A variational convergence result for functionals of Ginzburg-Landau type in any dimension (*)

Giovanni Alberti

Abstract. – *We describe an approach via Γ-convergence to the asymptotic behaviour of (minimizers of) complex Ginzburg-Landau functionals in any space dimension, summarizing some results of a joint research with S. Baldo and G. Orlandi [3], [4].*

Summary. – The asymptotic behaviour of solutions of certain variational elliptic equations depending on a parameter, or, better, of the minimizers of the associated energy functionals, can be studied by defining a suitable limit variational problem, e.g. using the notion of Γ-convergence. In collaboration with S. Baldo (**) and G. Orlandi (***) (cf. [3], [4]), we have followed this approach to study the asymptotics of minimizers of functionals of Ginzburg-Landau type in any dimension, namely

$$F_\varepsilon(u) := \int |\nabla u|^2 + \frac{1}{\varepsilon^2} W(u),$$

where u is defined on a regular domain in $\mathbb{R}^n$ and takes values in $\mathbb{R}^2$, and $W(u)$ is a positive potential vanishing when $|u| = 1$ only. We have obtained that in the limit $\varepsilon \to 0$ the minimizers of F_ε, or better the corresponding Jacobians, converge in a suitable sense to a minimal surface of codimension two.

The asymptotics of these functionals has been been object of extensive research in recent years, a from the results of F. Bethuel, H. Brezis, and F. Hélein in dimension two (cf. [9]), to the recent results of T. Riviére, F.-H. Lin, E. Sandier and others in higher dimension (cf. [21], [25], [26]). One of the relevant features of the variational approach we follow is the almost immediate reduction of the problem to two space dimensions, which underlines the essentially bidimensional nature of this problem. From this viewpoint,

(*) Note presented at the XVI Congress of the Italian Mathematical Union. English version of the paper "Un risultato di convergenza variazionale per funzionali di tipo Ginzburg-Landau in dimensione qualunque", appeared on *Boll. Un. Mat. It. B (8)*, 4 (2001), 289-310.

(**) Dipartimento di Matematica, Università della Basilicata, 85100 Potenza.

(***) Dipartimento Scientifico e Tecnologico, Università di Verona, 37100 Verona.

it is important to keep in mind the analogy with scalar Ginzburg-Landau functionals, or Cahn-Hilliard functionals, the equivalent of our convergence result in that case being the well-known theorem of L. Modica and S. Mortola (cf. [22], [23], [24]). Variational convergence results similar to ours have been also the object of an independent research by R. Jerrard and H.M. Soner [18].

THE BASIC PROBLEM. – We are interested in the asymptotic behaviour in the limit $\varepsilon \to 0$ of minimizers u_ε of

$$F_\varepsilon(u) := \int |\nabla u|^2 + \frac{1}{\varepsilon^2} W(u) \tag{1}$$

where

- $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^2$, $n \geq 2$, and $|\nabla u|$ is the euclidean norm of the matrix ∇u,
- $W(u)$ is a positive potential which vanishes for $|u| = 1$ only.

Notice that F_ε vanishes for all constant functions of modulus one, and therefore we get non-trivial minimizers only under some additional constraint, typically on the boundary values. However, it is usually assumed that the actual nature of these constraints is essentially unrelevant, at least for the qualitative features of solutions we are looking for.

SOME VARIANTS. – The above mentioned problem admits many variants and generalizations, not all of which can be directly reduced to the original problem. Among these, I would like to mention the following:

- $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^k$ with $n \geq k$, and $F_\varepsilon(u)$ is given by (1)

$$F_\varepsilon(u) := \int |\nabla u|^p + \frac{1}{\varepsilon^2} W(u); \tag{2}$$

- F_ε is defined for sections of fiber bundles on a manifold, instead of functions on a domain of $\mathbb{R}^n$ (cf. [6], [25]);
- the null set of W has a more complicated topology than the sphere S^{k-1};
- some additional terms appears in (1) (as for most functionals derived by physical models);
- the term $\int |\nabla u|^2$ in (1), or $\int |\nabla u|^p$ in (2), is replaced by an anisotropic integral of the gradient, or by a non-local term (interaction energy) like

$$\iint J(x'-x)|u(x') - u(x)|^2 \, dx' \, dx, \tag{3}$$

suitably rescaled in ε.

