# The hyperbolic half-plane

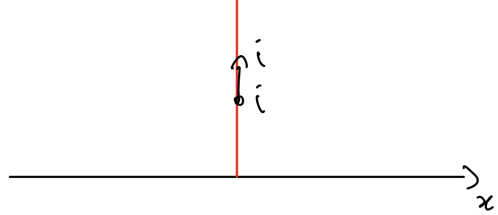
$\mathbb{H}^2 := \{ x+iy \,/\, y>0 \}$ , $T\mathbb{H}^2 \cong \mathbb{H}^2 \times \mathbb{C}$ , $K=-1$

$\forall \underset{x+iy}{\underset{\shortparallel}{z}} \in \mathbb{H}^2$ , $v, v' \in T_z\mathbb{H}^2$ $\quad \langle v, v' \rangle_z = \frac{1}{\Im(z)^2} \mathrm{Re}(v \bar{v}') = \frac{1}{y^2} \langle v, v' \rangle_{euclidean}$

- Find the geodesics on $\mathbb{H}^2$ , $\gamma_{(z,v)}$ , $(z,v) \in T^1\mathbb{H}^2$

<u>Lemma</u> $I = \{ x=0 \}$ is a geodesic with parameterization $\gamma_{(i,i)}(t) = ie^t$

$(i,i) \in T^1\mathbb{H}^2$




<u>proof</u> $\quad |\dot{\gamma}_{(i,i)}(t)|^2 = \langle ie^t, ie^t \rangle_{\gamma_{(i,i)}(t)} = \frac{1}{e^{2t}} \mathrm{Re}(ie^t \cdot \overline{ie^t}) = 1.$

$z_0 = iy_0$ , $z_1 = iy_1$ , $y_1 > y_0$ , $\alpha$ piecewise smooth curve $x(t) + iy(t)$ connecting $z_0$ and $z_1$ , $z_0 = \alpha(t_0)$, $z_1 = \alpha(t_1)$

$$\ell(\alpha) = \int_{t_0}^{t_1} |\dot{\alpha}(t)|\, dt = \int_{t_0}^{t_1} \sqrt{\frac{\dot{x}(t)^2 + \dot{y}(t)^2}{y(t)^2}}\, dt \geq \int_{t_0}^{t_1} \frac{\dot{y}(t)}{y(t)}\, dt = \log \frac{y_1}{y_0}$$

$\gamma_{(i,i)}(t) = ie^t$ , $\gamma(t_0) = z_0 \iff t_0 = \log y_0$ , $\gamma(t_1) = z_1 \iff t_1 = \log y_1$

$$\ell(\gamma|_{[\log y_0, \log y_1]}) = \int_{\log y_0}^{\log y_1} |\dot{\gamma}(t)|\, dt = \log \frac{y_1}{y_0} \qquad \square$$

<u>Def</u> $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_+(2,\mathbb{R})$ $(\det g > 0)$ , $M =$ <u>Möbius transformation</u> is the function

$$\mathbb{H}^2 \ni z \longmapsto \frac{az+b}{cz+d} \quad , \quad a,b,c,d \in \mathbb{R}, \; ad-bc>0$$

**Lemma** $\psi : GL_+(2,\mathbb{R}) \to \mathcal{M}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} = g \mapsto \psi_g : z \mapsto \frac{az+b}{cz+d}$ Then:

- $\mathcal{M}$ is a group under composition;
- $\psi$ is a group homomorphism $\left( \begin{matrix} \psi(gh) = \psi_g \circ \psi_h \quad \forall g, h \in GL_+(2,\mathbb{R}) \\ \psi(Id) = \text{identity} \end{matrix} \right)$
- $\psi(GL_+(2,\mathbb{R}))$ are isometries of $\mathbb{H}^2$;
- $\operatorname{Ker} \psi = \{ \lambda \cdot Id \,/\, \lambda \in \mathbb{R} \setminus \{0\} \}$.

**proof** - $\psi_g(z) = z \quad \forall z \in \mathbb{H}^2$. $\quad \frac{\lambda a z + \lambda b}{\lambda c z + d \lambda} = z \quad \forall z$, $\quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det g > 0$

- $\psi_g(z) \in \mathbb{H}^2 \quad \forall z \in \mathbb{H}^2 \iff \Im(\psi_g(z)) = \frac{\Im(z) \det g}{|cz+d|^2}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$v \in T_z\mathbb{H}^2$, $\quad |d_z\psi_g(v)|^2 = |\psi_g'(z) v|^2 = \langle \psi_g'(z) v, \psi_g'(z) v \rangle_{\psi_g(z)} =$

$$d_z\psi_g(v) = \psi_g'(z) v = \frac{\det g}{(cz+d)^2} v$$

$$= \frac{1}{\Im(\psi_g(z))^2} \operatorname{Re}\left( \psi_g'(z) v \cdot \overline{\psi_g'(z) v} \right) = \frac{1}{\Im(z)^2} \operatorname{Re}(v \bar{v}) = |v|^2 \qquad \square$$

