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Structure of null sets in the plane and applications

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Abstract: We describe a decomposition result for Lebesgue negligible sets in the plane, and outline some applications to real analysis and geometric measure theory. These results are contained in [2].

1. Introduction

This note is an extended version of a talk that the first author gave at the Fourth European Congress of Mathematics (Stockholm, June 27-July 2, 2004). As the talk, this paper is aimed to non-expert readers, with only a basic knowledge of measure theory and real analysis. Thus many theorems and definitions have been stated in a simplified form, while others of more technical nature have been entirely omitted. Without the burden of generality, certain proofs turned out to be relatively simple, and have therefore been included in a sketchy but hopefully clear form. The interested reader shall find general statements and detailed proofs in a forthcoming paper [2].

The starting point of our research was the observation that in the two-dimensional case the solutions of several problems of seemingly different nature can be derived by a simple covering result for null sets in the plane (Theorem 3.1). These problems include the so-called rank-one property of BV functions, the geometric structure of measures supporting normal currents, and the construction of Lipschitz maps with large non-differentiability sets. As shown below, this covering can be proved using a geometric version of a known combinatorial result (Dilworth's lemma, or Erdős-Szekeres theorem). Unfortunately, no equivalent combinatorial result is available in higher dimension, and it is still an open question whether the desired generalization of Theorem 3.1 holds even in the three-dimensional space (this issue is briefly discussed in Section 8).

Despite the fact the paper is mostly focused on the simplest—i.e., two-dimensional—situations, the reader should keep in mind that many results

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extend to higher dimension, too, although in that case they may be not as complete, and many questions are still unanswered.

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BASIC NOTATION AND TERMINOLOGY. - In this the paper, the word “measure” is only used for bounded or locally bounded measures on a Borel σ-algebra, with the only exception of the d-dimensional Hausdorff measure $\mathscr{H}^d$, which is not even σ-finite. Recall that if E is a subset of a d-dimensional surface of class $\mathscr{C}^1$ in the Euclidean space, then $\mathscr{H}^d(E)$ is the usual d-dimensional volume of E. The Lebesgue measure on $\mathbb{R}^d$ is denoted by $\mathscr{L}^d$.

Unless otherwise specified, sets and functions are assumed to be Borel measurable. We will conform as far as possible to the standard notation of measure theory, and just recall here some essential terminology: a set in $\mathbb{R}^d$ is *null* if it is Lebesgue negligible; a measure μ on $\mathbb{R}^d$ is *singular* if it is singular with respect to Lebesgue measure; the (upper/lower) *density* of a set $E \subset \mathbb{R}^d$ at a point $x \in \mathbb{R}^d$ is the (upper/lower) limit as $r \to 0$ of the ratio $\mathscr{L}^d(E \cap B_r(x))/\mathscr{L}^d(B_r(x))$, where $B_r(x)$ stands for the open ball with center x and radius r; if this limit exists and is equal to 1, then x is a called a *density point* of E.[1]

The term “curve” denotes connected 1-dimensional submanifolds of $\mathbb{R}^d$.

Given a positive real number L, a map f is called *L-Lipschitz* if it has Lipschitz constant $\mathrm{Lip}(f) \le L$.

2. A covering result for finite sets in the plane

As usual, we denote by x, y the coordinates of a point in the plane. We call *x-curve* the graph of a 1-Lipschitz function $y = y(x)$ defined for all x in $\mathbb{R}$. Similarly, a *y-curve* is the graph of a 1-Lipschitz function $x = x(y)$.

THEOREM 2.1. - *A set S of n points in the plane can be covered using at most $\sqrt{n}$ x-curves and $\sqrt{n}$ y-curves.*

REMARK 2.2. - (i) The argument in the proof of Theorem 2.1 can be used, with few modifications, to show that there exists an x- or a y-curve that contains at least $\sqrt{n}$ points of S. This statement is a particular case of Dilworth’s lemma (see [7]). It also implies, and indeed is equivalent to, the standard formulation of Erdős-Szekeres theorem: every finite sequence $(t_1, \dots, t_n)$ of real numbers

[1] When $\mathscr{L}^n$ is replaced by a positive measure μ we shall speak of μ-density.

contains a monotonic subsequence of length at least $\sqrt{n}$.[2] For a survey about the many variations of Erdős-Szekeres theorem, see [18].
(ii) The Lipschitz constant in the definition of x- and y-curves cannot be taken smaller than 1 (consider a set S contained in the line $y = x$). In general, both x- and y-curves are needed to cover S (consider a set S with $n/2$ points on the x-axis and $n/2$ points on the y-axis).
(iii) Theorem 2.1 can be stated in a slightly stronger form: given integers h, k such that $hk \ge n$, then S can be covered by h x-curves and k y-curves.

PROOF. - We define the following partial order in S: a point $p_1 = (x_1, y_1)$ is below a point $p_2 = (x_2, y_2)$, and we write $p_1 \preceq p_2$, if $y_2 - y_1 \ge |x_2 - x_1|$, that is, if p_2 belongs to the (one sided) cone with vertex p_1 and axis parallel to the y-axis shown in Figure 1, left.

We extract from S a chain (totally ordered subset) C_1 with $\sqrt{n}$ points or more. Then we extract from $S \setminus C_1$ a chain C_2 with $\sqrt{n}$ points or more, and we proceed in this way until every chain in $S' := S \setminus (C_1 \cup \dots \cup C_k)$ contains less than $\sqrt{n}$ points[3] —see Figure 1, right.

Now we extract from S' the set M_1 of all maximal points, that is, points that are below no other point of S'. Then we extract the set M_2 of all maximal points of $S' \setminus M_1$, and we repeat this operation until $S' \setminus (M_1 \cup \dots \cup M_h)$ is empty (thus the sets M_j are the *strata* of S).

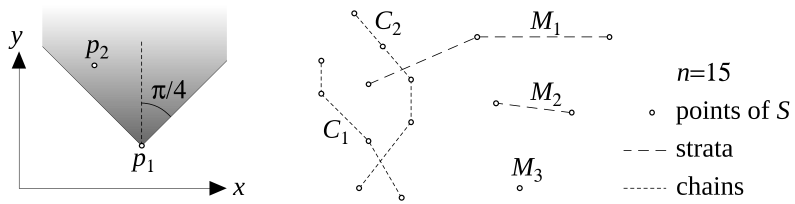

FIGURE 1

To conclude, it suffices to observe the following: (i) S is covered by the chains $C_1, \dots, C_k$ and the strata $M_1, \dots, M_h$; (ii) each chain is contained in a y-curve and each stratum is contained in a x-curve;[4] (iii) the number of chains, k, cannot exceed $\sqrt{n}$ because the chains are all disjoint subsets of S and contain at least $\sqrt{n}$ points. The number h of strata cannot exceed $\sqrt{n}$ either, because

[2] To prove Erdős-Szekeres theorem, consider the points $p_h := (h - t_h, h + t_h)$ with $h = 1, \dots, n$, and notice that any subset contained in an x-graph (resp., a y-graph) corresponds to a decreasing (resp., increasing) subsequence (t_k).

