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Reduction on characteristics for continuous solutions of a scalar balance law

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Abstract. We consider continuous solutions u to the balance equation

$$\partial_t u(t,x) + \partial_x [f(u(t,x))] = g(t,x)\,,$$

where f is of class C^2 and the source term g is bounded. Continuity improves to Hölder continuity when f is *uniformly* convex, but it is not more regular in general. We discuss the reduction to ODEs on characteristics, mainly based on the joint works [5, 1]. We provide here local Lipschitz regularity results holding in the region where $f'(u)f''(u) \neq 0$ and only in the simpler case of autonomous sources $g = g(x)$, but for solutions $u(t,x)$ which may depend on time. This corresponds to a local Lipschitz regularity result, in that region, for the system of ODEs

$$\begin{cases} \dot\gamma(t) = f'(u(t,\gamma(t)))\,, \\ \frac{d}{dt}u(t,\gamma(t)) = g(\gamma(t))\,. \end{cases}$$

Keywords: balance laws, method of characteristics, Lagrangian formulation, ordinary differential equations, Peano phenomenon.

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1. Introduction

In the context of classical solutions, the balance law

$$\partial_t u(t,x) + \partial_x [f(u(t,x))] = g(t,x) \tag{1.1}$$

with $f \in C^2(\mathbb{R})$ can be reduced to ordinary differential equations along characteristic curves, defined as those curves $t \mapsto (t,\gamma(t))$ satisfying $\dot\gamma(t) = f'(u(t,\gamma(t)))$. Indeed,

$$\begin{aligned} g(t,\gamma(t)) &= \partial_t u(t,\gamma(t)) + \partial_x [f(u(t,\gamma(t)))] \\ &= \partial_t u(t,\gamma(t)) + f'(u(t,\gamma(t)))\partial_x u(t,\gamma(t)) \\ &= \partial_t u(t,\gamma(t)) + \dot\gamma(t)\,\partial_x u(t,\gamma(t)) \\ &= \frac{d}{dt}u(t,\gamma(t))\,. \end{aligned}$$

This more generally allows a parallel between the Cauchy problem for a scalar quasilinear first order PDE and for a system of ODEs, which is known as the method of characteristics (see for instance [10], where it is also provided an application to determine local existence).

If one interprets $f'(u)$ as a velocity, this is just the change of variable from the Eulerian (PDE) to the Lagrangian (ODEs) formulation.

We discuss here what remains of this equivalence when u is just continuous and g is bounded. We prove then in Section 2 that when g depends only on x, but not on the time t, then $u(t,x)$ is locally Lipschitz continuous on the open set where $f'(u)f''(u)$ is nonvanishing. This is sensibly better than the general case, where u is only Hölder continuous. It is based on proving the corresponding result for the system of ODEs. As we are discussing local issues, we will fix for simplicity the domain $\mathbb{R}^2$ and we will assume u bounded.

A motivation for a different setting

The development of Geometric Measure Theory in the context of the sub-Riemannian Heisenberg group $\mathbb{H}^n$ brought the attention to *continuous* solutions to the equation

$$\partial_t u(t,x) + \partial_x \left[\frac{u^2(t,x))}{2}\right] = g(t,x)\,. \tag{1.2}$$

Continuity is natural from the fact that u parametrizes a surface. As one studies surfaces that have differentiability properties in the intrinsic structure of the Heisenberg group, but not in the Euclidean structure, then it is not natural assuming more regularity of u than continuity [13], which for bounded sources improves to 1/2-Hölder continuity [4, 5]. Notice that with u continuous the second term of the equation cannot even be rewritten as $u\partial_x u$, because $\partial_x u$ is only a distribution and u is not a suitable test function.

The PDE arises if one wants to show the equivalence between a pointwise, metric notion of differentiability and a distributional one: for $n=1$ the distributional definition is precisely (1.2), while for $n>1$ it is a related multi-D system of PDEs. The correspondence was introduced first in [3, 4] for intrinsic regular hypersurfaces, which are the analogue of what are C^1-hypersurfaces in the Euclidean setting. It was extended in [5, 7] when considering intrinsic Lipschitz hypersurfaces, analogue of Lipschitz hypersurfaces in the Euclidean setting. The source term g, in $\mathbb{H}^1$, turns out to be what is called the intrinsic gradient of u, which is the counterpart of the gradient in Euclidean geometry; in $\mathbb{H}^n$ it is one if its components: u locally parametrizes an intrinsic regular hypersurface if and only if (1.2) holds locally with g continuous; it parametrizes an intrinsic regular hypersurface if and only if (1.2) holds locally with g bounded. As the notion of differentiability they provide in the intrinsic structure of $\mathbb{H}^n$ is closer to the Lagrangian formulation, the equivalence between Lagrangian and Eulerian formulation arises as intermediate step of this characterization.

