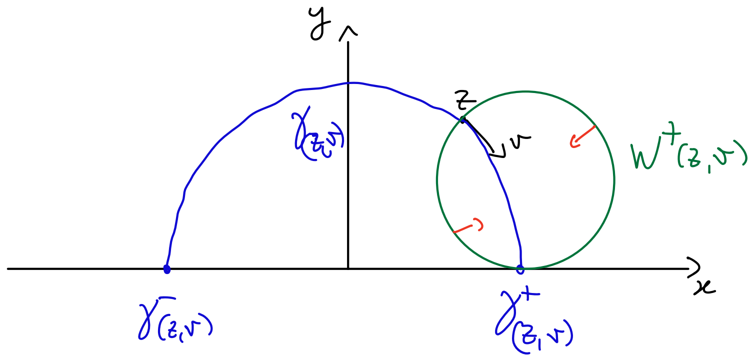
$$W^+(i,-i) = d\psi_S\left(W^+(i,i)\right)$$



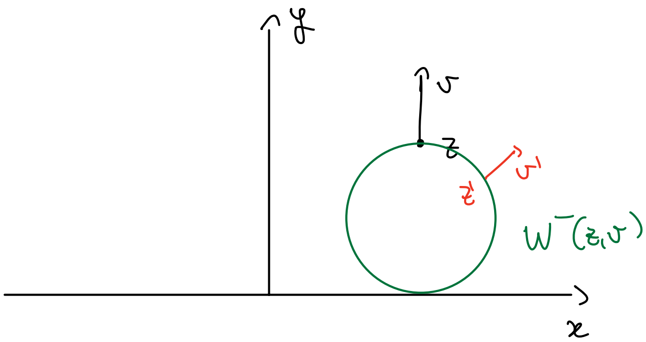
$$(i,-i) = d\psi_S(i,i)$$

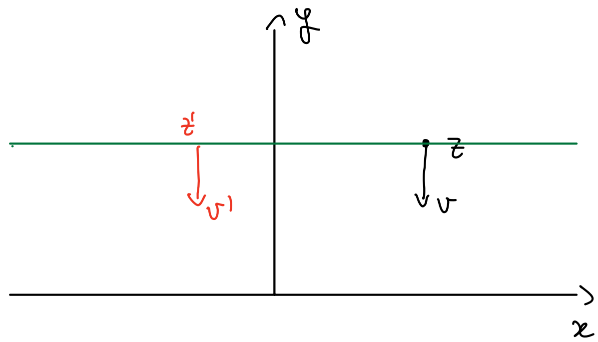
$$S=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$$

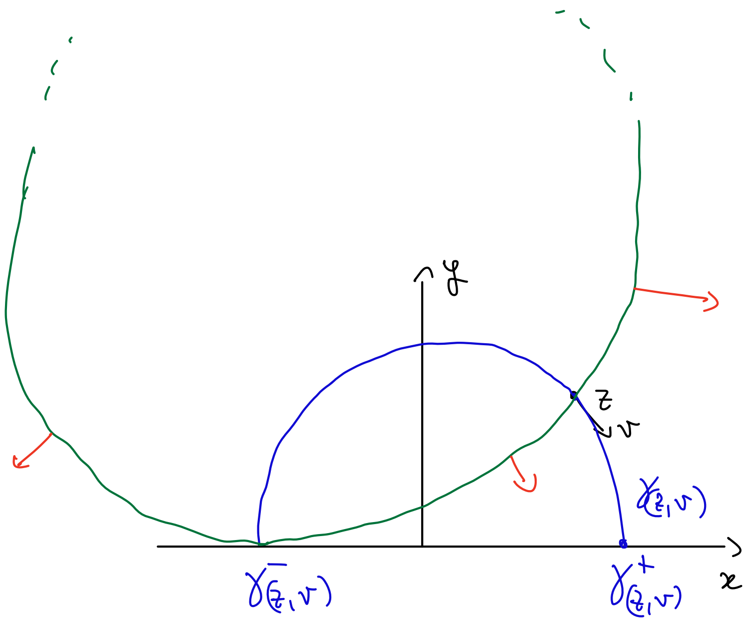
  - if $\vartheta\neq 0,\pm\pi$, $W^+(z,v)$ is the circle through z tangent to $\{y=0\}$ at $\gamma^+_{(z,v)}$, with vectors pointing inside.