[3] If S contains no chain with more than $\sqrt{n}$ points then $k = 0$ and $S' = S$.

[4] More precisely, every chain is the graph of a 1-Lipschitz function $x = x(y)$ defined for finitely many y, and can be extended to all $y \in \mathbb{R}$ using McShane’s extension lemma. A similar argument applies to the strata.

it agrees with the length of the maximal chain contained in S'.[5] □

3. A covering result for null sets in the plane

We call *x-strip* of thickness δ a subset T of the plane of the form

$$T = T^x(f,\delta) := \big\{(x,y) : \ |y - f(x)| \le \delta/2\big\}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a 1-Lipschitz function. The definition of *y-strip* $T^y(f,\delta)$ is the obvious one, one just swaps x and y.

THEOREM 3.1. - *Let E be a null set in the plane. Then E can be written as $E^x \cup E^y$ where E^x and E^y satisfy the following conditions:*
(a) *for every $\varepsilon > 0$, E^x can be covered by countably many x-strips T_i^x of thickness δ_i so that $\sum \delta_i \le \varepsilon$;*
(b) *for every $\varepsilon > 0$, E^y can be covered by countably many y-strips T_j^y of thickness η_j so that $\sum \eta_j \le \varepsilon$.*

REMARK 3.2. - (i) By Fubini's theorem, every null set E in the plane can be written as the union of two sets E^x and E^y such that all one-dimensional sections of E^x parallel to the y-axis and all sections of E^y parallel to the x-axis are null. This means that every such section can be covered by countably many intervals so that the sum of the lengths is smaller than any given $\varepsilon > 0$. Theorem 3.1 makes this statement more precise, by showing that these intervals can be chosen so that they depend in a Lipschitz way on the variable that parametrizes sections.
(ii) Conditions (a) and (b) imply the following: $\mathscr{H}^1(C \cap E^x) = 0$ for every y-curve C with Lipschitz constant $L < 1$ and $\mathscr{H}^1(C \cap E^y) = 0$ for every x-curve C with Lipschitz constant $L < 1$.
(iii) Adjusting the proof below, one easily deduces the following modification of Theorem 3.1: a set E with positive measure m can be covered by x-strips T_i^x of thickness δ_i and y-strips T_j^y of thickness η_i so that $\sum \delta_i \le 3\sqrt{m}$ and $\sum \eta_j \le 3\sqrt{m}$.[6]

PARTIAL PROOF. - We assume for simplicity that E is compact, and only prove that for every $\varepsilon > 0$ it can be covered by x-strips T_i^x and y-strips T_j^y so that $\sum \delta_i \le \varepsilon$ and $\sum \eta_j \le \varepsilon$.

We fix $\delta > 0$ and define the δ-discretization E_δ of E as the centers of all squares of the form $[h\delta, (h+1)\delta] \times [k\delta, (k+1)\delta]$, with k, h integers, which

[5] Take $P_1 \in M_h$. Then there exists $P_2 \in M_{h-1}$ such that $P_1 \preceq P_2$, otherwise P_1 too would belong to M_{h-1}. By the same argument we can find $P_3 \in M_{h-2}, \dots, P_h \in M_1$ such that $P_j \preceq P_{j+1}$ for every j, that is, a chain of length h in S'.

[6] By a different proof we can even obtain $\sum \delta_i \le a$ and $\sum \eta_j \le b$ where a and b are any two positive numbers that satisfy $ab > m$ (see [2]).

intersect E (see Figure 2). Since E is compact, it has Lebesgue measure zero if and only if

$$\#E_\delta = o(1/\delta^2) \ .$$

By Theorem 2.1, E_δ can be covered by $\sqrt{\#E_\delta}$ x-curves—the graphs of some functions f_i—and by $\sqrt{\#E_\delta}$ y-curves—the graphs of some functions g_j.

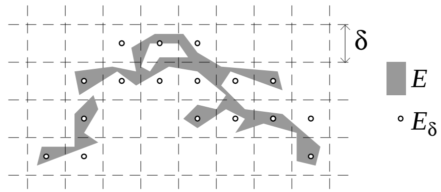

FIGURE 2

It is easy to check the x-strips $T^x(f_i, 2\delta)$ and the y-strips $T^y(g_j, 2\delta)$ cover E. Moreover the sum of the thicknesses for both families of strips is

$$\sqrt{\#E_\delta} \cdot (2\delta) = \sqrt{o(1/\delta^2)} \cdot (2\delta) = o(1)$$

i.e., it tends to 0 as $\delta \to 0$. To conclude, we choose δ so that $o(1) \le \varepsilon$. □

4. Tangent field to a null set in the plane

The first application of Theorem 3.1 is about a notion of tangent field for sets in the plane, and has some interesting consequences that will be explained in the next section.

DEFINITION 4.1. - Let $G(2,1)$ be the Grassmann manifold of lines in the plane. Given a Borel set $E \subset \mathbb{R}^2$, we say that a Borel map $\tau : E \to G(2,1)$ is a *weak tangent field* to E if

$$\tau_S(p) = \tau(p) \quad \text{for } \mathscr{H}^1\text{-a.e. } p \in S \cap E \tag{4.1}$$

for every curve S of class $\mathscr{C}^1$, where τ_S is the tangent field to S according to the usual definition.

REMARK 4.2. - (i) The notion of weak tangent field is compatible with the usual one: if E is a curve of class $\mathscr{C}^1$ then the tangent field τ_E is also a weak tangent field, and conversely, every weak tangent field τ agrees with τ_E up to an $\mathscr{H}^1$-negligible subset.[7]

[7] This is a corollary of the following lemma: given two curves S_1, S_2 of class $\mathscr{C}^1$, the corresponding tangent fields agree at $\mathscr{H}^1$-a.e. point of $S_1 \cap S_2$ (in fact, they agree at all points of $S_1 \cap S_2$ except a discrete subset).

(ii) A set E in the plane is *rectifiable* if it can be covered by countably many curves S_i of class $\mathscr{C}^1$ except an $\mathscr{H}^1$-negligible subset E_0.[8] A weak tangent field for such a set is constructed as follows: for $p \in E \setminus E_0$ we set $\tau(p) := \tau_{S_i}(p)$ where i is the smallest index such that p belongs to S_i, while for $p \in E_0$, $\tau(p)$ is taken arbitrarily.
(iii) If E is rectifiable, then the weak tangent field is unique up to $\mathscr{H}^1$-negligible sets, i.e., if τ_1 and τ_2 satisfy (4.1), then they agree outside a subset of E with $\mathscr{H}^1$ measure equal to zero. If in addition E has (locally) finite $\mathscr{H}^1$ measure, then the weak tangent field can be characterized in a pointwise way, and it is known as the *approximate tangent field* (or bundle) of E. For further details see [9], Section 3.2, or [17], Chapter 3.
(iv) A set E in the plane is *purely unrectifiable* (p.u.) if $\mathscr{H}^1(S \cap E) = 0$ for every curve S of class $\mathscr{C}^1$.[9] It is clear that for such sets every τ is tangent because (4.1) is automatically verified. Examples of p.u. sets are the products $E = F \times F$ where F is a null set in $\mathbb{R}$, and more generally all sets E which admit two different projections of measure zero. Essentially all examples of "fractals" in the plane are purely unrectifiable.
(v) The weak tangent field (if it exists) is unique up to p.u. sets; in other words, if τ_1 and τ_2 are tangent to E, then they agree outside a purely unrectifiable subset of E. In the following we shall denote any field in this equivalence class by τ_E.
(vi) A set E with positive Lebesgue measure admits no tangent field. Indeed, using as test curves S in (4.1) all lines parallel to a given line $\ell \in G(2,1)$, we deduce by Fubini's theorem that a tangent field τ should agree with ℓ for $\mathscr{L}^2$-a.e. point of E; since this should hold for every choice of ℓ, we have a contradiction.

