Σ surfaces, smooth connected Riemannian, $\langle \cdot , \cdot \rangle$

$z \in \Sigma$ points, $v \in T_z \Sigma$, tangent bundle $T\Sigma$

unit tangent bundle $T^1\Sigma = \{ (z,v) / z \in \Sigma, v \in T_z\Sigma, |v| = 1 \}$

$\mathfrak{X}(\Sigma)$ = vector fields

$D_X Y$

Def $D : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is an affine connection if: - it is bilinear; - $\forall f \in C^\infty(\Sigma)$, $D_{fX} Y = f D_X Y$;

- $D_X (fY) = f D_X Y + X(f) Y$

We say that D is the Levi-Civita connection if

$$Z(\langle X, Y \rangle) = \langle D_Z X, Y \rangle + \langle X, D_Z Y \rangle$$

$$D_X Y - D_Y X = [X, Y]$$

Def Riemann curvature tensor $R : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathrm{Hom}(\mathfrak{X}(\Sigma))$

$$R(X,Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z$$

Gaussian curvature

$$K := \frac{\langle X, R(X,Y)Y \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}$$

Def $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ smooth curve is a geodesic for $(z,v) \in T\Sigma$ if $\gamma(0) = z$, $\dot\gamma(0) = v$, $D_{\dot\gamma} \dot\gamma = 0$ $\forall t \in (-\varepsilon, \varepsilon)$

$\gamma_{(z,v)}(t)$ = geodesic for (z,v).

Prop $\forall (z,v) \in T\Sigma$, $v \neq 0$, $\exists!$ $\gamma_{(z,v)}(t)$ on $t \in (-\varepsilon, \varepsilon)$.

Def Σ is geodesically complete if $\forall (z,v) \in T\Sigma$, $v \neq 0$, $\gamma_{(z,v)}(t)$ is defined $\forall t \in \mathbb{R}$.

Teo (Hopf – Rinow) Let $\ell(\alpha) := \int_{-\varepsilon}^{\varepsilon} |\dot\alpha(t)|\, dt$ denote the length of a piecewise smooth curve $\alpha : (-\varepsilon, \varepsilon) \to \Sigma$, and let $\forall z, z' \in \Sigma$

$$d(z, z') := \inf \{ \ell(\alpha) \,/\, \alpha \text{ piecew. smooth curve connecting } z \text{ and } z' \}.$$

Then Σ is geodesically complete if and only if (Σ, d) is a complete metric space.

Prop $\forall z \in \Sigma \; \exists\, U(z)$ neigh. s.t. $\forall\, w, w' \in U(z) \; \exists!\, \gamma$ geodesic connecting w and w' for which $\ell(\gamma) \le \ell(\alpha)$ for any piecewise smooth α connecting w and w', and $\ell(\gamma) = \ell(\alpha)$ implies that γ and α have the same image.

Prop Let $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ be a geodesic for (z, v) with $v \neq 0$, then $|\dot\gamma(t)| = (\langle \dot\gamma(t), \dot\gamma(t) \rangle)^{1/2}$ is constant $\forall\, t \in (-\varepsilon, \varepsilon)$.

proof Let $s(t) := |\dot\gamma(t)|$ and $\varepsilon' > 0$ s.t. $s(t) > 0$ on $(-\varepsilon', \varepsilon')$.
Then $X := \dot\gamma / s$, $\langle X, X \rangle = 1$.

$$0 = X(\langle X, X \rangle) = 2 \langle D_X X, X \rangle$$

$$0 = D_{\dot\gamma} \dot\gamma = D_{(sX)}(sX) = s D_X (sX) = s^2 D_X X + s X(s) X$$

$$\Rightarrow D_X X = 0, \quad X(s) = 0$$

$\square$

Def Given $T^1\Sigma$, the geodesic flow on Σ, $\Phi = (\varphi_t)_{t \in \mathbb{R}}$, $\varphi_t : T^1\Sigma \to T^1\Sigma$, is defined by

$$T^1\Sigma \ni (z, v) \longmapsto \varphi_t(z, v) = \left(\gamma_{(z,v)}(t), \dot\gamma_{(z,v)}(t) \right) \in T^1\Sigma$$

In particular φ_t are smooth $\forall\, t$.