• <u>Negative Horocycle for (z,v)</u> is $W^-(z,v)$





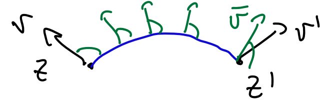


<u>Def</u> We consider the distance d_h on $T^1\mathbb{H}^2$ given by

$$d_h((z,v),(z',v')) = \sqrt{d^2_{\mathbb{H}^2}(z,z') + (v\hat{v}')^2} \qquad \forall (z,v),(z',v') \in T^1\mathbb{H}^2$$

[$v \in T_z\mathbb{H}^2$, $v' \in T_{z'}\mathbb{H}^2$, then $v\hat{v}'$ is the angle between v' and the vector $\bar{v}$ obtained by parallel transport from z to z', i.e. let γ be the geodesic through z and z', $z=\gamma(t_0)$, $z'=\gamma(t_1)$, then

$\bar{v} = d_z(\pi \circ \varphi_{(t_2 - t_0)})(v)$]



Prop Let $(x+i, i) \in W^+(i,i)$, then

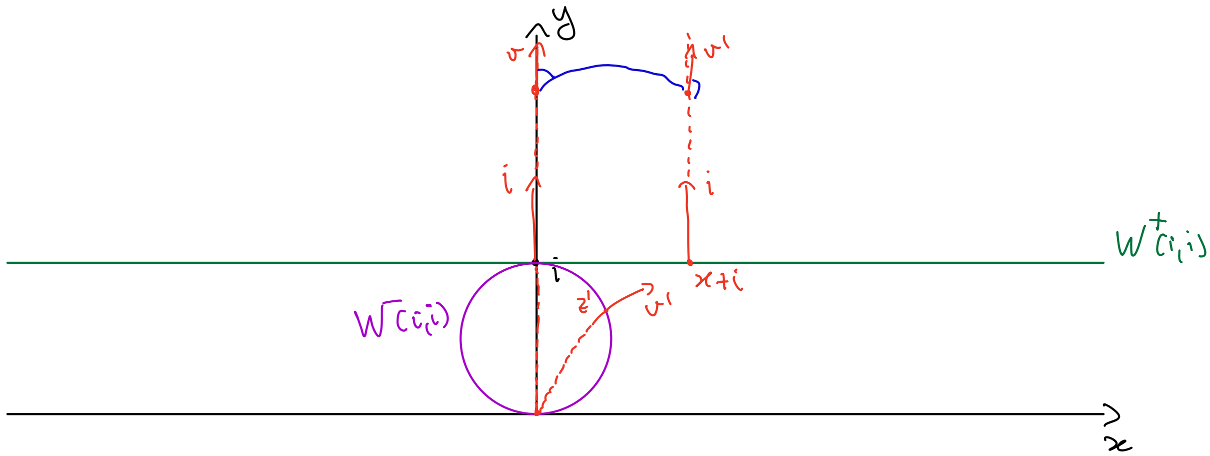
$$\lim_{t \to +\infty} e^{t}\, d_h(\varphi_t(i,i), \varphi_t(x+i,i)) = \sqrt{2}\,|x|.$$

$$d_h \underset{t \to +\infty}{\sim} e^{-t} \cdot \sqrt{2}\,|x|$$

Let $(z', v') \in W^-(i,i)$, then

$$\lim_{t \to -\infty} e^{-t}\, d_h(\varphi_t(i,i), \varphi_t(z',v')) = \sqrt{2} \left| \frac{Re(z')}{\Im(z')} \right|$$

$$d_h \underset{t \to -\infty}{\sim} e^{t} \cdot \sqrt{2} \left| \frac{Re(z')}{\Im(z')} \right|$$



Proof

· $W^+(i,i)$ $\qquad \varphi_t(i,i) = (ie^t, ie^t), \qquad \varphi_t(x+i,i) = (x+ie^t, ie^t)$

$$d(ie^t, x+ie^t) \sim e^{-t}|x| \sim \widehat{v v'} \quad \text{as } t \to +\infty$$