**Remark** $\mathcal{M}$ Möbius transformation $\cong SL(2,\mathbb{R})/_{\pm I} = PSL(2,\mathbb{R})$

**Def** $SL(2,\mathbb{R})$ can be given the following classification:

<u>elliptic</u> $g \in SL(2,\mathbb{R})$ s.t. $|\operatorname{tr} g| < 2$ (one fixed point in $\mathbb{H}^2$)

<u>ex</u> $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\psi_S(z) = -\frac{1}{z}$

<u>hyperbolic</u> " $|\operatorname{tr} g| > 2$ (two fixed points in $\partial\mathbb{H}^2$)

<u>ex</u> $\lambda_a = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$, $\psi_{\lambda_a}(z) = a^2 z$

$a \in \mathbb{R} \setminus \{0, \pm 1\}$ $\qquad \psi_{\lambda_a}(I) = I$.

<u>parabolic</u> " $|\operatorname{tr} g| = 2$ (one fixed point in $\partial\mathbb{H}^2$)

**ex** $\tau_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, $b \in \mathbb{R}$, $\quad \psi_{\tau_b}(z) = z + b$

**Rem** all parabolic and hyperbolic elements are conjugate to $\lambda_a$ or $\tau_b$.

**Lemma** Let $C$ be either a vertical line in $\mathbb{H}^2$ or a semicircle with center on $\{y=0\}$. Then there exists $g \in SL(2,\mathbb{R})$ s.t. $\psi_g(I) = C$.
On the contrary, $\psi_g(I)$ is a vertical line or a semicircle with center on $\{y=0\}$ $\forall g \in SL(2,\mathbb{R})$.

**proof** Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ then

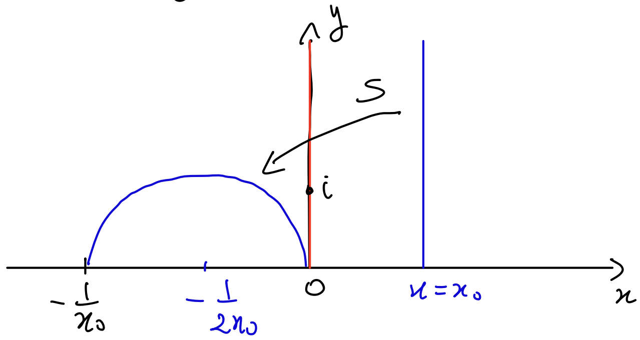
$$g = \begin{cases} \tau_{ba} \circ \lambda_a, & \text{if } c = 0 \\ \tau_{a/c} \circ S \circ \tau_{cd} \circ \lambda_c, & \text{if } c \neq 0. \end{cases}$$

$c=0$, $\quad \psi_g(I) = \psi_{\tau_{ba}}(\psi_{\lambda_a}(I)) = \psi_{\tau_{ba}}(I) = \{x = ba\}$ vertical line

$c \neq 0$, $\quad \psi_g(I) = \psi_{\tau_{a/c}} \psi_S(\{x = cd\})$, $\quad S(z) = -\frac{1}{z} \quad (S^2 = Id)$

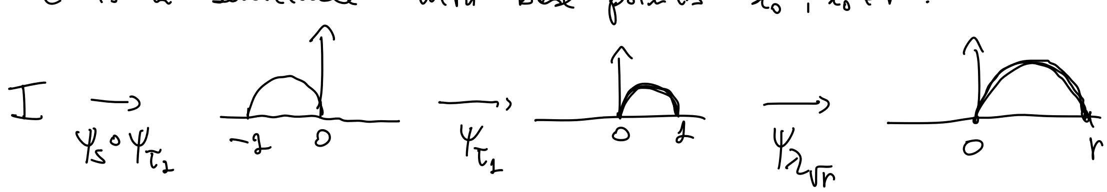
$S(i) = i$, $\; S(I) = I$

$\{S(x_0 + iy)\}_{y \in (0,+\infty)}$




- $C = \{x = x_0\}$ then $\psi_{\tau_{x_0}}(I) = C$.

$C$ is a semicircle with base points $x_0$, $x_0 + r$.



$$\xrightarrow[\psi_{\tau_{x_0}}]{} C$$

$\square$

<u>Prop</u> The geodesics of $\mathbb{H}^2$ are all and only the vertical lines $\{x = x_0\}$ and the semicircles with center on $\{y=0\}$.