(1) More precisely, for $p = 2$ and $k = 1$ we obtain the so-called scalar G.-L. functionals, or Cahn-Hilliard functionals, while for $p = 2$, $k = 2$ we obtain the complex G.-L. functionals, namely those given in (1).

MOTIVATIONS. – There are two different orders of reasons to consider this kind of singular perturbation problems. From a strictly mathematical viewpoint, it has been observed long ago that scalar G.-L. functionals, namely those defined in (2) for $k = 1$ and $p = 2$, approximate the perimeter functional, and the corresponding minimizers converge to sets of minimal perimeter (see Theorem 3 and following remarks). By analogy, one would expect that for arbitrary k the functionals in (2) approximate, in some sense, the area functionals for surfaces of codimension k, that is, of codimension two in the case of the complex G.-L. functionals in (1). A rigorous proof of this statement is the main goal of our research.

From a different viewpoint, minimizers of (1) in dimension two with prescribed boundary value g can be used to study harmonic maps from Ω into S^1 when there exist no maps from Ω into S^1 with trace g on the boundary and finite energy (2).

Finally, functionals of type (1) were used to model phase separation in certain fluids (cf. [11] (3)), phase transitions in superconductors or in certain superfluids (cf. [13], [14] (4), see also [15] for a more mathematical viewpoint), and several other physical phenomena as well (5).

(2) For instance, when Ω is the unit disk B^2 in the plane and the degree of $g : \partial B^2 = S^1 \to S^1$ is not 0. The obstruction is topological: the maps in the Sobolev space $W^{1,2}(B^2, S^1)$, although not necessarily continuous, behave like continuous ones under many aspects. The reason is that $p = 2$ is the critical exponent for Sobolev immersion in dimension two; indeed the picture changes completely for maps in $W^{1,2}(B^3, S^2)$, like, for example, $x/|x|$. For this and related questions, see [10], [8].

(3) In this case u is a scalar order parameter which represents the relative density of the two phases of the fluid at every point, so that the values $+1$ and -1 correspond to the presence of only one phase (pure states). Thus the part of the energy F_ε given by the potential W prefers the pure states ± 1 (phase separation), while the gradient term penalizes the variation of u (surface tension); the energy is minimized keeping the volumetric fraction of the phases fixed, that is, prescribing the average of u (not the boundary values).

(4) In both cases u is a complex order parameter; instead of a constraint on the boundary values, there is an additional term in the energy, which corresponds in the second case to the energy contribution due to an applied magnetic field. The behaviour of minimizers of these functionals cannot be really deduced form that of minimizers of (1), but has been studied by similar methods.

(5) Functionals like $\int |\nabla\nabla u|^2 + \varepsilon^{-2} W(\nabla u)$ have been recently proposed in a model for "blisters" induced in thin films by compression of the substrate;

In all these cases the parameter ε represents a length with a precise physical meaning, usually very small with respect to all other relevant parameters; for this reason it makes sense to study the behaviour of minimizers in the limit $\varepsilon \to 0$ instead of a fixed ε.

A SIMPLE COMPUTATION IN THE SCALAR CASE. – Now I present a simple computation which gives a certain understanding of the behaviour of minimizers of F_ε. We begin with the scalar G.-L. functionals, subject to the volume constraint $\fint u = m$, where m is a fixed number between -1 and 1 (cf. Note 3).

In this case, the second term in F_ε prefers functions u with values close to ± 1, while the first term penalizes the variation of u. For small ε, the second term prevails, and the minimizer u_ε will take values close to ± 1. On the other hand, both the phase $\{u_\varepsilon \simeq +1\}$ and the phase $\{u_\varepsilon \simeq -1\}$ are not empty because of the constraint on the average of u, and are separated by a thin layer T where u makes the transition of from -1 to $+1$.