· $W^-(i,i)$, $t<0$ $\qquad d_h(\varphi_t(i,i), \varphi_t(z,v)) = d_h(\varphi_{-t}(i,-i), \varphi_{-t}(z,-v)) =$

$$(z,-v) \in W^+(i,-i) = d\psi_s(W^+(i,i))$$

$$= d_h(\underbrace{d\psi_s(\varphi_{-t}(i,-i))}_{\varphi_{-t}(i,i)}, d\psi_s(\varphi_{-t}(z,-v))) = d_h(\varphi_{-t}(i,i), \underbrace{d\psi_s(\varphi_{-t}(z,-v))}_{\in W^+(i,i)})$$

$$\lim_{t \to -\infty} e^{-t} d_h(\varphi_t(i,i), \varphi_t(z,v)) = \lim_{t \to -\infty} e^{-t} d_h(\varphi_{-t}(i,i), d\psi_s(\varphi_{-t}(z,-v)))$$

$$= \lim_{t \to +\infty} e^{t} d_h(\varphi_t(i,i), d\psi_s(\varphi_t(z,v))) = \sqrt{2}\,|\tilde{x}| = \sqrt{2} \left| \frac{Re(z)}{\Im(z)} \right|$$

$$W^+(i,i)$$
$$d\psi_S(\varphi_t(z,v)) = (\tilde{x} + ie^t, ie^t)$$

$(z,v) \in W^-(i,i) \Rightarrow x^2 + (y - \frac{1}{2})^2 = \frac{1}{4} \Leftrightarrow x^2 + y^2 = y$ $\Uparrow$

$z = x + iy$

$$d\psi_S(\varphi_t(z,v)) = (\tilde{x} + ie^t, ie^t) \text{ with } \tilde{x} + i = Sz = -\frac{1}{z} =$$
$$= -\frac{x - iy}{x^2 + y^2} = -\frac{x}{y} + i$$

$\square$

Prop $\forall (z,v) \in T^1\mathbb{H}^2$, the $T_{(z,v)}(T^1\mathbb{H}^2)$ can be written as the sum of three one-dimensional spaces

$$T_{(z,v)}(T^1\mathbb{H}^2) = E^c \oplus E^s \oplus E^u$$

where:

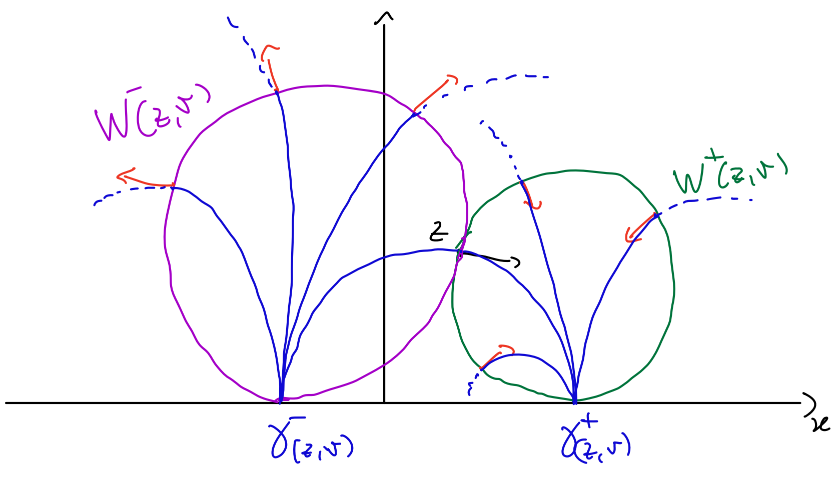
- $E^c = \mathrm{Span}\, X$, $E^s = \mathrm{Span}(H_+)$, $E^u = \mathrm{Span}(H_-)$

And if we move (z,v) along E^s we obtain (z',v') for which

$$d_h(\varphi_t(z,v), \varphi_t(z',v')) \sim e^{-t} \cdot \mathrm{const}(z,z') \text{ as } t \to +\infty$$

" " " E^u " " "

$$d_h(\varphi_t(z,v), \varphi_t(z',v')) \sim e^{t}\, \mathrm{const}(z,z') \text{ as } t \to -\infty.$$



Def Given $(z,v) \in T^1\mathbb{H}^2$, the positive horocycle flow $\{u_s^+(z,v)\}_{s\in\mathbb{R}}$ slides (z,v) along $W^+(z,v)$, and

$$g_{u_s^+(z,v)} = g_{(z,v)} \underbrace{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}}_{\exp(sH_+)}$$

Given $(z,v) \in T^1\mathbb{H}^2$, the negative horocycle flow $\{u_s^-(z,v)\}_{s\in\mathbb{R}}$ slides (z,v) along $W^-(z,v)$, and

$$g_{u_s^-(z,v)} = g_{(z,v)} \underbrace{\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}}_{\exp(sH_-)}$$