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BV has the bounded approximation property

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Abstract: We prove that the space $BV(\mathbb{R}^n)$ of functions with bounded variation on $\mathbb{R}^n$ has the bounded approximation property.

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Introduction

Given an integer $n \geq 1$, $BV(\mathbb{R}^n)$ denotes the Banach space of real functions with bounded variation on $\mathbb{R}^n$, namely the functions u in $L^1(\mathbb{R}^n)$ whose distributional gradient Du is a bounded (vector-valued) Radon measure. As usual, $\|u\|_{BV} := \|u\|_1 + \|Du\|$, where $\|Du\|$ is the total variation of the measure Du.

A Banach space F has the *bounded approximation property* if for every $\varepsilon > 0$ and every compact set $K \subset F$ one can find a finite-rank operator T from F into itself with norm $\|T\| \leq C$, where C is a finite constant which depends only on F, so that $\|Ty - y\|_F \leq \varepsilon$ for every $y \in K$ (cf. [4], Definition 1.e.11). In this definition, it is clearly equivalent to consider only sets K which are finite and contained in a prescribed dense subset of F.

We prove the following:

Theorem 1. - *The space $BV(\mathbb{R}^n)$ has the bounded approximation property.*

Remarks. - (i) The operators T given in our proof are projections.

(ii) The result can be extended to the space $BV(\mathbb{R}^n, \mathbb{R}^k)$ of BV functions valued in $\mathbb{R}^k$, as well as the corresponding Banach spaces on over the complex scalars. Moreover, it holds for the space $BV(\Omega, \mathbb{R}^k)$ of BV functions on an open subset Ω of $\mathbb{R}^n$ with the extension property, namely when there exists a bounded extension operator from $BV(\Omega)$ to $BV(\mathbb{R}^n)$. This class includes all bounded open sets Ω with Lipschitz boundary. We do not know if the result holds for Ω an arbitrary open set.

(iii) A careful analysis of the proof of Theorem 1 shows that for every separable subspace X of $BV(\mathbb{R}^n)$ there is a sequence (P_k) of commuting finite rank projections from $BV(\mathbb{R}^n)$ into itself with uniformly bounded norms such that $P_k f \to f$ in $BV(\mathbb{R}^n)$ for every $f \in X$. Moreover there is a separable subspace Y of $BV(\mathbb{R}^n)$ with a Schauder basis which contains X.

Specific aspects of Banach space theory and approximation theory of these spaces have been recently studied in [3], [7] (non linear approximation by Haar polynomial, boundedness of some averaging projections), and [6] (failure of lattice structure). Theorem 1 provides further information on Banach space properties of spaces of functions of bounded variation.

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Proof of the result

Let us fix some notation: $\mathscr{L}^n$ is the Lebesgue measure on $\mathbb{R}^n$ and $\mathscr{H}^d$ is the d-dimensional Hausdorff measure.

Let μ be a positive measure on $\mathbb{R}^n$, D a Borel set, and $f : \mathbb{R}^n \to \mathbb{R}^k$ a function whose restriction to D is μ-summable; if $0 < \mu(D) < \infty$, we denote by $f_{\mu,D} := (\int_D f \, d\mu)/\mu(D)$ the average of f on D with respect to the measure μ. We set $f_{\mu,D} := 0$ when $\mu(D) = 0$. If μ is the Lebesgue measure we simply write f_D. Since no confusion may arise, we denote by 1_A the characteristic function of a set A in $\mathbb{R}^n$ (and not the average of 1 on A).

Given a finite Borel measure λ on $\mathbb{R}^n$, real- or $\mathbb{R}^k$-valued, $|\lambda|$ is the variation of λ, while λ_a and λ_s are the absolutely continuous and the singular part of λ with respect to the Lebesgue measure. Given a positive locally finite measure μ on $\mathbb{R}^n$, $\frac{d\lambda}{d\mu}$ is the Radon-Nikodým derivative of λ with respect to μ; so $\frac{d\lambda}{d\mu}$ is defined even if λ is not absolutely continuous with respect to μ, in which case it is the Radon-Nikodým derivative of the absolutely continuous part of λ with respect to μ. Note that $\lambda \mapsto \lambda_a$ and $\lambda \mapsto \lambda_s$ are bounded linear operators from the space of measures $\mathscr{M}(\mathbb{R}^n)$ into itself, while $\lambda \mapsto \frac{d\lambda}{d\mu}$ is a bounded linear operator from $\mathscr{M}(\mathbb{R}^n)$ into $L^1(\mu)$.