The following result shows that there is nothing to add to Remark 4.2(vi):

THEOREM 4.3. - *Every null set E in the plane admits a weak tangent field.*

PROOF. - We need some additional notation: given a unit vector $e \in \mathbb{R}^2$ and an angle $\alpha \in [0, \pi]$, we denote by $C(e, \alpha)$ the two-sided closed cone of axis e and amplitude α, that is,

$$C(e, \alpha) := \big\{ v : \ |v \cdot e| \geq |v| \cos(\alpha/2) \big\} \ . \tag{4.2}$$

A map $\mathscr{C}$ from E to the class of all cones is a tangent cone-field to E if it satisfies the obvious analogue of (1) for every curve S of class $\mathscr{C}^1$:

$$\tau_S(p) \subset \mathscr{C}(p) \quad \text{for } \mathscr{H}^1\text{-a.e. } p \in S \cap E.$$

[8] In the terminology of Geometric Measure Theory these sets are called countably $(\mathscr{H}^1, 1)$-rectifiable or simply 1-rectifiable (cf. [9], [13], [17]). The standard definition, albeit equivalent, is different from this one.

[9] The standard terminology is $(\mathscr{H}^1, 1)$-purely unrectifiable, or 1-purely unrectifiable.

Step 1. We first establish the existence of a suitable tangent cone-field. Let $e := (1, 0)$ and $e' := (0, 1)$. Writing E as $E^x \cup E^y$ as in Theorem 3.1, then the cone-field

$$\mathscr{C}_\alpha(p) := \begin{cases} C(e, \alpha) & \text{if } p \in E^x, \\ \\ C(e', \alpha) & \text{if } p \in E^y \setminus E^x, \end{cases}$$

is tangent to E for every $\alpha > \pi/2$. This is an easy consequence of the property of E^x and E^y stated in Remark 3.2(ii).
Step 2. If we rotate the axes by an angle θ and perform the construction in Step 1, we obtain a new tangent cone-field $\mathscr{C}_{\theta,\alpha}$ which is equal either to $C(e_\theta, \alpha)$ or to $C(e'_\theta, \alpha)$ at every point of E, where $e_\theta := (\cos\theta, \sin\theta)$ and $e'_\theta := (-\sin\theta, \cos\theta)$.
Step 3. We observe that every countable intersection of tangent cone-fields is still a tangent cone-field, and then we set

$$\mathscr{C}(p) := \bigcap \mathscr{C}_{\theta,\alpha}(p) \quad \text{for every } p \in E,$$

where the intersection is taken over all α in a given countable dense subset of $(\pi/2, \pi)$ and all θ in a given countable dense subset of $[0, 2\pi]$. It is not difficult to check that $\mathscr{C}(p)$ is either a line or a point for every $p \in E$, and if in the latter occurrence we change it to an (arbitrarily chosen) line, we obtain a weak tangent field to E. □

5. The rank-one property of BV functions

Given an open set Ω in $\mathbb{R}^d$, the space of functions of bounded variation $BV(\Omega)$ consist of all $u \in L^1(\Omega)$ whose distributional derivative Du is (represented by) a bounded measure on Ω with values in $\mathbb{R}^d$.

Let μ be a measure in the plane and E a null set. By Theorem 4.3, E admits a tangent field τ_E in the sense of Definition 4.1. Then the following holds:

PROPOSITION 5.1. - *For every function $u \in BV(\mathbb{R}^2)$, the Radon-Nikodym density of the vector measure Du with respect to μ is a map valued in $\mathbb{R}^2$ which satisfies*

$$\frac{d(Du)}{d\mu}(x) \perp \tau_E(x) \quad \textit{for } \mu\textit{-a.e. } x \in E. \tag{5.1}$$

PROOF. - It is not difficult to see that it suffices to prove (5.1) when μ is equal to $|Du|$, the total variation of the vector measure Du.

We need the following results about BV functions: the positive measure $|Du|$ can be disintegrated as

$$|Du| = \int_{\mathbb{R}} \mathscr{H}^1 \llcorner S_t \, d\mathscr{L}^1(t) \ , \tag{5.2}$$

where each S_t is a rectifiable set with finite $\mathscr{H}^1$-measure and $\mathscr{H}^1 \llcorner S_t$ denotes the restriction of $\mathscr{H}^1$ to the set S_t.[10] Moreover, denoting by τ_t the approximate tangent field to S_t (see Remarks 4.2(ii) and (iii)), the Radon-Nikodym density of Du with respect to $|Du|$ satisfies

$$\frac{d(Du)}{d(|Du|)} \perp \tau_t(x) \quad \text{for } \mathscr{H}^1\text{-a.e. } x \in S_t \text{ and } \mathscr{L}^1\text{-a.e. } t \in \mathbb{R}. \tag{5.3}$$

More precisely, one takes S_t equal to the reduced boundary of the sublevel $\{x : u(x) \geq t\}$, thus S_t is rectifiable by De Giorgi's theorem (see [3], Theorem 3.59), and (5.2) is a reformulation of the coarea formula for BV functions (see [3], Theorem 3.40). Formula (5.3), like identity (5.2), can be derived with a little extra work from the coarea formula (cf. [1], Theorem 1.12).

Let t be fixed. Since S_t is rectifiable, then it can be covered by countably many curves of class $\mathscr{C}^1$, and the definition of weak tangent field yields

$$\tau_t(x) = \tau_E(x) \quad \text{for } \mathscr{H}^1\text{-a.e. } x \in S_t \cap E. \tag{5.4}$$

Finally, (5.3) and (5.4) imply $\frac{d(Du)}{d(|Du|)}(x) \perp \tau_E(x)$ for $\mathscr{H}^1$-a.e. $x \in S_t \cap E$ and $\mathscr{L}^1$-a.e. $t \in \mathbb{R}$. By (5.2), the same is true for $|Du|$-a.e. $x \in E$, and we have proved (5.1) for $\mu = |Du|$. $\square$

Proposition 5.1 implies the so-called rank-one property of BV functions, which was first proved by the first author, in a completely different way, in [1], Corollary 4.6. Recall that given a BV map $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^m$, the derivative Du is a measure valued in $m \times d$ matrices, and the Radon-Nikodym density of Du with respect to a positive measure μ is a map valued in $m \times d$ matrices.

