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Reduction on characteristics for continuous solutions of a scalar balance law

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ABSTRACT. We consider continuous solutions u to the balance equation

$$\partial_t u(t, x) + \partial_x [f(u(t, x))] = g(t, x),$$

where f is of class C^2 and the source term g is bounded. Continuity improves to Hölder continuity when f is *uniformly* convex, but it is not more regular in general. We discuss the reduction to ODEs on characteristics, mainly based on the joint works [5, 1]. We provide here local Lipschitz regularity results holding in the region where $f'(u)f''(u) \neq 0$ and only in the simpler case of autonomous sources $g = g(x)$, but for solutions $u(t, x)$ which may depend on time. This corresponds to a local Lipschitz regularity result, in that region, for the system of ODEs

$$\begin{cases} \dot{\gamma}(t) = f'(u(t, \gamma(t))), \\ \frac{d}{dt} u(t, \gamma(t)) = g(\gamma(t)). \end{cases}$$

KEYWORDS: balance laws, method of characteristics, Lagrangian formulation, ordinary differential equations, Peano phenomenon.

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1. INTRODUCTION

In the context of classical solutions, the balance law

$$\partial_t u(t, x) + \partial_x [f(u(t, x))] = g(t, x) \tag{1.1}$$

with $f \in C^2(\mathbb{R})$ can be reduced to ordinary differential equations along characteristic curves, defined as those curves $t \mapsto (t, \gamma(t))$ satisfying $\dot{\gamma}(t) = f'(u(t, \gamma(t)))$. Indeed,

$$\begin{aligned} g(t, \gamma(t)) &= \partial_t u(t, \gamma(t)) + \partial_x [f(u(t, \gamma(t)))] \\ &= \partial_t u(t, \gamma(t)) + f'(u(t, \gamma(t))) \partial_x u(t, \gamma(t)) \\ &= \partial_t u(t, \gamma(t)) + \dot{\gamma}(t) \partial_x u(t, \gamma(t)) \\ &= \frac{d}{dt} u(t, \gamma(t)). \end{aligned}$$

This more generally allows a parallel between the Cauchy problem for a scalar quasi-linear first order PDE and for a system of ODEs, which is known as the method of characteristics (see for instance [10], where it is also provided an application to determine local existence).

If one interprets $f'(u)$ as a velocity, this is just the change of variable from the Eulerian (PDE) to the Lagrangian (ODEs) formulation.

We discuss here what remains of this equivalence when u is just continuous and g is bounded. We prove then in Section 2 that when g depends only on x , but not on the time t , then $u(t, x)$ is locally Lipschitz continuous on the open set where $f'(u)f''(u)$ is nonvanishing. This is sensibly better than the general case, where u is only Hölder continuous. It is based on proving the corresponding result for the system of ODEs. As we are discussing local issues, we will fix for simplicity the domain \mathbb{R}^2 and we will assume u bounded.

A motivation for a different setting

The development of Geometric Measure Theory in the context of the sub-Riemannian Heisenberg group \mathbb{H}^n brought the attention to *continuous* solutions to the equation

$$\partial_t u(t, x) + \partial_x \left[\frac{u^2(t, x)}{2} \right] = g(t, x). \quad (1.2)$$

Continuity is natural from the fact that u parametrizes a surface. As one studies surfaces that have differentiability properties in the intrinsic structure of the Heisenberg group, but not in the Euclidean structure, then it is not natural assuming more regularity of u than continuity [13], which for bounded sources improves to 1/2-Hölder continuity [4, 5]. Notice that with u continuous the second term of the equation cannot even be rewritten as $u\partial_x u$, because $\partial_x u$ is only a distribution and u is not a suitable test function.

The PDE arises if one wants to show the equivalence between a pointwise, metric notion of differentiability and a distributional one: for $n = 1$ the distributional definition is precisely (1.2), while for $n > 1$ it is a related multi- D system of PDEs. The correspondence was introduced first in [3, 4] for intrinsic regular hypersurfaces, which are the analogue of what are C^1 -hypersurfaces in the Euclidean setting. It was extended in [5, 7] when considering intrinsic Lipschitz hypersurfaces, analogue of Lipschitz hypersurfaces in the Euclidean setting. The source term g , in \mathbb{H}^1 , turns out to be what is called the intrinsic gradient of u , which is the counterpart of the gradient in Euclidean geometry; in \mathbb{H}^n it is one if its components: u locally parametrizes an intrinsic regular hypersurface if and only if (1.2) holds locally with g continuous; it parametrizes an intrinsic regular hypersurface if and only if (1.2) holds locally with g bounded. As the notion of differentiability they provide in the intrinsic structure of \mathbb{H}^n is closer to the Lagrangian formulation, the equivalence between Lagrangian and Eulerian formulation arises as intermediate step of this characterization.