When considering intrinsic Lipschitz hypersurfaces the fact that g is only bounded gives rise to new subtleties. In particular, one already knows by an intrinsic Rademacher theorem [11] that the intrinsic differential exists and it is unique $\mathscr{L}^2$-a.e. However, for the ODE formulation this is not enough: as one needs to restrict this L^∞ function on curves, a precise representative is needed also at points where u is not intrinsically differentiable. Viceversa, if one chooses badly the representative of the source of the ODE formulation a priori it differs on a positive measure set from the source of the ODE. There is however a canonical choice for defining the two sources, which makes the formulations equivalent when the inflection points of f are negligible.

Summary of the equivalence

When u is Lipschitz, the ODE

$$\begin{cases}\dot\chi(t,x) = f'(u(t,\chi(t,x)))\\ \chi(0,x)=x\end{cases}$$

with $x\in\mathbb{R}$ and f of class C^2, provides a local diffeomorphism by the classical theory on ODE. If u is instead continuous, Peano's theorem ensures local existence of solutions, but more characteristics may start at one point and characteristics from different points may collapse (see in [5] the classical example of the square-root). This makes clearly impossible to have a local diffeomorphism, or even having a Lagrangian flow in the sense of Ambrosio-DiPerna-Lions [9, 2]. A recent result about this can be obtained for u not depending on time [6], but it is clearly not our assumption. Dropping out injectivity, it is however possible to construct a continuous change of variables with bounded variation.

Let u be a continuous, bounded function.

Lemma 1.1. *There exists a continuous function $\chi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that*

- *$\tau\mapsto\chi(t,\tau)$ is nondecreasing for every t and surjective;*
- *$\partial_t\chi(t,\tau)=f'(u(t,\tau))$.*

We call it Lagrangian parameterization. This function is not unique.

See [1, 5] for the proof. See also [12] for a similar change of variable, for a $1D$-system. In general one cannot have that χ is SBV [1].

Consider now u continuous distributional solution to (1.1) with g bounded.

Lemma 1.2. *Assume that $\mathscr{L}^1(\text{clos}(\{\text{inflection points of } f\}))=0$. Then u is Lipschitz continuous along every characteristic curve.*

The proof follows a computation by Dafermos [8]. For general fluxes, there are cases when u is not Lipschitz along some Lagrangian parameterization [1]. The counterexample holds also for continuous autonomous sources $g(t,x)=g_0(x)$. What we find more striking is the following.

Theorem 1.3. *Assume that $\mathscr{L}^1(\text{clos}(\{\text{inflection points of } f\}))=0$. Then there exists a pointwise defined function $\hat g(t,x)$*

$$\frac{d}{dt}u(t,\gamma(t)) = \hat g(t,\gamma(t)) \quad \textit{in } \mathcal{D}'(\mathbb{R}) \textit{ for every characteristic curve } \gamma.$$

The proof is based on a selection theorem as a technical device, but $\hat g$ is essentially uniquely defined as the derivative of u along some characteristic.

Remark 1.4. There is a substantial difference between the uniformly convex and the strictly convex cases: in the former at almost every (t,x) there exists a unique value for $\frac{d}{dt}u(t,\gamma(t))$, $\gamma(t)=x$, and it does not depend on which characteristic $\gamma(s)$ one has chosen. That value is the most natural choice of $\hat g$ at those points, and this a.e. defined function $\hat g$ identifies the same distribution as the source term g. Without uniform convexity $\frac{d}{dt}u(t,\gamma(t))$ may not exist on a set of positive $\mathscr{L}^2$-measure, independently of which characteristic γ one chooses through the point. The correspondence between distributional and Lagrangian sources gets more complicated with non-convexity.

The converse also holds. We give here a weaker statement without the negligibility condition on the inflection points. As mentioned identifying sources is delicate, we refer for it to the more extensive work [1].

Theorem 1.5. *Assume that a continuous function u has a Lagrangian parameterization χ for which there exists a bounded function $\tilde{g}$ s.t. it satisfies*

$$\frac{d}{dt}u(t,\chi(t,\tau)) = \tilde{g}(t,\chi(t,\tau)) \quad \text{in } \mathcal{D}'(\mathbb{R}) \text{ for every } \tau \in \mathbb{R}. \tag{1.3}$$

Then there exists a function $g(t,x)$ s.t. (1.1) holds.