THEOREM 5.2. - *Let u be a map in $BV(\Omega, \mathbb{R}^m)$, μ a positive measure on Ω, and E a null set in Ω. Then*

$$\operatorname{rank}\left[\frac{d(Du)}{d\mu}(x)\right] \leq 1 \quad \textit{for } \mu\textit{-a.e. } x \in E. \tag{5.5}$$

In particular, the density of Du with respect to any singular measure μ is valued in matrices of rank one or zero.

PROOF. - For $d = 2$, this statement is an immediate consequence of Proposition 5.1: denoting by u_i, $i = 1, \dots, n$, the components of u, the rows of the matrix $\frac{d(Du)}{d\mu}(x)$ are the vectors $\frac{d(Du_i)}{d\mu}(x)$; since all these vectors are orthogonal to $\tau_E(x)$, they are co-linear, which means that the matrix has rank one or zero.

[10] Identity (5.2) should be read as follows: $|Du|(B)$ is equal to the integral $\int \mathscr{H}^1(S_t \cap B)\, d\mathscr{L}^1(t)$ for every Borel set $B \subset \mathbb{R}^2$. Clearly, a certain Borel regularity of the map $t \mapsto S_t$ is assumed.

For general d, the statement can be proved by reduction to the previous case. Indeed, the distributional derivative of u can be reconstructed from the distributional derivatives of its restrictions to the planes parallel to the coordinate planes using a natural "slicing" formula (cf. [1], Proposition 1.10), and since the rank of an $m \times d$ matrix is one or zero if (and only if) the same holds for all $m \times 2$ minors, the rank-one property of Du is implied by the rank-one property of its restrictions to planes. $\square$

6. Mapping sets of positive measure onto balls

Among the problems meant to explore the geometric structure of sets with positive Lebesgue measure, the following one, proposed by M. Laczkovich, is particularly interesting:

QUESTION 6.1. - *Given a compact set K in $\mathbb{R}^d$ of positive Lebesgue measure, is there a Lipschitz map $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ which takes K onto a closed ball?*

It is clearly equivalent to assume that K is Borel, or require that $f(K)$ contains a ball, that is, it has non-empty interior.

Looking at a density point of K, it is possible to find a ball B such that the measure of $B \setminus K$ is extremely small compared to that of B. Thus one would expect that a perturbation Φ of the identity can be found, which maps $B \setminus K$ into a set with empty interior, so that $\Phi(K)$ contains $\Phi(B)$, and hopefully the latter set has nonempty interior. However, after few attempts one realizes that, in dimension larger than one, making Φ Lipschitz and the interior of $\Phi(B)$ non-empty at the same time is quite difficult.

PROPOSITION 6.2. - *The answer to Question 6.1 is positive for $d = 1$.*

PROOF. - Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a primitive of the characteristic function 1_K, that is

$$\Phi(x) := \mathscr{L}^1\big(K \cap (-\infty, x)\big) \quad \text{for every } x \in \mathbb{R}.$$

Then Φ is constant on each connected component of the complement of K, and $\Phi(K)$ is equal to $\Phi(\mathbb{R})$, which is a non-trivial interval because Φ is not constant. $\square$

THEOREM 6.3. - *The answer to Question 6.1 is positive for $d = 2$.*

This theorem was first proved by the third author (a version of this proof will appear in [2]); a proof based on Erdős-Szekeres theorem was then given by J. Matoušek in [11]. Question 6.1 is still open for $d \geq 3$.

Before giving the proof of this result, we briefly review some naïve solutions, and explain why they do not work.

ATTEMPT OF SOLUTION FOR $d > 1$. - A way to extend the construction in the proof of Proposition 6.2 is to solve the equation

$$\det(\nabla\Phi) = 1_K \tag{6.1}$$

on some smooth bounded domain Ω of $\mathbb{R}^d$ which contains K, imposing a Dirichlet boundary condition which guarantees that $\Phi(\Omega)$ contains a ball B. Because of (6.1), $\Phi(\Omega \setminus K)$ must be a null set, and therefore has empty interior, which implies that $\Phi(K)$ agrees with $\Phi(\Omega)$, and in particular contains B.

The difficulty is that in general the equation $\det(\nabla\Phi) = g$ admits no Lipschitz solution even if the datum g is continuous and strictly positive (see [16], [5]), and the situation gets no better when g is discontinuous and takes the value zero.

AN ITERATIVE CONSTRUCTION FOR $d = 1$. - The function Φ in the proof of Proposition 6.2 can also be obtained by an iterative construction that might be extended to higher dimension. Given an interval $I = (a, b)$ in $\mathbb{R}$, we denote by Φ_I the function

$$\Phi_I(x) := \begin{cases} x & \text{if } x \le a \\ a & \text{if } a < x < b \\ x - (b-a) & \text{if } b \le x \end{cases} .$$

Thus Φ_I maps I into a point and is measure-preserving in the complement of I. By composing maps of this type we can "remove" one by one all connected components I in the complement of K. More precisely, we take Φ to be the limit of the functions Φ_n defined by induction on n as follows: $\Phi_0(x) := x$ is the identity map, and $\Phi_n(x) := \Phi_{I_n}(\Phi_{n-1}(x))$ where I_n is a bounded connected component of maximal length in the complement of $\Phi_{n-1}(K)$. It is easy to check that Φ is 1-Lipschitz, maps the complement of K into a set of measure 0, and is measure preserving on K. In particular $\Phi(K)$ agrees with $\Phi(\mathbb{R})$ and has the same measure as K, and therefore is an interval of positive length.

SECOND ATTEMPT OF SOLUTION FOR $d > 1$. - One way of adapting the previous construction to higher dimension is the following: for every open ball B in $\mathbb{R}^d$ we construct a Lipschitz map $\Phi_B : \mathbb{R}^d \to \mathbb{R}^d$ such that $\Phi_B(B)$ has measure zero, we choose a bounded open set Ω which contains K, and then let Φ be the limit of the maps Φ_n defined as follows: $\Phi_n(x) := \Phi_{B_n}(\Phi_{n-1}(x))$ where B_n is, say, the largest open ball contained in $\Phi_{n-1}(\Omega) \setminus \Phi_{n-1}(K)$.

In order to ensure that the limit Φ exists and is Lipschitz, the maps Φ_B must be (asymptotically) 1-Lipschitz. Now, the difficulty is that a 1-Lipschitz map which takes a ball into a null set is far from being measure-preserving on the complement. In other words, there is no easy way to prevent the sets $\Phi_n(\Omega)$ from collapsing to a set $\Phi(\Omega)$ of measure zero, and therefore with empty interior.

PROOF OF THEOREM 6.3. - The iterative construction described in the previous paragraphs can be made work in dimension $d = 2$ by removing suitably chosen strips that cover the complement of K.