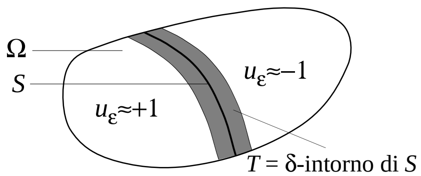

FIGURE 1

Assuming that $W(u_\varepsilon)$ is negligible out of T, T is a δ-neighborhood of an hypersurface S depending on ε (cf. Fig. 1), and thus the gradient of u_ε is of order $1/\delta$ in T, we can give a rough estimate of $F_\varepsilon(u_\varepsilon)$:

$$\int |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2} W(u_\varepsilon) \sim \Big(\frac{1}{\delta^2} + \frac{1}{\varepsilon^2}\Big) \cdot \mathrm{vol}(T) \sim \Big(\frac{1}{\delta^2} + \frac{1}{\varepsilon^2}\Big) \delta \cdot \mathrm{area}(S)$$

(where “vol” and “area” stand for the n- and $(n-1)$-dimensional measures, respectively). The last term can be optimized by taking $\delta \sim \varepsilon$, so that

$$F_\varepsilon(u_\varepsilon) \sim \frac{1}{\varepsilon} \mathrm{area}(S). \tag{4}$$

We deduce that for $\varepsilon \to 0$ the transition layer of the minimizer u_ε has thickness of order ε, while S minimizes the area among all admissible surfaces, namely those which divide Ω into two parts Ω^+ and Ω^- which satisfy the volume

---

there are moreover strong similarities with the energies used to model certain liquid crystals, or in micromagnetics, in particular for ferromagnetic films (cf. [19]).

constraint $\mathrm{vol}(\Omega^+) - \mathrm{vol}(\Omega^-) = m\,\mathrm{vol}(\Omega)$. In other words, the minimizers u_ε converge to a function u with values ± 1 which satisfy the volume constraint $\fint u = m$, and minimizes the area of the interface between the phases $\{u = +1\}$ and $\{u = -1\}$.

Even if obtained through very rough approximations, this conclusion is essentially correct, and can be easily confirmed by a more accurate formal asymptotic expansion (6). However, even if more accurate, this methods is still based on some a priori assumption (e.g., that the transition layer T is more or less a δ-neighborhood of some regular surface S). On the other hand, a rigorous proof of this statement was given by L. Modica and S. Mortola (cf. [22], [23], [24]), who found the limit in the sense of Γ-convergence of the functionals F_ε.

Γ-CONVERGENCE AND MODICA-MORTOLA THEOREM. – It is now the right moment to briefly recall recall the notion of Γ-convergence, in a suitably simplified version (cf. [2], see [12] for a more detailed exposition).

DEFINITION 1. *Let X be a metric space, and $F_\varepsilon : X \to [0, +\infty]$ a sequence*

---

(6) Indeed formula (4) suggest an expansion $F_\varepsilon(u_\varepsilon) = \varepsilon^{-1}\Psi(S) + o(\varepsilon^{-1})$, where $\Psi(S)$ is of the order of the area of S. This does not necessarily imply that $\Psi(S)$ agree up to a constant factor with area(S)—even though this looks extremely reasonable for isotropy reasons—and therefore it is not correct to deduce the minimality of S from this expansion. The lack of accuracy in (4) is essentially due to the fact that we replaced $|\nabla u|$ by a rough estimate of the Lipschitz constant of u inside the transition layer T. We can obtain a more precise expansion assuming that u_ε is of the form $u_\varepsilon(x) = \phi(\varepsilon^{-1} d_S(x))$, where $\phi : \mathbb{R} \to [-1, 1]$ is an unknown function with limits ± 1 at $\pm\infty$, d_S is the oriented distance from an unknown surface S, while the term ε^{-1} takes into account the fact that the thickness of the transition layer T is of order ε. Then

$$F_\varepsilon(u_\varepsilon) = \frac{1}{\varepsilon}\Big[\int_{\mathbb{R}} |\phi'|^2 + W(\phi)\Big] \cdot \mathrm{area}(S) + O(1).$$