- $\forall (z,v) \in T^1\mathbb{H}^2$, the geodesic $\gamma_{(z,v)}(t)$ is $\qquad z = x+iy$
  - a vertical line, if $v = \pm i \cdot \Im(z)$ $\quad [\gamma^+_{(z,iy)} = \infty,\ \gamma^-_{(z,iy)} = x]$
  - semicircle with base points $\gamma^{\pm}_{(z,v)} = \lim_{t\to\pm\infty} \gamma_{(z,v)}(t) \in \{y=0\}$, otherwise

  Then there are no closed geodesics on $\mathbb{H}^2$ and all geodesics are unbounded.

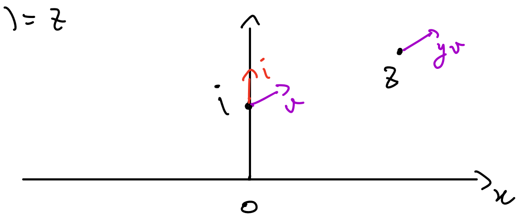
  On quotients $\Gamma / \mathbb{H}^2$ with $\Gamma <$ (isometries of $\mathbb{H}^2$) we can have closed geodesics and dense geodesics on $T^1\mathbb{H}^2$.

<u>Prop</u> $\forall g \in SL(2,\mathbb{R})$, $d\psi_g : T^1\mathbb{H}^2 \to T^1\mathbb{H}^2$, $(z,v) \xmapsto{\Psi_g} (\psi_g(z), d_z\psi_g(z))$

This action $SL(2,\mathbb{R}) \mapsto \Psi_g$ is simply transitive. In particular $PSL(2,\mathbb{R}) \cong T^1\mathbb{H}^2$.

<u>proof</u> $\cdot$ $\forall (z,v) \in T^1\mathbb{H}^2$ $\exists g \in SL(2,\mathbb{R})$ s.t. $\Psi_g(i,i) = (z,v)$

$\forall z \in \mathbb{H}^2$ $\exists g_0$ s.t. $\psi_{g_0}(i) = z$




$z = x+iy$

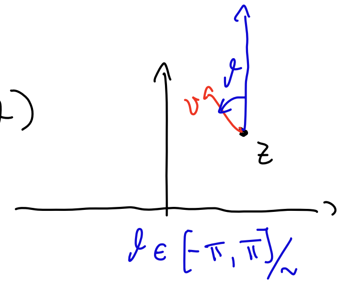
$$g_0 = \tau_x \lambda_{\sqrt{y}} = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$$

$$d\psi_{g_0}(i,v) = (z, yv) \quad , \quad \psi_{g_0}(i)' v = \frac{1}{(ci+d)^2} v = yv$$

$\cdot$ $\rho_\vartheta = \begin{pmatrix} \cos\frac{\vartheta}{2} & \sin\frac{\vartheta}{2} \\ -\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \in SL(2,\mathbb{R}) \quad \forall \vartheta \in \mathbb{R} \quad [S = \rho_{-\pi}]$

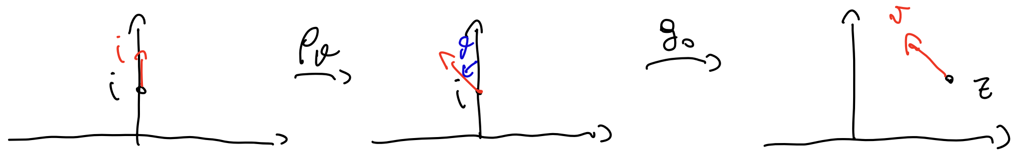
$$d\psi_{\rho_\vartheta}(i,i) = \left(i,\ i(\cos\vartheta + i\sin\vartheta)\right), \quad \psi_{\rho_\vartheta}(i)'\, i = \frac{1}{\left(-\sin\frac{\vartheta}{2}\, i + \cos\frac{\vartheta}{2}\right)^2}\, i = i(\cos\vartheta + i\sin\vartheta)$$

- $v \in T^1_z \mathbb{H}^2$, $\quad z = x+iy$, $\quad v = iy(\cos\vartheta + i\sin\vartheta)$




- Given $(z,v) = \left(x+iy,\ iy(\cos\vartheta + i\sin\vartheta)\right)$ then

$$d\psi_g(i,i) = (z,v) \quad \text{for} \quad g = g_0 \cdot \rho_\vartheta$$




$\square$

<u>Rem</u> $\quad T^1\mathbb{H}^2 \ni (z,v) \longmapsto g_0\rho_\vartheta \in SL(2,\mathbb{R})$

$$SL(2,\mathbb{R}) \ni \underset{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\overset{}{g}} \longmapsto d\psi_g(i,i) = (z,v) = \left(\frac{ai+b}{ci+d},\ i\,\frac{1}{c^2+d^2}(\cos\vartheta + i\sin\vartheta)\right)$$

$$\Im(z) = \frac{1}{c^2+d^2}$$

$$\vartheta = \begin{cases} -2\arctan\frac{c}{d}, & d \neq 0 \\ \pi \pmod{2\pi}, & d = 0 \end{cases}$$