Given an x-strip $T = T^x(f, \delta)$, we define $\Phi_T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi_T(x,y) := \begin{cases} (x, y) & \text{if } y \le f(x) - \delta/2 \\ (x, f(x) - \delta/2) & \text{if } f(x) - \delta/2 < y < f(x) + \delta/2 \\ (x, y - \delta) & \text{if } f(x) + \delta/2 \le y \end{cases}$$

Thus Φ_T maps T into a null set, is measure preserving in the complement of T, and is 1-Lipschitz provided that $\mathbb{R}^2$ is endowed with the ℓ^∞-norm

$$\|(x,y)\| := \sup\{|x|, |y|\} \tag{6.2}$$

instead of the Euclidean norm. Moreover Φ_T maps any x-strip into (but not necessarily onto) another x-strip with same thickness.

Now we choose an open square Ω with side-length r and parallel to the coordinate axes so that the set $A := \Omega \setminus K$ is small (the precise requirement will be made explicit later). By Remark 3.2(iii), we can cover A using countably many x- or y-strips T_n with thickness δ_n so that

$$\sum_{n=1}^{\infty} \delta_n \le 6\sqrt{\mathscr{L}^2(A)} .$$

We assume for the time being that all T_n are x-strips, and take Φ equal to the limit of the maps Φ_n defined as follows: $\Phi_0(x) := x$ is the identity map, and $\Phi_n(x) := \Phi_{T'_n}(\Phi_{n-1}(x))$ where T'_n is a strip of thickness δ_n which contains $\Phi_{n-1}(T_n)$. Thus Φ is 1-Lipschitz with respect to the norm (6.2) and maps A into a null set, and therefore $\Phi(K)$ contains $\Phi(\Omega)$. Moreover $\Phi(\Omega)$ contains a rectangle Ω' with width r and height

$$r - \sum \delta_n \ge r - 6\sqrt{\mathscr{L}^2(A)} , \tag{6.3}$$

and has non-empty interior provided that $\mathscr{L}^2(A) < r^2/36$. Note that this inequality is verified by all squares Ω centered at a density point of K and sufficiently small.

This proof works only if the strips T_n are of the same type. In general, using only strips of one type we cannot cover all of A, but we can cover at least half of it, that is, a subset B such that $\mathscr{L}^2(B) \ge \mathscr{L}^2(A)/2$. Hence the map Φ given above takes B into a null set, and therefore $\Phi(K)$ contains $\Omega' \setminus A'$ where $A' := \Phi(A \setminus B)$ satisfies $\mathscr{L}^2(A') \le \mathscr{L}^2(A)/2$. This estimate, in combination with (6.3), allows to iterate this construction countably many times, and finally obtain a map Φ such that $\Phi(K)$ contains a non-trivial rectangle. □

REMARK 6.4. - The proof described above is closer to that in [11]. The proof presented in [2] gives a stronger result: the set $\Phi(\Omega \setminus K)$ is one-dimensional and rectifiable, and not just Lebesgue negligible. This proof uses maps $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ that remove at once countable unions of x-strips (or y-strips). Although it does not rely directly on the covering result proved in Theorem 3.1, the basic argument is close in spirit to the proof of Theorem 2.1.

7. Differentiability of Lipschitz maps on the plane

A large part of [2] is devoted to the structure of differentiability sets of Lipschitz maps. In this section, we address one of the basic questions about differentiability of Lipschitz maps, and state the results which have been obtained in dimension two. Some of the results in higher dimension are briefly mentioned in Subsection 8e.

A classical theorem of Rademacher states that a Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^m$ is differentiable $\mathscr{L}^d$-almost everywhere. Thus the question naturally arises, about what happens if the Lebesgue measure $\mathscr{L}^d$ is replaced by a positive measure μ. There are obvious examples of singular measures μ for which Rademacher theorem does not hold: for instance, if μ is the restriction of the Hausdorff measure $\mathscr{H}^k$ to a k-dimensional surface M, then $f(x) := \operatorname{dist}(x, M)$ is differentiable μ-almost nowhere. On the other hand, if μ is absolutely continuous with respect to Lebesgue measure, then every Lipschitz map is differentiable μ-almost everywhere. So the question becomes: are there other measures for which Rademacher theorem holds besides the absolutely continuous ones?

This question can be refined by asking for which sets E in $\mathbb{R}^d$ there exists a Lipschitz map which is nowhere differentiable on E. By Rademacher theorem, all these sets must be Lebesgue-negligible, but is this condition also sufficient?

These two questions can be restated as follows:

QUESTION 7.1. - Weak formulation*: given a singular measure μ in $\mathbb{R}^d$, is there a Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^m$ which is differentiable μ-almost nowhere?*

Strong formulation*: given a null set E in $\mathbb{R}^d$, is there a Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^m$ which is differentiable at no point of E?*

REMARK 7.2. - (i) In the weak formulation, it does not matter whether f is scalar or vector-valued. The reason is the following lemma: given a positive measure μ on $\mathbb{R}^d$ and a sequence of functions $f_n : \mathbb{R}^d \to \mathbb{R}$ which are uniformly Lipschitz, and uniformly bounded at one point, there exist $\alpha_n \in [0, 2^{-n}]$ such that the non-differentiability set of $f := \sum \alpha_n f_n$ agrees with the union of the non-differentiability sets of f_n, up to a μ-negligible subset. In fact, this holds for almost every choice of the coefficients α_n.
(ii) Whether f is scalar or vector-valued does matter for the strong formulation of Question 7.1. Indeed, the third author showed in [15], Corollary 6.5, that there exist null sets E in the plane such that every scalar Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable in at least one point of E (but for the same sets there also exist Lipschitz maps $f : \mathbb{R}^2 \to \mathbb{R}^2$ which are nowhere differentiable on E).

PROPOSITION 7.3. - *The answer to Question 7.1 in the strong formulation is positive for $d = 1$.*

REMARK 7.4. - (i) Proposition 7.3 is an immediate corollary of the following lemma: given a null set E in $\mathbb{R}$, there exists a set of positive and finite Lebesgue measure F with upper density 1 and lower density 0 at every point of E. Then a primitive of 1_F—e.g., $f(x) := \mathscr{L}^1(F \cap (-\infty, x))$—is a Lipschitz function that is not differentiable at any point of E.
(ii) A more precise statement is proved in [19]: a set E in the line is the non-differentiability set of a Lipschitz function if and only if it is a $G_{\delta\sigma}$ set (a countable union of countable intersections of open sets) and has Lebesgue measure zero.

THEOREM 7.5 (see [2]). - *The answer to Question 7.1 in the strong formulation is positive for $d = 2$.*

REMARK 7.6. - Given a null set E, the construction in [2] yields a Lipschitz map $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is not differentiable at each $x \in E$ in the sense—stronger than the usual one—that the directional derivative $D_e f(x)$ does not exist for at least one direction $e \in \mathbb{R}^2$ (depending on x).

Question 7.1 is open for $d \geq 3$, both in the weak and strong formulation.

In the rest of this section we will recall an important class of Lipschitz functions with "large" non-differentiability sets—the distance functions—then describe a direct construction to prove Proposition 7.3, and briefly discuss its extension to dimension two.