When considering intrinsic Lipschitz hypersurfaces the fact that g is only bounded gives rise to new subtleties. In particular, one already knows by an intrinsic Rademacher theorem [11] that the intrinsic differential exists and it is unique \mathcal{L}^2 -a.e. However, for the ODE formulation this is not enough: as one needs to restrict this L^∞ function on curves, a precise representative is needed also at points where u is not intrinsically differentiable. Viceversa, if one chooses badly the representative of the source of the ODE formulation a priori it differs on a positive measure set from the source of the ODE. There is however a canonical choice for defining the two sources, which makes the formulations equivalent when the inflection points of f are negligible.

Summary of the equivalence

When u is Lipschitz, the ODE

$$\begin{cases} \dot{\chi}(t, x) = f'(u(t, \chi(t, x))) \\ \chi(0, x) = x \end{cases}$$

with $x \in \mathbb{R}$ and f of class C^2 , provides a local diffeomorphism by the classical theory on ODE. If u is instead continuous, Peano's theorem ensures local existence of solutions, but more characteristics may start at one point and characteristics from different points may collapse (see in [5] the classical example of the square-root). This makes clearly impossible to have a local diffeomorphism, or even having a Lagrangian flow in the sense of Ambrosio-DiPerna-Lions [9, 2]. A recent result about this can be obtained for u not depending on time [6], but it is clearly not our assumption. Dropping out injectivity, it is however possible to construct a continuous change of variables with bounded variation.

Let u be a continuous, bounded function.

Lemma 1.1. *There exists a continuous function $\chi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ such that*

- $\tau \mapsto \chi(t, \tau)$ is nondecreasing for every t and surjective;
- $\partial_t \chi(t, \tau) = f'(u(t, \tau))$.

We call it Lagrangian parameterization. This function is not unique.

See [1, 5] for the proof. See also [12] for a similar change of variable, for a 1D-system. In general one cannot have that χ is SBV [1].

Consider now u continuous distributional solution to (1.1) with g bounded.

Lemma 1.2. *Assume that $\mathcal{L}^1(\text{clos}(\{\text{inflection points of } f\})) = 0$. Then u is Lipschitz continuous along every characteristic curve.*

The proof follows a computation by Dafermos [8]. For general fluxes, there are cases when u is not Lipschitz along some Lagrangian parameterization [1]. The counterexample holds also for continuous autonomous sources $g(t, x) = g_0(x)$. What we find more striking is the following.

Theorem 1.3. *Assume that $\mathcal{L}^1(\text{clos}(\{\text{inflection points of } f\})) = 0$. Then there exists a pointwise defined function $\hat{g}(t, x)$*

$$\frac{d}{dt}u(t, \gamma(t)) = \hat{g}(t, \gamma(t)) \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ for every characteristic curve } \gamma.$$

The proof is based on a selection theorem as a technical device, but \hat{g} is essentially uniquely defined as the derivative of u along some characteristic.

Remark 1.4. There is a substantial difference between the uniformly convex and the strictly convex cases: in the former at almost every (t, x) there exists a unique value for $\frac{d}{dt}u(t, \gamma(t))$, $\gamma(t) = x$, and it does not depend on which characteristic $\gamma(s)$ one has chosen. That value is the most natural choice of \hat{g} at those points, and this a.e. defined function \hat{g} identifies the same distribution as the source term g . Without uniform convexity $\frac{d}{dt}u(t, \gamma(t))$ may not exist on a set of positive \mathcal{L}^2 -measure, independently of which characteristic γ one chooses through the point. The correspondence between distributional and Lagrangian sources gets more complicated with non-convexity.

The converse also holds. We give here a weaker statement without the negligibility condition on the inflection points. As mentioned identifying sources is delicate, we refer for it to the more extensive work [1].

Theorem 1.5. *Assume that a continuous function u has a Lagrangian parameterization χ for which there exists a bounded function \tilde{g} s.t. it satisfies*

$$\frac{d}{dt}u(t, \chi(t, \tau)) = \tilde{g}(t, \chi(t, \tau)) \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ for every } \tau \in \mathbb{R}. \quad (1.3)$$

Then there exists a function $g(t, x)$ s.t. (1.1) holds.

Viceversa, if (1.1) holds then there exists a Lagrangian parameterization χ and function \tilde{g} s.t. (1.3) holds.