Viceversa, if (1.1) holds then there exists a *Lagrangian parameterization χ and function $\tilde{g}$ s.t. (1.3) holds.*

We finally mention that continuous distributional solutions to this simple equation do not dissipate entropy.

Theorem 1.6. *Let u be a continuous distributional solution to (1.1) with bounded source g. Then for every smooth function η and q satisfying $q' = \eta' f'$*

$$\partial_t\left[\eta(u(t,x))\right] + \partial_x\left[q(u(t,x))\right] = \eta'(u(t,x))g(t,x)\,.$$

2. Some local regularity with autonomous sources

We mention a local regularity result holding in the case of autonomous sources: the continuous function $u(t,x)$ is locally Lipschitz continuous in the (open) complementary of the 0-level set of the product $f'(u)\,f''(u)$. For $f(u) = u^2/2$, this means $u \neq 0$. When the source is not autonomous, then this fails to be true, indeed characteristics may bifurcate also at points where u is not vanishing.

We remind [1] that when f has inflection points of positive measure, then a priori u may not be Lipschitz along some characteristics, even with $g = g(x)$.

Lemma 2.1. *There may be locally multiple solutions to the ordinary differential equation*

$$\begin{cases} \dot\gamma(t) = u(t,\gamma(t))\,, \\ \ddot\gamma(t) = g(\gamma(t))\,, \\ \gamma(\bar t) = \bar x\,, \end{cases}$$

with $u(t,x)$ continuous, $g(x)$ bounded, only if $u(\bar t,\bar x) = 0$ but it does not identically vanish in a whole neighborhood.

Remark 2.2. We are not stating existence. The lemma is however still not obvious because we do not have differentiability properties of u, which follow a posteriori by the next corollary in the region where u does not vanish. As a consequence, we do not have now the differentiability of the map $\gamma(t)$ w.r.t. the initial data of the ODE. The lemma asserts indeed the continuity in this variable in that region, provided it exists. We remind that when g depends on t bifurcations may easily occur also if $u \neq 0$.

Proof. We just prove that if u does not vanish at some point $(\bar t,\bar x)$, at that point there is at most one solution of the ODE, as an effect of the autonomous source. The reason is that if $u(\bar t,\bar x)$ does not vanish, then any Lipschitz characteristic $x = \gamma(t)$, with $\bar x = \gamma(\bar t)$, is a diffeomorphism in some neighborhood of $(\bar t,\bar x)$, and we can invert it. This allows to have the space variable as a parameter: the characteristic can be expressed

as $t = \theta(x)$. However, the second order relation $\ddot\gamma(t) = g(\gamma(t))$, once expressed in the x variable, can be integrated determining the function θ.

By elementary arguments, it suffices to show that there exists (locally) only one characteristic passing through $(\bar t,\bar x) = (0,0)$ with slope $u(0,0) = 1$. Focus the attention on a neighborhood U of the origin where u is bigger than some $\varepsilon > 0$. Let $x = \gamma(t)$ be any Lipschitz continuous solution of the ODE. Since $\dot\gamma(0) = u(0,0) > 0$, by the inverse function theorem there exists $\delta > 0$ and a function

$$\theta : (\gamma(-\delta),\gamma(\delta)) \to (-\delta,\delta) \quad \text{s.t.} \quad \theta(\gamma(t)) = t\,, \quad \gamma(\theta(x)) = x\,.$$

Moreover, it is continuously differentiable with derivative

$$\dot\theta(x) = \frac{1}{\dot\gamma(\theta(x))} = \frac{1}{u(\theta(x),x)} \in \left[\frac{1}{\max|u|},\frac{1}{\varepsilon}\right]. \tag{2.1}$$

From the Lipschitz continuity of $u(t,\gamma(t))$ and the fact that γ is a local diffeomorphism with inverse θ we deduce that the composite function $u(\theta(x),x)$ is Lipschitz continuous. At points $X \subset U$ of differentiability by the classical chain rule

$$\begin{aligned}\lim_{h\downarrow 0}&\frac{\dot\gamma(\theta(x+h)) - \dot\gamma(\theta(x))}{h} = \\ &= \frac{\dot\gamma(\theta(x+h)) - \dot\gamma(\theta(x))}{\theta(x+h)-\theta(x)}\cdot\frac{\theta(x+h)-\theta(x)}{h} = \ddot\gamma(\theta(x))\,\dot\theta(x)\,,\end{aligned}$$

and by (2.1) we have that $\dot\theta$ is differentiable at $x \in X$ with derivative

$$\ddot\theta(x) = -\frac{\ddot\gamma(\theta(x))\dot\theta(x)}{[\dot\gamma(\theta(x))]^2} = -\frac{g(\theta(x))}{u^3(\theta(x),x)} \quad \Leftrightarrow \quad -\frac{\ddot\theta(x)}{[\dot\theta(x)]^3} = g(x)\,.$$