If u is a BV function, we write $D_a u$ and $D_s u$ for the absolutely continuous and the singular part of the measure Du. We denote by $BV_{\text{loc}}(\mathbb{R}^n)$ the class of all functions on $\mathbb{R}^n$ which belong to $BV(A)$ for every bounded open set $A \subset \mathbb{R}^n$; in this case $|Du|$ is a locally finite Borel measure on $\mathbb{R}^n$. We will need the following scaled version of Poincaré inequality (cf. [2], Remark 3.50): for every open cube Q in $\mathbb{R}^n$ and every $u \in BV(Q)$ there holds

$$\int_Q |u - u_Q| \, d\mathscr{L}^n \leq C \operatorname{diam}(Q) \, |Du|(Q) \ . \tag{1}$$

The letter C denotes any constant that might depend only on the dimension of the space n; the actual value of C may change at every occurrence.

LEMMA 2. - *Let $\mathscr{P}$ be a locally finite family of pairwise disjoint open cubes Q in $\mathbb{R}^n$ whose closures cover $\mathbb{R}^n$. If u is a function in $BV_{\mathrm{loc}}(\mathbb{R}^n)$ with average* 0 *on each cube $Q \in \mathscr{P}$ then*

$$\|Du\| \leq C|Du|(\Omega)$$

where Ω is the the union of all $Q \in \mathscr{P}$.

PROOF. - We must show that $|Du|(E) \leq C|Du|(\Omega)$ for $E := \mathbb{R}^n \setminus \Omega$. Since the covering $\mathscr{P}$ is locally finite, E is the union of all boundaries ∂Q with $Q \in \mathscr{P}$. Thus E is an $(n-1)$-rectifiable set, and the restriction of the measure $|Du|$ to E is given by $|Du| \llcorner E = |\mathrm{tr}^+_E(u) - \mathrm{tr}^-_E(u)| \, \mathscr{H}^{n-1} \llcorner E$, where $\mathrm{tr}^+_E(u)$ and $\mathrm{tr}^-_E(u)$ are the traces of u on the two sides of E with respect to some choice of orientation for E, and $\mathscr{H}^{n-1} \llcorner E$ denotes the restriction of the measure $\mathscr{H}^{n-1}$ to the set E (see [2], Theorem 3.77). Hence

$$\begin{aligned}
|Du|(E) &= \int_E |\mathrm{tr}^+_E(u) - \mathrm{tr}^-_E(u)| \, d\mathscr{H}^{n-1} \\
&\leq \int_E |\mathrm{tr}^+_E(u)| + |\mathrm{tr}^-_E(u)| \, d\mathscr{H}^{n-1} \\
&= \sum_{Q \in \mathscr{P}} \int_{\partial Q} |\mathrm{tr}^-_{\partial\Omega}(u)| \, d\mathscr{H}^{n-1} \\
&\leq \sum_{Q \in \mathscr{P}} C \, |Du|(Q) = C \, |Du|(\Omega) \ .
\end{aligned}$$

The inequality in the fourth line is obtained as follows. For every open cube Q with size 1 in $\mathbb{R}^n$ and every $u \in BV(Q)$ there holds

$$\|\mathrm{tr}^-_{\partial\Omega}(u)\|_{L^1(\partial Q)} \leq C\|u\|_{BV(Q)}$$

by the continuity of the trace operator, cf. [2], Theorem 3.87 (here $L^1(\partial Q)$ stands, as usual, for $L^1(\mathscr{H}^{n-1} \llcorner \partial Q)$). If in addition u has average 0 on Q then $\|u\|_{L^1(Q)} \leq C|Du|(Q)$ by (1), and therefore

$$\|\mathrm{tr}^-_{\partial\Omega}(u)\|_{L^1(\partial Q)} \leq C|Du|(Q) \ .$$

Since the last inequality is scaling invariant, it holds for cubes of any size. □

DEFINITION 3. - Let μ be a locally finite, positive measure on $\mathbb{R}^n$, and let $\mathscr{P}$ be a countable family of pairwise disjoint, bounded Borel sets. We denote by Ω the union of all $D \in \mathscr{P}$ and set

$$\sigma := \sup\big\{\mathrm{diam}(D) : \ D \in \mathscr{P}\big\} \tag{2}$$

and, for every map $f \in L^1(\mu, \mathbb{R}^k)$,

$$Uf := \sum_{D \in \mathscr{P}} f_{\mu,D} \, 1_D \ . \tag{3}$$

LEMMA 4. - *(a) The operator U defined in* (3) *is a bounded linear operator from $L^1(\mu, \mathbb{R}^k)$ into $L^1(\mu, \mathbb{R}^k)$ with norm $\|U\| = 1$.*