DISTANCE FUNCTIONS OF POROUS SETS. - A typical example of non-smooth Lipschitz function on $\mathbb{R}^d$ is the distance function of a closed set E, namely

$$d_E(x) := \operatorname{dist}(x, E) \ .$$

It is not difficult to see that d_E is not differentiable at $x \in E$ if (and only if) there exists a sequence of open balls $B_{r_n}(x_n)$ contained in the complement of E, such that x_n converge to x and $|x_n - x| \leq O(r_n)$. A set E which satisfies this condition at every point is called *porous*; in this case the function d_E is not differentiable at any point of E.

Let μ be a positive measure on $\mathbb{R}^d$, and assume that there are countably many porous sets E_n which cover μ-almost every point. Then there exists a linear combination of the distance functions d_{E_n} which is differentiable μ-almost nowhere (cf. Remark 7.2(i)). Unfortunately, this construction does not settle Question 7.1, because for every $d \geq 1$ there are singular measures μ on $\mathbb{R}^d$ such that every porous set is μ-negligible.[11]

[11] It is not difficult to prove that given a positive measure μ and a point $x \in \mathbb{R}^d$ such that the support of every tangent measure to μ at x is $\mathbb{R}^d$, then x cannot be a μ-density point for any porous set E. On the other hand, there are examples of singular measures μ on $\mathbb{R}^d$ whose tangent measures at x are all multiples of the Lebesgue measure for μ-a.e. x (cf. [14], Example 5.9(1)); hence the set of μ-density points of any porous set E is μ-negligible, which implies that E itself is μ-negligible. For further details on the notion of tangent measure see [14], Chapter 2, or [13], Chapter 14.

A direct construction for $d = 1$. - Let E be a compact null set in $\mathbb{R}$. Then we can find a decreasing sequence of bounded open sets A_n which contain E and satisfy the following property:

$$\mathscr{L}^1(A_n) \le 2^{-n}\mathscr{L}^1(I) \tag{7.1}$$

for every connected component I of A_{n-1} (since E is compact, we can assume that A_{n-1} has only finitely many connected components).

We denote by g_n a primitive of the characteristic function of A_n, and set

$$f_n(x) := \sum_{m=1}^{n} (-1)^{m-1} g_m(x) \quad \text{and} \quad f(x) := \lim_{n\to+\infty} f_n(x) \ . \tag{7.2}$$

Using that the sets A_n are decreasing, it is not difficult to show that each f_n is 1-Lipschitz, and so is the limit f (cf. Figure 3 below).

We claim that f is not differentiable at any $x \in E$. Fix an *odd* integer n, and denote by I the closure of the connected component of A_n which contains x. Then f_n is affine with derivative 1 on I, and therefore for every $y \in I$ there holds

$$\begin{aligned} \frac{f(y)-f(x)}{y-x} &\ge \frac{f_n(y)-f_n(x)}{y-x} - \sum_{m=n+1}^{\infty} \frac{|g_m(y)-g_m(x)|}{|y-x|} \\ &= 1 - \sum_{m=n+1}^{\infty} \frac{\mathscr{L}^1(A_m \cap [x,y])}{|y-x|} \ge 1 - \sum_{m=n+1}^{\infty} \frac{\mathscr{L}^1(A_m)}{|y-x|} \ . \end{aligned}$$

Now, (7.1) implies $\mathscr{L}^1(A_m) \le 2^{-m}\mathscr{L}^1(I)$ for every $m > n$, and choosing $y_n \in I$ such that $|y_n - x| \ge \mathscr{L}^1(I)/2$ we obtain

$$\frac{f(y_n)-f(x)}{y_n-x} \ge 1 - \sum_{m=n+1}^{\infty} \frac{2^{-m}\mathscr{L}^1(I)}{|y_n-x|} = 1 - \frac{2^{-n}\mathscr{L}^1(I)}{|y_n-x|} \ge 1 - 2^{1-n} \ .$$

Thus the upper derivative of f at x is 1.

If n is *even*, the function f_n is affine with derivative 0 on I, and choosing y_n as above we obtain a sequence which shows that the lower derivative of f at x is 0.

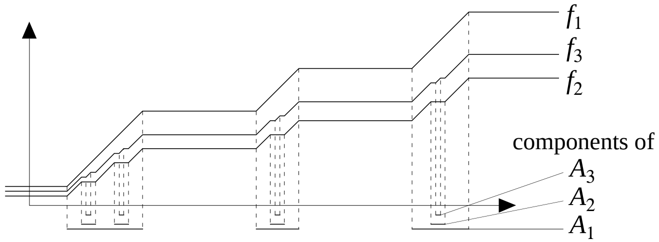

Figure 3

If the set E is not compact, condition (7.1) may not be satisfied by any sequence of open sets A_n. To make the proof work in this case, one has to replace inequality (7.1) by $\mathscr{L}^1(A_n \cap I) \le 2^{-n}\mathscr{L}^1(I)$.

Extension to dimension $d = 2$. - A naïve way to extend the construction in the previous paragraph to the plane would be the following: given a null set E, we write E as $E^x \cup E^y$ as in Theorem 3.1, and construct f for E^x and E^y separately. To construct a Lipschitz function f which is not differentiable on E^x, we take a decreasing sequence of open sets A_n so that each A_n is a union of x-strips which cover E^x and satisfy a suitable counterpart of (7.1); then we define f as in (7.2), where now g_n is a Lipschitz function on the plane whose partial derivative $D_y g_n$ is the characteristic function of A_n. Then f is not differentiable in the y direction at any point of E^x.

There is, however, a serious problem: the partial derivatives $D_x g_n$ are all of order one, but, unlike the partial derivatives $D_y g_n$, do not cancel each other when summed, that is, the partial derivatives $D_x f_n$ may be not uniformly bounded, and f may be not Lipschitz. Since $|D_x g_n|$ is bounded by the Lipschitz constant of the x-strips which cover A_n, this difficulty can be by-passed using strips with smaller and smaller Lipschitz constant; in turn, this requires a suitable refinement of Theorem 3.1, and a careful truncation-and-localization argument. The price to pay is that the resulting function f could still be differentiable at some point of E. However, it is possible to tune the construction parameters so that these "bad" points are μ-negligible with respect to a prescribed singular measure μ, and this suffices to answer the weak formulation of Question 7.1 in the positive.

The construction required for the strong formulation is considerably more complicated.

8. Further results and open problems

In this section we briefly discuss the extension to higher dimension of Theorem 3.1, and of other results from the previous sections. Since many relevant questions are still unanswered even in dimension three, the following discussion will be sometimes restricted to this case.

8a. Covering of finite sets. - As usual, x, y, z denote the coordinates of points in the space. Given $L > 0$, an *x-surface* of constant L in the space is the graph of an L-Lipschitz function $x = x(y, z)$ defined for all $(y, z) \in \mathbb{R}^2$, while an *x-curve* of constant L is the graph of an L-Lipschitz map $(y = y(x), z = z(x))$ defined for all $x \in \mathbb{R}$. The definitions of y- and z-surfaces and curves are the obvious ones.