We finally mention that continuous distributional solutions to this simple equation do not dissipate entropy.

Theorem 1.6. *Let u be a continuous distributional solution to (1.1) with bounded source g . Then for every smooth function η and q satisfying $q' = \eta' f'$*

$$\partial_t [\eta(u(t, x))] + \partial_x [q(u(t, x))] = \eta'(u(t, x))g(t, x).$$

2. SOME LOCAL REGULARITY WITH AUTONOMOUS SOURCES

We mention a local regularity result holding in the case of autonomous sources: the continuous function $u(t, x)$ is locally Lipschitz continuous in the (open) complementary of the 0-level set of the product $f'(u) f''(u)$. For $f(u) = u^2/2$, this means $u \neq 0$. When the source is not autonomous, then this fails to be true, indeed characteristics may bifurcate also at points where u is not vanishing.

We remind [1] that when f has inflection points of positive measure, then a priori u may not be Lipschitz along some characteristics, even with $g = g(x)$.

Lemma 2.1. *There may be locally multiple solutions to the ordinary differential equation*

$$\begin{cases} \dot{\gamma}(t) = u(t, \gamma(t)), \\ \dot{\gamma}(t) = g(\gamma(t)), \\ \gamma(\bar{t}) = \bar{x}, \end{cases}$$

with $u(t, x)$ continuous, $g(x)$ bounded, only if $u(\bar{t}, \bar{x}) = 0$ but it does not identically vanish in a whole neighborhood.

Remark 2.2. We are not stating existence. The lemma is however still not obvious because we do not have differentiability properties of u , which follow a posteriori by the next corollary in the region where u does not vanish. As a consequence, we do not have now the differentiability of the map $\gamma(t)$ w.r.t. the initial data of the ODE. The lemma asserts indeed the continuity in this variable in that region, provided it exists. We remind that when g depends on t bifurcations may easily occur also if $u \neq 0$.

Proof. We just prove that if u does not vanish at some point (\bar{t}, \bar{x}) , at that point there is at most one solution of the ODE, as an effect of the autonomous source. The reason is that if $u(\bar{t}, \bar{x})$ does not vanish, then any Lipschitz characteristic $x = \gamma(t)$, with $\bar{x} = \gamma(\bar{t})$, is a diffeomorphism in some neighborhood of (\bar{t}, \bar{x}) , and we can invert it. This allows to have the space variable as a parameter: the characteristic can be expressed

as $t = \theta(x)$. However, the second order relation $\ddot{\gamma}(t) = g(\gamma(t))$, once expressed in the x variable, can be integrated determining the function θ .

By elementary arguments, it suffices to show that there exists (locally) only one characteristic passing through $(\bar{t}, \bar{x}) = (0, 0)$ with slope $u(0, 0) = 1$. Focus the attention on a neighborhood U of the origin where u is bigger than some $\varepsilon > 0$. Let $x = \gamma(t)$ be any Lipschitz continuous solution of the ODE. Since $\dot{\gamma}(0) = u(0, 0) > 0$, by the inverse function theorem there exists $\delta > 0$ and a function

$$\theta : (\gamma(-\delta), \gamma(\delta)) \rightarrow (-\delta, \delta) \quad \text{s.t.} \quad \theta(\gamma(t)) = t, \quad \gamma(\theta(x)) = x.$$

Moreover, it is continuously differentiable with derivative

$$\dot{\theta}(x) = \frac{1}{\dot{\gamma}(\theta(x))} = \frac{1}{u(\theta(x), x)} \in \left[\frac{1}{\max |u|}, \frac{1}{\varepsilon} \right]. \quad (2.1)$$

From the Lipschitz continuity of $u(t, \gamma(t))$ and the fact that γ is a local diffeomorphism with inverse θ we deduce that the composite function $u(\theta(x), x)$ is Lipschitz continuous. At points $X \subset U$ of differentiability by the classical chain rule

$$\begin{aligned} \lim_{h \downarrow 0} \frac{\dot{\gamma}(\theta(x+h)) - \dot{\gamma}(\theta(x))}{h} &= \\ &= \frac{\dot{\gamma}(\theta(x+h)) - \dot{\gamma}(\theta(x))}{\theta(x+h) - \theta(x)} \cdot \frac{\theta(x+h) - \theta(x)}{h} = \ddot{\gamma}(\theta(x)) \dot{\theta}(x), \end{aligned}$$

and by (2.1) we have that $\dot{\theta}$ is differentiable at $x \in X$ with derivative

$$\ddot{\theta}(x) = -\frac{\ddot{\gamma}(\theta(x)) \dot{\theta}(x)}{[\dot{\gamma}(\theta(x))]^2} = -\frac{g(\theta(x))}{u^3(\theta(x), x)} \Leftrightarrow -\frac{\ddot{\theta}(x)}{[\dot{\theta}(x)]^3} = g(x).$$