(b) Given a sequence of coverings $\mathscr{P}_i$, define Ω_i, σ_i, and U_i accordingly. If $\sigma_i \to 0$ as $i \to +\infty$, then $(U_i f - 1_{\Omega_i} f) \to 0$ in $L^1(\mu, \mathbb{R}^k)$ for every f in $L^1(\mu, \mathbb{R}^k)$, that is

$$\sum_{D \in \mathscr{P}_i} \int |f - f_{\mu,D}| \, d\mu \to 0 \ . \tag{4}$$

PROOF. - Statement (a) is immediate. To prove statement (b), we remark that since the operators U_i and the truncation operators $f \mapsto 1_{\Omega_i} f$ are uniformly bounded, it suffices to show that $(U_i f - 1_{\Omega_i} f) \to 0$ for f in a suitable a dense subset of $L^1(\mu, \mathbb{R}^k)$. For instance, we can take f continuous and compactly supported, and then (4) is an immediate consequence of the uniform continuity of f. □

DEFINITION 5. - Fix $\bar{u}$ in $BV(\mathbb{R}^n)$ and set $\mu := |D\bar{u}|$. Given a locally finite family $\mathscr{P}$ of mutually disjoint open cubes Q whose closures cover $\mathbb{R}^n$, we define σ as in (2), and for every $u \in BV(\mathbb{R}^n)$ we set

$$\begin{aligned}
T^1 u(x) &:= \sum_{Q \in \mathscr{P}} u_Q \, 1_Q(x) \\
T^2 u(x) &:= \sum_{Q \in \mathscr{P}} \Big[\frac{dDu}{d\mathscr{L}^n}\Big]_Q \cdot (x - x_Q) \, 1_Q(x) \\
T^3 u(x) &:= \sum_{Q \in \mathscr{P}} \Big[\frac{dD_s u}{d\mu} \cdot \frac{dD\bar{u}}{d\mu}\Big]_{\mu,Q} (\bar{u}(x) - \bar{u}_Q) \, 1_Q(x)
\end{aligned}$$

(here x_Q is the center of Q, i.e., the average of the identity map $x \mapsto x$ over Q, and the dot " $\cdot$ " stands for the usual scalar product in $\mathbb{R}^n$). Finally we set

$$Tu := T^1 u + T^2 u + T^3 u \ . \tag{5}$$

Lemma 6. - *(a) The operator T defined in* (5) *is a bounded linear operator from $BV(\mathbb{R}^n)$ into itself with norm $\|T\| \leq C(1+\sigma)$. If in addition $D_a\bar{u} = 0$, then T is a projection.*

(b) Given a sequence of families $\mathscr{P}_i$ of cubes which satisfy the assumptions of Definition 5, we define σ_i and T_i accordingly, and if $\sigma_i \to 0$ as $i \to +\infty$, then $T_i u \to u$ in $L^1(\mathbb{R}^n)$ for every $u \in BV(\mathbb{R}^n)$. If in addition $D_s u$ is of the form $D_s u = f \cdot D\bar{u}$ with f a scalar function, then $T_i u \to u$ in $BV(\mathbb{R}^n)$.

Proof. - We first prove (a). The following estimates are immediate:

$$\|T^1 u\|_1 \leq \|u\|_1 \tag{6}$$

$$\|T^2 u\|_1 \leq \sigma \|D_a u\| \tag{7}$$

$$\|T^3 u\|_1 \leq C\sigma \|D_s u\| \tag{8}$$

(for the second estimate we use that $|x - x_Q| \leq \operatorname{diam}(Q) \leq \sigma$ and for the third that $\int_Q |\bar{u} - \bar{u}_Q| \leq C\,\sigma\,|D\bar{u}|(Q)$ by estimate (1)). Estimates $(6,7,8)$ imply

$$\|Tu\|_1 \leq \|u\|_1 + C\sigma\|Du\| \ . \tag{9}$$

In order to estimate $\|D(Tu)\|$ we notice that Tu and u have the same average on every cube $Q \in \mathscr{P}$, and therefore we can use Lemma 2 to estimate the total variation of $D(Tu - u)$; denoting by Ω the union of the open cubes $Q \in \mathscr{P}$ we obtain

$$\begin{aligned} \|D(Tu)\| &\leq \|Du\| + \|D(Tu-u)\| \\ &\leq \|Du\| + C\,|D(Tu-u)|(\Omega) \\ &\leq (1+C)\|Du\| + C\,|D(Tu)|(\Omega) \\ &\leq (1+C)\|Du\| + C \sum_{Q\in\mathscr{P}} |D(T^2u)|(Q) + |D(T^3u)|(Q) \ . \end{aligned} \tag{10}$$