The minimum of the integral between square brackets is achieved when ϕ agrees up to translations with the solution of the Cauchy problem $\phi(0) = 0$, $\phi' = W^{1/2}(\phi)$—namely the Euler-Lagrange equation $2\phi'' = W'(\phi)$, suitably integrated—and is equal to $\sigma := 2\int_{\mathbb{R}} W^{1/2}(\phi)\,\phi' = 2\int_{-1}^{1} W^{1/2}$; hence

$$F_\varepsilon(u_\varepsilon) = \frac{\sigma}{\varepsilon} \mathrm{area}(S) + O(1).$$

This particular ϕ is called *optimal profile* for the phase transition, and plays a fundamental rôle also in the proof of Theorem 3 (cf. [2]).

of lower-semicontinuous functions. We say that F_ε Γ-converges to F when the following conditions are verified (7)*:*

- $\forall x \in X$, $\forall x_\varepsilon \to x$ *there holds* $\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \geq F(x)$*;*
- $\forall x \in X$, $\exists x_\varepsilon \to x$ *such that* $\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = F(x)$*.*

Moreover, we say that the functions F_ε are equi-coercive when the following compactness condition is verified:

- *if $F_\varepsilon(x_\varepsilon) \leq C < +\infty$, then the sequence (x_ε) is pre-compact in X.*

One easily derive from this definition the following fundamental property of Γ-convergence.

PROPOSITION 2. *If $F_\varepsilon \xrightarrow{\Gamma} F$ on X, also the corresponding minimal values (or the infima of values) converge. If in addition x_ε is a minimizer of F_ε for every ε, and x is a limit point of x_ε, then x is a minimizer F.*

Hence the asymptotic behaviour of minimizers of a sequence of functionals F_ε (on some function space X) can be at least partly understood by mean of the Γ-limit of F_ε. It must be pointed out that the knowledge of the Γ-limit gives some information on the behaviour of minimizers only if they are pre-compact in X. This requirement plays an essential rôle in the choice of the topology on X, and consequently also on the shape of the Γ-limit (8), and explains the importance, together with Γ-convergence, of the compactness condition given in Definition 1.

Finally, the Γ-limit F must be non-trivial: if it is, for instance, identically equal to some constant, then every point would be a minimizer, which gives no information on the limit points of minimizers of F_ε. To this regard, notice that scaling the functionals F_ε by suitably chosen positive constants (depending on ε) does not affect minimizers, but can change significantly the Γ-limit. One typically looks for a rescaling such that the minimal values converge neither to 0 nor to $+\infty$.

Going back to scalar G.-L. functionals, in the previous paragraph we obtained an expansion like $F_\varepsilon(u) \sim \varepsilon^{-1} \cdot \text{area}(S)$ (cf. formula (4) and Note 6); this suggests that, provided we rescale F_ε by a factor ε, the Γ-limit is finite

(7) Called respectively Γ-liminf and Γ-limsup inequalities. Together, they are equivalent to the convergence in the sense of Kuratowski of the epigraphs of F_ε, viewed as closed sets in $X \times \mathbb{R}$.

(8) For instance, let ϕ be a periodic function on $\mathbb{R}$ bounded between two positive constants, then the Γ-limit of $F_\varepsilon(u) := \int_0^1 \phi(t/\varepsilon)|\dot u|^2 dt$ on the space $W^{1,2}_g(0,1)$ of Sobolev functions with boundary values g depends on whether the metric on $W^{1,2}_g(0,1)$ is induced by the $W^{1,2}$-norm or by the L^2-norm. Yet, only the second one ensures the required compactness of minimizers.

only on those functions $u =$ such that $|u| = 1$ a.e., and in that case is proportional to the area of the interface between the phases $\{u = +1\}$ and $\{u = -1\}$, that is, the area of the singular set Su. Indeed the following theorem holds (cf. [22], [23], [24]):

THEOREM 3. *The rescaled functionals $\varepsilon F_\varepsilon$ are equi-coercive on $L^1(\Omega)$, and Γ-converge to*

$$F(u) := \begin{cases} \sigma \cdot \text{area}(Su) & \text{if } |u| = 1 \text{ a.e.,} \\ +\infty & \text{otherwise,} \end{cases} \tag{5}$$

where σ is the constant $2\int_{-1}^1 W^{1/2}$ (cf. Note 6).