For those $x \in X$, the differential equation may be rewritten as

$$\frac{d}{dx} \left[\frac{1}{2[\dot{\theta}(x)]^2} \right] = g(x) \quad \Leftrightarrow \quad \frac{d}{dx} \frac{u^2(\theta(x), x)}{2} = g(x).$$

The explicit ODE for $\theta(x)$, with initial data $\theta(0) = 0$, $[\dot{\theta}(0)]^{-1} = u(0, 0) = 1$, is easily solved locally by

$$u^2(\theta(x), x) = \frac{1}{\dot{\theta}^2(x)} = 1 + 2 \int_0^x g(z) dz. \quad (2.2)$$

This shows that the slope of every characteristic through the origin, which is a local diffeomorphism, is fixed at each x independently of the characteristic we have chosen: therefore there can be only one characteristic, precisely (in the space parameterization)

$$\theta(x; \bar{t}, \bar{x}) = \bar{t} + \int_{\bar{x}}^x \frac{1}{\sqrt{u^2(\bar{t}, \bar{x}) + 2 \int_{\bar{x}}^w g(z) dz}} dw. \quad (2.3)$$

Notice finally that if u vanishes in a neighborhood, being $\dot{\gamma}(t) \equiv 0$ there characteristics must be vertical (in that region of the (x, t) -plane). \square

Lemma 2.3. *Under the hypothesis of Lemma 2.1, if $g(x)$ is continuous it should also vanish at points where there are more characteristics, but it must not identically vanish in a neighborhood.*

Proof. We show that not only u , but also g must vanish. The argument shows that when two characteristics meet and have both second derivative with the same value, this value must be 0. For simplifying notations, consider two characteristics $\gamma_1(t) \leq \gamma_2(t)$ for arbitrarily small $t > 0$ with $\gamma_1(0) = \gamma_2(0) = 0$. If $\gamma_1(t_k)\gamma_2(t_k) \leq 0$ for $t_k \downarrow 0$, then

$$0 \leq \ddot{\gamma}_2(0) = g(0) = \ddot{\gamma}_1(0) \leq 0,$$

thus g vanishes. If instead e.g. $g > 0$ near the origin, having excluded the above case there exists $\delta > 0$ such that $0 < \gamma_1(t) \leq \gamma_2(t)$ for $t \in [0, \delta]$. Then (2.2) implies that the two curves coincide: having $\dot{\gamma}_1(t_k) = 0$ or $\dot{\gamma}_2(t_k) = 0$ for a sequence $|t_k| \downarrow 0$ would contradict the positivity of g , therefore for small $t > 0$ necessarily $\dot{\gamma}_1(t) > 0$, $\dot{\gamma}_2(t) > 0$ and therefore

$$\begin{aligned} u^2(\gamma_1^{-1}(x), x) + 2 \int_x^0 g(z) dz &= \dot{\gamma}_1^2(0) \\ &= 0 = \dot{\gamma}_2^2(0) = u^2(\gamma_2^{-1}(x), x) + 2 \int_x^0 g(z) dz. \end{aligned}$$

Being $\dot{\gamma}_i(t) = u(t, \gamma_i(t))$, $i = 1, 2$, by the differential relation, this shows that $\dot{\gamma}_1(t) \equiv \dot{\gamma}_2(t)$ for small times. This implies that the two curves coincide.

Finally, suppose g vanishes in a neighborhood. Then, as $\ddot{\gamma}(t) = 0$ in that neighborhood, characteristics are straight lines. As by the continuity of u characteristics may only intersect with the same derivative, they must be parallel lines and therefore bifurcation of characteristics does not occur. \square

We now show that in case u does not vanish, in the above lemma much more regularity holds.

Lemma 2.4. *If for every $(\bar{t}, \bar{x}) \in \Omega$ open in \mathbb{R}^2 there exists a curve γ s.t.*

$$\begin{cases} \dot{\gamma}(t) = u(t, \gamma(t)), \\ \ddot{\gamma}(t) = g(\gamma(t)), \\ \gamma(\bar{t}) = \bar{x}, \end{cases}$$

with $u(t, x)$ continuous, $g(x)$ bounded, then $u(t, x)$ is locally Lipschitz in the open set $\{(t, x) : u(t, x) \neq 0\} \subset \Omega$.