For those $x \in X$, the differential equation may be rewritten as

$$\frac{d}{dx}\left[\frac{1}{2[\dot\theta(x)]^2}\right] = g(x) \quad \Leftrightarrow \quad \frac{d}{dx}\frac{u^2(\theta(x),x)}{2} = g(x)\,.$$

The explicit ODE for $\theta(x)$, with initial data $\theta(0) = 0$, $[\dot\theta(0)]^{-1} = u(0,0) = 1$, is easily solved locally by

$$u^2(\theta(x),x) = \frac{1}{\dot\theta^2(x)} = 1 + 2\int_0^x g(z)dz\,. \tag{2.2}$$

This shows that the slope of every characteristic through the origin, which is a local diffeomorphism, is fixed at each x independently of the characteristic we have chosen: therefore there can be only one characteristic, precisely (in the space parameterization)

$$\theta(x;\bar t,\bar x) = \bar t + \int_{\bar x}^x \frac{1}{\sqrt{u^2(\bar t,\bar x) + 2\int_{\bar x}^w g(z)dz}}dw\,. \tag{2.3}$$

Notice finally that if u vanishes in a neighborhood, being $\dot\gamma(t) \equiv 0$ there characteristics must be vertical (in that region of the (x,t)-plane). □

Lemma 2.3. *Under the hypothesis of Lemma 2.1, if $g(x)$ is continuous it should also vanish at points where there are more characteristics, but it must not identically vanish in a neighborhood.*

Proof. We show that not only u, but also g must vanish. The argument shows that when two characteristics meet and have both second derivative with the same value, this value must be 0. For simplifying notations, consider two characteristics $\gamma_1(t) \le \gamma_2(t)$ for arbitrarily small $t > 0$ with $\gamma_1(0) = \gamma_2(0) = 0$. If $\gamma_1(t_k)\gamma_2(t_k) \le 0$ for $t_k \downarrow 0$, then

$$0 \le \ddot\gamma_2(0) = g(0) = \ddot\gamma_1(0) \le 0\,,$$

thus g vanishes. If instead e.g. $g > 0$ near the origin, having excluded the above case there exists $\delta > 0$ such that $0 < \gamma_1(t) \le \gamma_2(t)$ for $t \in [0,\delta]$. Then (2.2) implies that the two curves coincide: having $\dot\gamma_1(t_k) = 0$ or $\dot\gamma_2(t_k) = 0$ for a sequence $|t_k| \downarrow 0$ would contradict the positivity of g, therefore for small $t > 0$ necessarily $\dot\gamma_1(t) > 0$, $\dot\gamma_2(t) > 0$ and therefore

$$\begin{aligned} u^2(\gamma_1^{-1}(x),x) + 2\int_x^0 g(z)\,dz &= \dot\gamma_1^2(0) \\ &= 0 = \dot\gamma_2^2(0) = u^2(\gamma_2^{-1}(x),x) + 2\int_x^0 g(z)\,dz\,. \end{aligned}$$

Being $\dot\gamma_i(t) = u(t,\gamma_i(t))$, $i = 1,2$, by the differential relation, this shows that $\dot\gamma_1(t) \equiv \dot\gamma_2(t)$ for small times. This implies that the two curves coincide.

Finally, suppose g vanishes in a neighborhood. Then, as $\ddot\gamma(t) = 0$ in that neighborhood, characteristics are straight lines. As by the continuity of u characteristics may only intersect with the same derivative, they must be parallel lines and therefore bifurcation of characteristics does not occur. □

We now show that in case u does not vanish, in the above lemma much more regularity holds.

Lemma 2.4. *If for every $(\bar t,\bar x) \in \Omega$ open in $\mathbb{R}^2$ there exists a curve γ s.t.*

$$\begin{cases} \dot\gamma(t) = u(t,\gamma(t))\,, \\ \ddot\gamma(t) = g(\gamma(t))\,, \\ \gamma(\bar t) = \bar x\,, \end{cases}$$

with $u(t,x)$ continuous, $g(x)$ bounded, then $u(t,x)$ is locally Lipschitz in the open set $\{(t,x):\ u(t,x) \neq 0\} \subset \Omega$.