Since $T^2 u$ is affine on Q and its gradient is the average of $\frac{dDu}{d\mathscr{L}^n}$ over Q, we have

$$|D(T^2u)|(Q) = \left| \left[\frac{dDu}{d\mathscr{L}^n} \right]_Q \right| \mathscr{L}^n(Q) \leq \int_Q \left| \frac{dDu}{d\mathscr{L}^n} \right| d\mathscr{L}^n \leq |Du|(Q) \ .$$

The gradient of $T^3 u$ on Ω agrees with $D\bar{u}$ times the average of the scalar product of $\frac{dD_s u}{d\mu}$ and $\frac{dD\bar{u}}{d\mu}$ with respect to the measure $\mu = |D\bar{u}|$, and since $\left|\frac{dD\bar{u}}{d\mu}\right| = 1$ μ-almost everywhere,

$$|D(T^3u)|(Q) = \left| \left[\frac{dD_s u}{d\mu} \cdot \frac{dD\bar{u}}{d\mu} \right]_{\mu,Q} \right| \mu(Q) \leq \int_Q \left| \frac{dD_s u}{d\mu} \right| d\mu \leq |Du|(Q) \ .$$

Hence (10) becomes

$$\|D(Tu)\| \le C\|Du\| \ . \tag{11}$$

Estimates (9) and (11) imply the first part of statement (a).

Concerning the second part, assume that $D_a\bar{u} = 0$. Notice that T^1, T^2, T^3 are mutually commuting projections, thus T is a projection, and its range is the direct sum of the ranges of T^1, T^2, T^3, namely the space of all BV functions whose restriction to each $Q \in \mathscr{P}$ agrees with a linear combination of $\bar{u}$ and an affine function.

Let us prove statement (b). Inequality (1) yields

$$\|T_i^1 u - u\|_1 = \sum_{Q\in\mathscr{P}_i} \int_Q |u - u_Q|\, d\mathscr{L}^n \le \sum_{Q\in\mathscr{P}_i} C\sigma_i\, |Du|(Q) \le C\sigma_i\, \|Du\| \ ,$$

and recalling estimates (7), (8) we get

$$\|T_i u - u\|_1 \le \|T_i^1 u - u\|_1 + \|T_i^2 u\|_1 + \|T_i^3 u\|_1 \le C\sigma_i\, \|Du\| \ ,$$

which tends to 0 as $\sigma_i \to 0$. To prove the second part of statement (b), we denote by Ω_i the union of all cubes $Q \in \mathscr{P}_i$, and estimate $\|D(T_i u - u)\|$ using Lemma 2 again and taking into account that $D(T_i^1 u)(Q) = 0$ for every $Q \in \mathscr{P}_i$:

$$\begin{aligned} \|D(T_i u - u)\| &\le C\, |D(T_i u - u)|(\Omega_i) \\ &= C \sum_{Q\in\mathscr{P}_i} |D(T_i u) - Du|(Q) \\ &\le C \sum_{Q\in\mathscr{P}_i} |D(T_i^2 u) - D_a u|(Q) + |D(T_i^3 u) - D_s u|(Q) \ . \end{aligned} \tag{12}$$

Setting $g := \frac{dDu}{d\mathscr{L}^n} = \frac{dD_a u}{d\mathscr{L}^n}$, we have $D(T_i^2 u) = g_Q \mathscr{L}^n$ on each $Q \in \mathscr{P}_i$, and then

$$|D(T_i^2 u) - D_a u|(Q) = \int_Q |g_Q - g|\, d\mathscr{L}^n \ . \tag{13}$$

Since $D_s u = f D\bar{u}$, a simple computation yields $D(T_i^3 u) = f_{\mu,Q} D\bar{u}$ on each $Q \in \mathscr{P}_i$, and then

$$|D(T_i^3 u) - D_s u|(Q) = \int_Q |f_{\mu,Q} - f|\, d|D\bar{u}| \ . \tag{14}$$

Finally $(12, 13, 14)$ yield

$$\|D(T_i u - u)\| \le C \sum_{Q\in\mathscr{P}_i} \left[\int_Q |g - g_Q|\, d\mathscr{L}^n + \int_Q |f - f_{\mu,Q}|\, d\mu \right] .$$