It follows from this result (9) and Proposition 2 that the minimizers of $F_\varepsilon$under the usual volume constraint converge to minimizers of F, that is, functions u with values ± 1 which satisfy the volume constraint $⨍ u = m$ and minimize the area of the interface between the phases.

REMARK 4. A comment is required at this point: as F is formulated in (5), it is well-defined on functions u with values ± 1, the singular set of which is a sufficiently regular surface of codimension one. On the other hand, our variational approach is based on compactness in L^1, and therefore assumes that F is defined for every measurable function $u : \Omega \to \pm 1$; but it is not that clear what the area of the singular set of such a function should be, or, equivalently, what the perimeter of a generic measurable set (like $\{u = +1\}$) should be (10). In this case, the right concept is the perimeter in the sense of Caccioppoli of $\{u = +1\}$ within Ω, that is, half of the total variation $\int |Du|$ of the distributional derivative of u. This also corresponds to the (Hausdorff) $(n-1)$-dimensional measure of the set of *essential* singularities of u, namely the points where u has no approximate limit.

(9) More precisely, from the same Γ-convergence statement on the space X of all $u \in L^1(\Omega)$ which satisfy the volume constraint $⨍ u = m$. Notice that this is not an immediate corollary of Theorem 3, since in general Γ-convergence is not inherited to subspaces (the Γ-limsup inequality may no longer holds). However, in this case switching from one space to the other requires only minimal modifications in the proof.

(10) Indeed, whichever notion of $(n-1)$-dimensional measure we adopt, if we define the perimeter of a set in $\mathbb{R}^n$ simply as the $(n-1)$-dimensional measure of the topological boundary, there would exist sets with infinite perimeter which can be approximated by regular sets with uniformly bounded perimeters (for instance, a disjoint union of balls B_i with radius 2^{-i} the topological boundary of which has positive Lebesgue measure). In other words, this perimeter is not lower semicontinuous (e.g., with respect to the L^1 metric), while we know that F must be lower semicontinuous, as every Γ-limit.

A SIMPLE COMPUTATION IN THE COMPLEX CASE. – We consider now the minimizers of complex G.-L. functionals when Ω is a simply connected domain in the plane, and the boundary values are given by a prescribed smooth function $g : \partial\Omega \to S^1$ of degree $d > 0$. For small ε, the second term in F_ε forces the minimizers u_ε to take values close to 1 in modulus. On the other hand, for any positive $\rho < 1$, the set T where $|u_\varepsilon| \leq \rho$ cannot be empty: if it were, the map $u_\varepsilon/|u_\varepsilon|$ would be well-defined and continuous on Ω, and once superimposed to an homotopy valued in Ω which takes $\partial\Omega$ to a point (which must exist, because Ω is simply connected),it would produce an homotopy of g to a constant (as maps from $\partial\Omega$ into S^1), thus violating the invariance of degree under homotopy ([11]).

Since the rôle of T is, in a sense, to make $\Omega \setminus T$ not simply connected (but not necessarily disconnected, as it happens in the scalar case), it is reasonable to assume at first that T is the union of a certain number of disks T_i with radius δ_i on the boundary of which the degree of u_ε is d_i (cf. Fig. 2). Clearly $\sum d_i = d$.