Corollary 2.5. *If u is not locally Lipschitz where nonvanishing then the system in Lemma 2.1 cannot have solutions through each point of the plane. In particular, u cannot be a continuous solution to*

$$\partial_t u(t,x) + \partial_x [f(u(t,x))] = g(x)\,.$$

Proof. By Lemma 2.1 there is a unique characteristic starting at each point $(\bar t,\bar x) \in \Omega = \{(t,x):\ u(t,x) \neq 0\}$, which is given by (2.3). We start comparing the value of u at two points $(0,0)$, $(-t,0)$, $t > 0$, in a ball B compactly contained in Ω. In particular, there exists $\delta(B)$ s.t. the two characteristics starting from the points we have chosen do not intersect if $0 < x < \delta(B)$, as there u does not vanish. For such small x one has by (2.3)

$$\int_0^x \frac{1}{\sqrt{\lambda_1^2 + 2\int_0^w g(z)dz}}\,dw > -t + \int_0^x \frac{1}{\sqrt{\lambda_2^2 + 2\int_0^w g(z)dz}}\,dw\,, \tag{2.4}$$

where we defined $\lambda_1 = u(0,0)$ and $\lambda_2 = u(-t,0)$. Equivalently

$$t > \int_0^x \frac{1}{\sqrt{\lambda_2^2 + 2\int_0^w g(z)\,dz}} - \frac{1}{\sqrt{\lambda_1^2 + 2\int_0^w g(z)\,dz}}\,dw.$$

Suppose $\lambda_1 > \lambda_2$. By the convexity of $r \mapsto 1/\sqrt{r}$, the right-hand side is larger than

$$\begin{aligned} \int_0^x \left[\frac{d}{dr}\left(\frac{1}{\sqrt r}\right)\Big|_{r=\lambda_1^2+2\int_0^w g(z)dz} (\lambda_2^2 - \lambda_1^2) \right] dw &= \\ &= \left[\frac{\lambda_2+\lambda_1}{-2} \int_0^x \frac{1}{\left(\lambda_1^2 + 2\int_0^w g(z)dz\right)^{3/2}}\,dw \right] (\lambda_2 - \lambda_1) \\ &\ge \left[\frac{\lambda_2+\lambda_1}{2(\lambda_1^2 + 2Gx)^{3/2}} x \right] (\lambda_1 - \lambda_2)\,. \end{aligned}$$

The argument within square brackets in the last line is uniformly continuous and as $t \downarrow 0$ it is larger than x/λ_1^2. As the inequalities hold for every positive t, $x < \delta = \delta(B)$, the non-intersecting condition (2.4) gives

$$t > \left(\frac{\lambda_1^2}{\delta} + \varepsilon\right)^{-1} (\lambda_1 - \lambda_2) \quad \Rightarrow \quad u(0,0) - u(t,0) = \lambda_1 - \lambda_2 \le \left(\frac{\lambda_1^2}{\delta} + \varepsilon\right) t\,,$$

which is half the Lipschitz inequality at the points $(0,0)$, $(-t,0)$. The other half, for $\lambda_1 < \lambda_2$ is similarly obtained considering small negative x.

For comparing two generic close points (t,x) and $(0,0)$, by the finite speed of propagation one can combine the Lipschitz regularity along characteristics and the Lipschitz regularity along vertical lines. □

Corollary 2.6. *Let $u(t,x)$ be a continuous solution to the balance equation*

$$\partial_t u(t,x) + \partial_x [f(u(t,x))] = g(x)\,, \quad g \in L^\infty(\mathbb{R})\,.$$

Then the function $u(t,x)$ is locally Lipschitz in the open set

$$\{(t,x):\ f'(u(t,x)) \cdot f''(u(t,x)) \neq 0\}\,.$$

Proof. We first consider the case of quadratic flux $f(u) = u^2/2$. By Theorem 1.3, there exists a function $\hat g(t,x)$ such that we can apply Lemma 2.1, which gives the thesis. If $g \in L^\infty$ they may a priori differ on an $\mathscr{L}^2$-negligible set, but one can prove that $\hat g(t,x) = \hat g(x)$.

Being u an entropy solution by Theorem 1.6, $f'(u)$ solves the equation

$$[f'(u)]_t + \left[\frac{f'(u)^2}{2}\right]_x = f''(u) g\,.$$

By the previous case then $f'(u)$ is Lipschitz in the open set where it does not vanish. If moreover $f''(u)$ does not vanish, then the regularity of u can be proved just by inverting f'. □

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