Proposition 8.1. - *Every set S of n points in the space can be covered by $n^{1/3}$ x-surfaces with constant 1 and $n^{2/3}$ x-curves with constant 1.*

The proof of Proposition 8.1 is a straightforward adaptation of that of Theorem 2.1. However, this result has limited applications, and the generalization of Theorem 2.1 with wider impact would be another:

QUESTION 8.2 (see [11], [12]). - *Are there finite positive constants L, M such that any set S of n points in the space can be covered by $Mn^{1/3}$ x-, y-, or z-surfaces with constant L?*

In [2], we answer this question in the negative. To do this, we first show that a positive answer is *equivalent* to the following statement: there exists a finite constant L such that, for every set S of n points in the space,

$$\max\left\{\sigma_{x,L}(S), \sigma_{y,L}(S), \sigma_{z,L}(S)\right\} \geq n^{2/3} \ , \tag{8.1}$$

where $\sigma_{x,L}(S)$ is the largest number of points of S covered by a single x-surface of constant L, and so on. Now, equality holds in (8.1) if S is the product of three sets S_x, S_y, S_z in $\mathbb{R}$ with same cardinality, and we could show that (8.1) fails for suitable "perturbations" of these product sets.

A weaker, but very interesting version of Question 8.2 is still open:

QUESTION 8.3. - *Are there finite positive constants L, M such that the following holds: for every set S of n points in the space it is possible to choose the coordinate axes so that S can be covered by $Mn^{1/3}$ x-, y-, or z-surfaces with constant L?*

8b. Covering of nulls sets. - Theorem 2.1 implies, via a discretization argument, Theorem 3.1. We can use Proposition 8.1 in the same way and prove the following:

PROPOSITION 8.4. - *Every null set E in the space can be written as $E^x \cup \widehat{E}^x$ where E^x can be covered by δ_i-neighbourhoods of x-surfaces S_i of constant 1 with $\sum \delta_i$ arbitrarily small, and $\widehat{E}^x$ can be covered by η_i-neighbourhoods of x-curves C_i of constant 1 with $\sum \eta_i^2$ arbitrarily small.*

But again, the covering result which would be most useful is another:

QUESTION 8.5. - *Is there a constant L such that every null set E in the space can be covered by δ_i neighbourhoods of x-, or y-, or z-surfaces S_i of constant L so that $\sum \delta_i$ is arbitrarily small?*

A positive answer to this question would imply positive answers to all open questions listed in this paper, with the notable exception of Question 6.1 for $d = 3$, for which this covering result may not be sufficient.

8c. Tangent field to a null set. - We consider here a possible generalization of the notion of weak tangent field to higher dimension. Let E be a set in $\mathbb{R}^d$, τ a map from E into the Grassmann manifold $G(d, d-1)$ of hyperplanes in $\mathbb{R}^d$, and k an integer between 1 and $d-1$. We say that τ is *k-weakly tangent* to E if for every k-dimensional surface S of class $\mathscr{C}^1$ there holds

$$\tau_S(x) \subset \tau(x) \quad \text{for } \mathscr{H}^k\text{-a.e. } x \in S \cap E. \tag{8.2}$$

If τ is k-weakly tangent to E, then it is also h-weakly tangent for h greater than k, but not necessarily for h smaller.[12]

Using Proposition 8.4, we can show that every null sets in $\mathbb{R}^3$ admits a 2-weak tangent field, but we do not know if every null set in $\mathbb{R}^3$ admits a 1-weak tangent field. Of course, this would be the case if Question 8.5 were given a positive answer.

8d. Geometric structure of one-dimensional normal currents. - A one-dimensional normal current in $\mathbb{R}^d$ is an $\mathbb{R}^d$-valued, bounded measure T on $\mathbb{R}^d$ whose distributional divergence is (represented by) a finite measure.[13]

Since T is a bounded measure, it can be written as $T = \tau \cdot \mu$ where μ is a positive measure and τ is an $\mathbb{R}^d$-valued density. It is proved in [2] that if E is a null set and τ_E is a 1-weak tangent field to E (see the previous subsection), then $\tau(x)$ belongs to the hyperplane $\tau_E(x)$ at μ-a.e. $x \in E$. An immediate consequence of this observation, and of the fact that every null set E in $\mathbb{R}^2$ admits a 1-weak tangent field (Theorem 4.3), is the following:

PROPOSITION 8.6. - *Let $T_1 = \tau_1 \cdot \mu_1$ and $T_2 = \tau_2 \cdot \mu_2$ be 1-dimensional normal currents on $\mathbb{R}^2$, and let μ be a positive measure absolutely continuous with respect to μ_1 and μ_2, such that $\tau_1(x)$ and $\tau_2(x)$ span $\mathbb{R}^2$ for μ-a.e. x. Then μ is absolutely continuous with respect to the Lebesgue measure.*

REMARK 8.7. - (i) Since every gradient rotated by $\pi/2$ is a divergence-free vector-field, Proposition 8.6 implies the rank-one property for BV functions on $\mathbb{R}^2$ (cf. Theorem 5.2).
(ii) The following definition of tangent bundle of a positive measure μ on $\mathbb{R}^d$ has been used in the framework of shape optimization problems (see [10], [4], and references therein): given $p \in [1, +\infty]$, the tangent bundle $T^p_\mu(x)$ is the μ-essential span of all vector-fields $v \in L^q(\mu)$ such that the (distributional) divergence of $v \cdot \mu$ belongs to $L^q(\mu)$, where q denotes as usual the conjugate exponent to p. If μ is a singular measure on $\mathbb{R}^2$, then μ is supported on a null

[12] Indeed, the notion k-tangent field is stable under arbitrary modifications of τ in a $\mathscr{H}^k$-negligible set, including h-dimensional surfaces of class $\mathscr{C}^1$, while this clearly not true for the notion h-tangent field.

[13] The usual definition of k-dimensional normal current looks quite different from this one, but turns out to be equivalent for $k = 1$ (for more details see [17], Chapter 6, or [9], Section 4.1). We have not included in this paper the results about general normal currents, because they are too technical.

set E (i.e., $\mu(\mathbb{R}^2 \setminus E) = 0$), and therefore $T^p_\mu(x)$ is contained in the tangent field to E, which exists by Theorem 4.3. In particular $T^p_\mu(x)$ has dimension at most 1 for μ-a.e. x. In the plane, this answers in the positive a questions raised in [10].

It is not known if Proposition 8.6 holds to higher dimension: let $T_i = \tau_i \cdot \mu_i$, $i = 1, 2, 3$, be 1-dimensional normal currents on $\mathbb{R}^3$, and let μ be a positive measure absolutely continuous with respect to all μ_i, such that the vectors $\tau_i(x)$ span $\mathbb{R}^3$ for μ-a.e. x. Is μ absolutely continuous with respect to Lebesgue measure?