Corollary 2.5. *If u is not locally Lipschitz where nonvanishing then the system in Lemma 2.1 cannot have solutions through each point of the plane. In particular, u cannot be a continuous solution to*

$$\partial_t u(t, x) + \partial_x [f(u(t, x))] = g(x).$$

Proof. By Lemma 2.1 there is a unique characteristic starting at each point $(\bar{t}, \bar{x}) \in \Omega = \{(t, x) : u(t, x) \neq 0\}$, which is given by (2.3). We start comparing the value of u at two points $(0, 0)$, $(-t, 0)$, $t > 0$, in a ball B compactly contained in Ω . In particular, there exists $\delta(B)$ s.t. the two characteristics starting from the points we have chosen do not intersect if $0 < x < \delta(B)$, as there u does not vanish. For such small x one has by (2.3)

$$\int_0^x \frac{1}{\sqrt{\lambda_1^2 + 2 \int_0^w g(z) dz}} dw > -t + \int_0^x \frac{1}{\sqrt{\lambda_2^2 + 2 \int_0^w g(z) dz}} dw, \quad (2.4)$$

where we defined $\lambda_1 = u(0, 0)$ and $\lambda_2 = u(-t, 0)$. Equivalently

$$t > \int_0^x \frac{1}{\sqrt{\lambda_2^2 + 2 \int_0^w g(z) dz}} - \frac{1}{\sqrt{\lambda_1^2 + 2 \int_0^w g(z) dz}} dw.$$

Suppose $\lambda_1 > \lambda_2$. By the convexity of $r \mapsto 1/\sqrt{r}$, the right-hand side is larger than

$$\begin{aligned} & \int_0^x \left[\frac{d}{dr} \left(\frac{1}{\sqrt{r}} \right) \Big|_{r=\lambda_1^2 + 2 \int_0^w g(z) dz} (\lambda_2^2 - \lambda_1^2) \right] dw = \\ & = \left[\frac{\lambda_2 + \lambda_1}{-2} \int_0^x \frac{1}{(\lambda_1^2 + 2 \int_0^w g(z) dz)^{3/2}} dw \right] (\lambda_2 - \lambda_1) \\ & \geq \left[\frac{\lambda_2 + \lambda_1}{2(\lambda_1^2 + 2Gx)^{3/2}} x \right] (\lambda_1 - \lambda_2). \end{aligned}$$

The argument within square brackets in the last line is uniformly continuous and as $t \downarrow 0$ it is larger than x/λ_1^2 . As the inequalities hold for every positive t , $x < \delta = \delta(B)$, the non-intersecting condition (2.4) gives

$$t > \left(\frac{\lambda_1^2}{\delta} + \varepsilon \right)^{-1} (\lambda_1 - \lambda_2) \quad \Rightarrow \quad u(0, 0) - u(t, 0) = \lambda_1 - \lambda_2 \leq \left(\frac{\lambda_1^2}{\delta} + \varepsilon \right) t,$$

which is half the Lipschitz inequality at the points $(0, 0)$, $(-t, 0)$. The other half, for $\lambda_1 < \lambda_2$ is similarly obtained considering small negative x .

For comparing two generic close points (t, x) and $(0, 0)$, by the finite speed of propagation one can combine the Lipschitz regularity along characteristics and the Lipschitz regularity along vertical lines. \square

Corollary 2.6. *Let $u(t, x)$ be a continuous solution to the balance equation*

$$\partial_t u(t, x) + \partial_x [f(u(t, x))] = g(x), \quad g \in L^\infty(\mathbb{R}).$$

Then the function $u(t, x)$ is locally Lipschitz in the open set

$$\{(t, x) : f'(u(t, x)) \cdot f''(u(t, x)) \neq 0\}.$$

Proof. We first consider the case of quadratic flux $f(u) = u^2/2$. By Theorem 1.3, there exists a function $\hat{g}(t, x)$ such that we can apply Lemma 2.1, which gives the thesis. If $g \in L^\infty$ they may a priori differ on an \mathcal{L}^2 -negligible set, but one can prove that $\hat{g}(t, x) = \hat{g}(x)$.

Being u an entropy solution by Theorem 1.6, $f'(u)$ solves the equation

$$[f'(u)]_t + \left[\frac{f'(u)^2}{2} \right]_x = f''(u)g.$$

By the previous case then $f'(u)$ is Lipschitz in the open set where it does not vanish. If moreover $f''(u)$ does not vanish, then the regularity of u can be proved just by inverting f' . \square

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