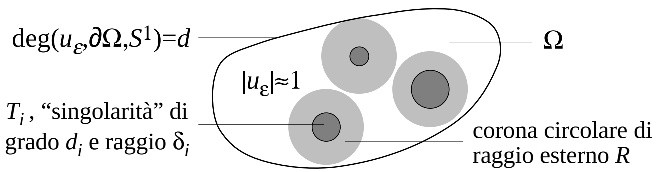

FIGURE 2

To estimate the value of $F_\varepsilon(u_\varepsilon)$ we assume that $|\nabla u_\varepsilon|$ is of order $1/\delta_i$ within each T_i, and that the contribution of $W(u_\varepsilon)$ in $\Omega \setminus T$ is negligible. Since the restriction of u_ε to any circle of radius r concentric with T_i is an S^1-valued map of degree d_i, the integral of the square of its tangential derivative must be larger than $2\pi d_i^2/r$. Therefore we take the maximal R such that the disks of radius R concentric with each T_i are disjoint and contained in Ω (cf. Fig. 2), and get

$$F_\varepsilon(u_\varepsilon) \sim \sum_i \pi\delta_i^2\Big(\frac{1}{\delta_i^2} + \frac{1}{\varepsilon^2}\Big) + \int_{\delta_i}^R \frac{2\pi d_i^2}{r}\,dr \sim \pi \sum_i 1 + \frac{\delta_i^2}{\varepsilon^2} - 2d_i^2 \log\delta_i .$$

Each addendum in the last term can be optimized by taking $\delta_i \sim \varepsilon$, so that

$$F_\varepsilon(u_\varepsilon) \sim 2\pi|\log\varepsilon| \sum_i d_i^2; \tag{6}$$

finally, the sum $\sum d_i^2$ can be optimized with respect to the constraint $\sum d_i = d$ by taking $d_i = 1$ for every i. We infer that in the limit $\varepsilon \to 0$ the transition

([11]) The classical formulation of this argument requires the continuity of u_ε, but it works as well for maps of class $W^{1,2}$ (cf. Note 2).

set of u_ε consists of d "singularities" of degree 1 and radius of order ε. In other words, the minimizer u_ε converge to functions $u : \Omega \to S^1$ which are smooth (harmonic) out of d singular points of degree 1.

Even if these conclusions have been drawn on the basis of very rough estimates ([12]), they are essentially correct; a detailed analysis of the behaviour of this minima, including rigorous proofs of these and other statements ([13]), was provided in the fundamental monograph by F. Bethuel, H. Brezis, and F. Hélein [9], to which the reader is also referred for an extensive bibliography on the subject.

PASSING TO HIGHER DIMENSION. – We consider now the three-dimensional case. Given a convex domain Ω, one typically consider minimizers of F_ε with prescribed boundary values g_ε, where $g_\varepsilon : \Omega \to B^2$ are smooth functions with modulus equal to 1 out of a finite number of "singularities" U_i of radius ε, with center x_i and degree $d_i = \pm 1$ independent of ε (since $\partial\Omega$ is simply connected, the number of $+1$ singularities must equals that of -1's).

Proceeding as in the previous paragraph, one quickly convince himself that for small ε the minimizers u_ε have modulus close to 1 out of a transition set T which looks approximately like an ε-neighborhood of a 1-manifold S (a finite union of oriented curves) the boundary of which corresponds to the points x_i (cf. Fig. 3), and

$$F_\varepsilon(u_\varepsilon) \sim 2\pi|\log\varepsilon| \cdot \mathrm{length}(S) \tag{7}$$

(for the time being we do not consider the multiplicity which may arise in connection with the degree of "winding" of u around S).

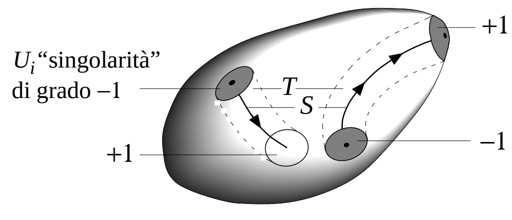

FIGURE 3

([12]) However, the expansion in (6), unlike the one in (4), is extremely precise; in fact, one could show that $F_\varepsilon(u_\varepsilon) = 2\pi|\log\varepsilon| \sum d_i^2 + O(1)$. This surprising accuracy is due to the fact that the energy does not concentrate in the transition set—that is, the disks T_i, where we used very drastic estimates—but rather in the annuli around, where the estimates we used turn out to be very precise. Thus in the complex case W does not appear in the expansion (6), and for much the same reason there is not optimal profile to be found (cf. Note 6).