The answer would be clearly yes if every null set in $\mathbb{R}^3$ admitted a 1-weak tangent field. This is probably the weakest of all corollaries that a positive answer to Question 8.5 would yield (thus the most interesting to disprove).

8e. Differentiability of Lipschitz maps in higher dimension. - The problem of characterizing those sets E in $\mathbb{R}^d$ such that there exists a Lipschitz map on $\mathbb{R}^d$ which is nowhere differentiable on E (cf. Section 7) has also been solved in [2] for any dimension d. However, the characterization for $d > 2$ is not as simple as that in the planar case; whether it can be simplified or not is an open problem. We begin with a definition:

DEFINITION 8.8. - Given a unit vector e in $\mathbb{R}^d$ and an angle $\alpha \in (0, \pi)$, $C = C(e, \alpha)$ denotes the two-sided closed cone with axis e and amplitude α (cf. formula (4.2)). A set $E \subset \mathbb{R}^d$ is called *C-null* if for every $\varepsilon > 0$ there exists an open set A such that $E \subset A$ and

$$\mathscr{H}^1(A \cap S) \leq \varepsilon$$

for every curve S of class $\mathscr{C}^1$ which satisfies $\tau_S \subset C$ in every point.[14] Finally, we denote by $\mathscr{N}$ the σ-ideal of all sets $E \subset \mathbb{R}^d$ which satisfies the following condition: for every $\alpha < \pi$, E can be covered by countably many sets E_i so that each E_i is C_i-null for some cone C_i with amplitude α.

THEOREM 8.9 (see [2]). - *Given a set $E \subset \mathbb{R}^d$, there exists a Lipschitz map $f : \mathbb{R}^d \to \mathbb{R}^m$, $m \geq d$, which is differentiable at no point of E if and only if $E \in \mathscr{N}$.*

REMARK 8.10. - (i) Theorem 8.9 characterizes the subsets of non-differentiability sets of Lipschitz maps. We do not have a complete characterization of non-differentiability sets.
(ii) The map f constructed in [2] is not differentiable at each $x \in E$ in the sense that there exists at least one direction $e \in \mathbb{R}^d$ of non-differentiability, that is, the directional derivative $D_e f(x)$ does not exist.

[14] It is essential that S is of class $\mathscr{C}^1$ and connected: were we to consider Lipschitz curves, the class of admissible S should be defined more carefully.

(iii) In the construction in [2] we need that $m \geq d$. On the other hand, from Remark 3.2(ii) we know that m cannot be 1 and the results of [6] give a strong indication that m must be at least d.
(iv) Theorem 3.1 shows that every null set $E \subset \mathbb{R}^2$ can be written as $E_1 \cup E_2$ so that each E_i is $C(e_i, \alpha)$-null where $\{e_1, e_2\}$ is any orthonormal base of $\mathbb{R}^2$ and α is any angle such that $\alpha < \pi/2$ (cf. Remark 3.2(ii)). It can be proved with some additional work that E belongs to $\mathscr{N}$. This remark and Theorem 8.9 imply Theorem 7.5.
(v) Theorem 8.9 leaves many questions open. The most important one is: does every null set E in $\mathbb{R}^d$ belong to $\mathscr{N}$? This would be the case if Question 8.5 were given a positive answer. In fact, we do not even know if a set which is C-null for one cone belongs to $\mathscr{N}$.
(vi) If the set E is C-null, then $\mathscr{H}^1(E \cap S) = 0$ for every curve S such that $\tau_S \subset C$ in every point. The converse is true if E is compact, but we do not know if the same holds when E is a G_δ set (countable intersection of open sets); if so, the definition of $\mathscr{N}$ would become significantly simpler.

The notion of non-differentiability of a map f at a point $x \in \mathbb{R}^d$ can be strengthened by requiring more than one direction of non-differentiability. For instance, a natural generalization of Question 7.1 is the following: for which sets $E \subset \mathbb{R}^d$ there exists a Lipschitz map on $\mathbb{R}^d$ which is not differentiable in any direction (i.e., there exists no directional derivatives) at every point of E?

Rademacher theorem in dimension 1 implies that every Lipschitz map on $\mathbb{R}^d$ is differentiable in the direction τ_S for $\mathscr{H}^1$-a.e. point of every curve S of class $\mathscr{C}^1$. Hence such a set E must satisfy $\mathscr{H}^1(E \cap S) = 0$ for every curve S, that is, E is purely unrectifiable (cf. Remark 4.2(iv)). With a little more work one can show that E must be C-null with respect to every cone C (cf. Definition 8.8), what we call a *uniformly purely unrectifiable* (u.p.u.) set. This condition turns out to be also sufficient:

THEOREM 8.11 (see [2]). - *Given a set E in $\mathbb{R}^d$, there exists a Lipschitz function $f : \mathbb{R}^d \to \mathbb{R}$ which is not differentiable in any direction at every point of E if (and only if) E is uniformly purely unrectifiable.*

REMARK 8.12. - A u.p.u. set E is also purely unrectifiable, and the converse is true if E is compact. We do not know if the same holds when E is a Borel set, or even a G_δ set (cf. Remark 8.10 (vi)).

THE MYSTERIOUS VECTOR-FIELD. - Let f be a Lipschitz map on $\mathbb{R}^2$. As pointed out before Theorem 8.11, the set of points where f is not differentiable in any direction is u.p.u. Moreover, it can be proved that the set of points where f admits at least two different directions of differentiability but is not differentiable is u.p.u., too.

This remark and Theorem 7.5 imply the existence, for every null set E in the plane, of a map $\tau : E \to G(2, 1)$ with the following property: every

Lipschitz map $f : \mathbb{R}^2 \to \mathbb{R}^m$ is differentiable in the direction τ at every point of E except a u.p.u. subset. Moreover τ is unique up to a u.p.u. subset of E.[15]

It is not difficult to show that τ must agree with the weak tangent field to E (see Definition 4.1 and Theorem 4.3) except in a p.u. subset of E.[16]

As it happens, the definition of τ came before that of weak tangent field, and since we found this object quite puzzling, we referred to it as the "mysterious vector-field". Were the class of u.p.u. sets strictly contained in that of p.u. sets, the definition of τ would not be equivalent to that of weak tangent field, but—in a still mysterious way—more precise.

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[15] To construct τ, we take a map $\bar{f}$ which is nowhere differentiable on E (Theorem 7.5) and set $\tau(x)$ equal to the direction of differentiability of $\bar{f}$ at x—as remarked above, such a direction exists and is unique for all $x \in E$ except a u.p.u. set. Given any other Lipschitz map f, we know that $(f, \bar{f})$ must be differentiable in at least one direction at every point of E except a u.p.u. set, and since the unique direction of differentiability of $\bar{f}$ is τ, f must be differentiable in the direction τ for all points of E except a u.p.u. subset. The uniqueness of τ follows by the existence of $\bar{f}$.

[16] If not, we could find a curve S of class $\mathscr{C}^1$ and a Lipschitz map f such that f is not differentiable in the direction τ_S for a subset with positive length of $S \cap E$, and this contradicts Rademacher theorem in dimension 1.

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