Constant curvature cases

<u>Teo</u> (Killing - Hopf) Σ simply connected s.c.R. surface with constant Gaussian curvature K. Then: If $K=0$, then Σ is isometric to $\mathbb{R}^2$ with flat metric; If $K>0$, then Σ is isometric to S^2 with constant positive curvature; If $K<0$, then Σ is isometric to $\mathbb{H}^2$, hyperbolic Poincaré half-plane.

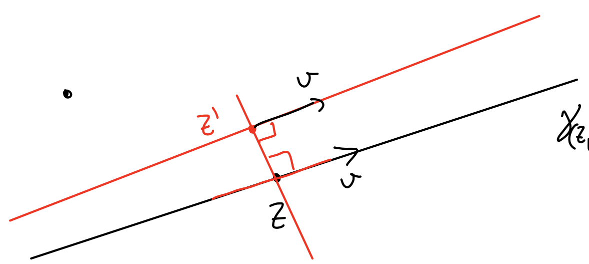
If Σ is not simply connected, then Σ is isometric to a quotient of $\mathbb{R}^2$, S^2, $\mathbb{H}^2$ wrt a group Γ which acts freely and properly discontinuously ($\forall z \ \exists U(z)$ s.t. $gU \cap U = \emptyset \ \ \forall g \in \Gamma \setminus \{e\}$).

<u>Flat $\mathbb{R}^2$</u> If $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ smooth curve then $D_{\dot\gamma}\dot\gamma = \ddot\gamma = 0$

$$\gamma_{(z,v)}(t) = z + vt \qquad \forall\ (z,v) \in T^1\mathbb{R}^2$$

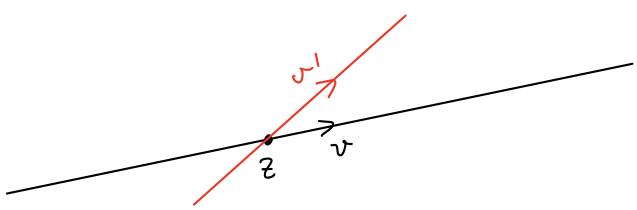
- No closed geodesics on $\mathbb{R}^2$, all geodesics are unbounded on $\mathbb{R}^2$.

On $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ we find closed geodesics. Are there dense geodesics on $\mathbb{T}^2$? Yes on $\mathbb{T}^2$, no on $T^1\mathbb{T}^2$.

- 

$T_{(z,v)}(T^1\mathbb{R}^2)$

$$d_{\mathbb{R}^2}(\gamma_{(z,v)}(t), \gamma_{(z',v)}(t)) = d_{\mathbb{R}^2}(z, z')$$
$$\parallel$$
$$d_{T^1\mathbb{R}^2}(\varphi_t(z,v), \varphi_t(z',v)) \qquad \varphi_t \text{ isometry } \forall t.$$



$$d_{T^1\mathbb{R}^2}(\varphi_t(z,v), \varphi_t(z,v'))$$
$$\parallel$$
$$\cos_{\widehat{vv'}} t + 2t \sin\vartheta \quad , \quad \widehat{vv'} = 2\vartheta$$

Sphere S^2 $\quad S^2 = \{ x^2 + y^2 + w^2 = 1 \}$

Prop The geodesics on S^2 are all and only the great circles parameterised with constant velocity.

proof The great circle $\Gamma = S^2 \cap \{w = 0\}$ with $\gamma(t) = (\cos t, \sin t, 0)$ $t \in [0, 2\pi]$ is a geodesic.

Since $\dot\gamma(t) = (-\sin t, \cos t, 0)$, $\ddot\gamma(t) = (-\cos t, -\sin t, 0) \in \{w=0\}$

$\dot\gamma \perp \ddot\gamma$, $\Rightarrow D_{\dot\gamma}\dot\gamma = 0$.

All great circles are obtained from Γ by an isometry, hence they are geodesics.

Given $z, z' \in S^2$, there exists a great circle connecting z and z', hence the geodesic connecting z and z' is a great circle. $\square$

- All geodesics are closed.
- $\forall (z, v) \in T^1 S^2$ any variation leads to another great circle