([13]) Cf. the comments following Theorem 10.

Hence, we expect that u_ε converge (up to subsequences) to functions $u : \Omega \to S^1$ smooth (harmonic) out of a one-dimensional singular set S which minimizes the length among all 1-manifolds with boundary given by the points x_i (with multiplicity taken into account).

Consequently one can't help conjecturing that in any dimension the minimizers u_ε of the complex G.-L. functionals (under suitable boundary constraints) converge to functions $u : \Omega \to S^1$ smooth (harmonic) out of a codimension-two singular set S which minimizes the $(n-2)$-dimensional measure.

Indeed a result in this direction has been proved by F.H. Lin and T. Rivière [21] (cf. also [25], [26]). The approach followed in this paper, like for almost all papers on complex G.-L. functionals, consists essentially of a direct analysis of the behaviour of minimizers u_ε.

THE VARIATIONAL APPROACH. – In collaboration with S. Baldo and G. Orlandi, we have followed a different way, based on the analysis of the asymptotic behaviour of the functionals F_ε rather than that of minimizers—that is, we have given a theorem *à la* Modica-Mortola for the complex G.-L. functionals (14). If several are the analogies between Γ-convergence in the complex and in the scalar case, the differences are at least as many, and unfortunately the relatively simple statement and proof of the Modica-Mortola theorem are mirrored in the complex case by a less effective statement (compare Theorems 9 and 10 to Theorem 3) together with a definitely more complicated proof—which perhaps partly explains the over twenty years since the first papers by Modica and Mortola [22], [23].

In the following I will try to describe the main difficulties in giving a correct formulation of the Γ-convergence theorem for complex G.-L. functionals, and briefly sketch the crucial points of the proof. However, I will first quickly comment upon our methodological choice.

It is worth noticing that once compactness and lower bound have been proved in dimension two, they can be extended to higher dimension by a very general "slicing" method; this underlines the intrinsically bidimensional nature of this problem (see [2] for a description of this method for scalar G.-L. functionals). On the other hand, no dimension-reduction principle (by slicing or whatever) can be proved just for minimizers. More generally, unlike the direct analysis of minimizers, which is usually based on very sharp and specific elliptic estimates, the study of the Γ-limit of the functionals makes only use of estimates which are very general, and under many regards more elementary. For this reason proofs are less dependent on the specific form of

(14) The same approach has been proposed independently by R. Jerrard and H.M. Soner [18].

the functional, and can be (more or less easily) adapted to different variants of the original problem.

If Γ-convergence results are in a sense more "robust" than those obtained by a direct analysis of minimizers, yet they are also inevitably weaker; for instance, the compactness properties of minimizers is usually stronger than what can be deduced by the equi-coercivity of functionals (cf. Note 22).

OPTIMAL RESCALING. – Prior to any Γ-convergence result is the choice of optimal rescaling for the functionals under consideration. Estimates (6) and (7) clearly suggest for F_ε the rescaling $|\log\varepsilon|^{-1}$. Therefore we define F_ε anew as:

$$F_\varepsilon(u) := \frac{1}{|\log\varepsilon|}\int |\nabla u|^2 + \frac{1}{\varepsilon^2}W(u), \tag{8}$$

with $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^2$, as usual.

IDENTIFYING THE Γ-LIMIT. – We already know that the Γ-limit F should be finite only for functions $u : \Omega \to S^1$, and in this case should agree up to a factor 2π, with the $(n-2)$-dimensional measure of the singular set Su (15), counted with the right multiplicity. Thus F is clearly defined for functions u which are smooth out of a codimension-two surface, but since this class is not large enough to prove a compactness result for the minimizers u_ε, we face a problem similar that already discussed in the scalar case (cf. Remark 4), namely, which is the appropriate notion of measure of the singular set for maps from Ω into S^1. At least in dimension two, a solution is provided by the *distributional* Jacobian (cf. [7]).