By (4) both sums at the right-hand side of this inequality vanish as $\sigma_i \to 0$, and the proof of statement (b) is complete. $\square$

LEMMA 7. - *Let u_j be a sequence of functions from $BV(\mathbb{R}^n)$. Then there exists $\bar{u} \in BV(\mathbb{R}^n)$ such that $D_a\bar{u} = 0$, and for every j, $D_s u_j$ can be written in the form $D_s u_j = f_j D\bar{u}$ for some scalar function $f_j \in L^1(|D_s\bar{u}|)$.*

PROOF. - This result is essentially contained in [1], but not explicitly stated there. In the following proof, all numbered statements and definitions are taken from [1]; we omit to recall the full statements.

Let $\mu := \sum_j \alpha_j |D_s u_j|$, where the α_j are positive numbers chosen so to make the series converge. Let $E(\mu, x) \subset \mathbb{R}^n$ be the normal space to μ at every $x \in \mathbb{R}^n$ in the sense of Definition 2.3, namely a Borel map which takes every $x \in \mathbb{R}^n$ into a linear subspace $E(x)$ of $\mathbb{R}^n$, satisfies $\frac{dDu}{d\mu}(x) \in E(x)$ for μ-a.e. x and every $u \in BV(\mathbb{R}^n)$, and is μ-minimal with respect to inclusion.

It follows immediately that $\frac{dD_s u_j}{d\mu}(x) \in E(\mu, x)$ for μ-a.e. x and every j, and then $E(\mu, x)$ contains non-zero vectors for μ a.e. x. Then we can choose a Borel map $f \in L^1(\mu, \mathbb{R}^n)$ so that $f(x) \in E(\mu, x)$ and $f(x) \neq 0$ for μ-a.e. x (cf. Proposition 2.11), and since μ is a singular measure, we can also assume that $f(x) = 0$ for $\mathscr{L}^n$-a.e. x. Set now $\mu' := \mu + \mathscr{L}^n$. By Proposition 2.6(iii), $f(x) \in E(\mu', x)$ for μ'-a.e. x, and we can apply Theorem 2.12 to f and μ' to obtain a BV function $\bar{u}$ such that

$$\frac{dD\bar{u}}{d\mu'}(x) = f(x) \quad \text{for } \mu'\text{-a.e. } x.$$

Since $f(x) = 0$ for $\mathscr{L}^n$-a.e. x, then $D\bar{u}$ is singular, and since $f(x) \neq 0$ for μ-a.e. x, then $\mu \ll |D\bar{u}|$. In particular $|D_s u_j| \ll |D\bar{u}|$ for every j.

Finally, by Theorem 3.1 the space $E(\mu, x)$ has dimension at most 1 for every x because μ is a singular measure, and since both $\frac{dDu_j}{d\mu}(x)$ and $\frac{dD\bar{u}}{d\mu}(x)$ belong $E(\mu, x)$, then they must be parallel for μ-a.e. x. $\square$

The following lemma is well-known (see, for instance, [2, Corollary 3.89], [5, Section I.8, Theorem 2], [7 ,Proposition 5]):

LEMMA 8. - *Let Q be any open cube in $\mathbb{R}^n$ with edge-length $r \geq 1$, and let U_Q be the associated truncation operator, that is, $U_Q u := 1_Q\, u$. U_Q is a linear projection on $BV(\mathbb{R}^n)$ with norm bounded by some universal constant C.*

PROOF OF THEOREM 1. - Let be given $\varepsilon > 0$ and a finite set $K = \{u_j\}$ of BV functions with compact support in $\mathbb{R}^n$.

Take $\bar{u}$ as in Lemma 7. Take a sequence of families $\mathscr{P}_i$ of cubes which satisfy the assumptions of Definition 5, so that $\sigma_i \leq 1$ and $\sigma_i \to 0$ as $i \to +\infty$ (cf. (2)), and consider the corresponding operators T_i. By Lemma 6 these operators are projections, their norms $\|T_i\|$ are bounded by a universal constant, and there is i so large that $\|T_i u_j - u_j\|_{BV} < \varepsilon$ for every j.

However, T_i has not finite rank. To solve this problem, we choose an open cube Q which contains the support of all u_j, take U_Q as in Lemma 8, and set $T := U_Q T_i$. If we have chosen Q so that every cube in $\mathscr{P}_i$ which intersects Q is actually contained in Q, then T is a projection, too. □

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