DEFINITION 5. *Given a bounded function $u = (u^1, u^2) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ of class $W^{1,1}$, the Jacobian of u is the distribution*

$$Ju := \frac{\partial}{\partial x_1}\Big(u^1\frac{\partial u^2}{\partial x_2}\Big) - \frac{\partial}{\partial x_2}\Big(u^1\frac{\partial u^2}{\partial x_1}\Big). \tag{9}$$

One easily verify that Ju agrees with the usual $\det(\nabla u)$ for every function of class C^2, and indeed the same holds even for functions of class $W^{1,2}$. On the other hand, for functions of class $W^{1,1}$, the determinant may be not well-defined, while Ju always is, at least as a distribution; typical example is the function $x/|x|$, which belongs to $W^{1,p}_{\rm loc}(\mathbb{R}^2)$ for every $p < 2$, and has Jacobian equal to the Dirac mass $\pi \cdot \delta_0$.

(15) The singular set cannot be empty because of the boundary conditions imposed on F_ε, which in the limit $\varepsilon \to 0$ become a boundary constraint on Su.

REMARK 6. If $u : \Omega \to S^1$ is smooth out of a finite singular set $S = \{x_i\}$ ([16]), then Ju is a measure of the form

$$Ju = \sum_i \pi d_i \cdot \delta_{x_i}, \tag{10}$$

where d_i is the degree of the restriction of u to any circle which contains x_i and no other point of S. This yields the following general formula: given a regular domain A relatively compact in Ω and such that $\partial A \cap S = \emptyset$

$$\int_A Ju = \pi \deg(u, \partial A, S^1) = \pi \operatorname{link}(\partial A, S), \tag{11}$$

where $\operatorname{link}(\partial A, S)$ denotes the winding number of the curve ∂A around the set of all points x_i, counted with multiplicity d_i. Conversely, for a function $u : \Omega \to S^1$ such that Ju is a finite measure, Ju can re written as in (10) for suitable $x_i \in \Omega$ and $d_i \in \mathbb{Z}$. Formula (11) still holds, provided that the middle term is correctly interpreted.

This remark suggests that a good candidate for the Γ-limit $F(u)$ in dimension two, is, up to a constant factor, $\|Ju\|$, that is, the mass of Ju when it is a finite measure, and $+\infty$ otherwise. This functional is indeed lower semicontinuous ([17]) and agrees for sufficiently regular maps with the total number of singularities, (counted with multiplicity).

Definition 5 can be generalized to higher dimension introducing the notions of form and current—cf. [27], Chapter 6 ([18]).

([16]) Like, for instance, $u(x) := \prod_i (x - x_i)/|x - x_i|$, where the product is induced by the identification of $\mathbb{R}^2$ and the complex field.

([17]) With respect to the weak convergence in $W^{1,1}$; indeed it immediately follows from (9) that Ju is a weakly continuous operator from $W^{1,1}$ into the space of distributions.

([18]) k-currents generalize oriented k-surfaces in much the same way distributions generalize functions. In fact, they are usually defined as the dual of the space of k-forms of class C_c^∞, the action of a regular oriented k-surface on such a form being given by integration. Are called integral all currents T of the form

$$\langle T; \omega \rangle := \int_S \xi \cdot \langle \omega; \tau \rangle \cdot d\mathscr{H}^k,$$

where S is a k-dimensional rectifiable set, i.e., it is covered by countably many k-surfaces of class C^1, τ is an orientation of S, namely a unitary (simple) k-vector which identifies the tangent space to S at (almost) every point of S, ξ is an integer multiplicity, and finally $\mathscr{H}^k$ denotes the (Hausdorff) k-dimensional

DEFINITION 7. *For a bounded function $u = (u^1, u^2) : \Omega \subset \mathbb{R}^n \to \mathbb{R}^2$ of class $W^{1,1}$, the two-dimensional Jacobian is the 2-form (with distributional coefficients)*

$$Ju := d(u^1 \cdot du^2) \tag{12}$$

Moreover, we denote by $\star Ju$ the $(n-2)$-current without boundary ([19]) which is obtained by the canonical identification $\star$ of 2-covectors and $(n-2)$-vectors in $\mathbb{R}^n$ ([20]).

The class of all functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^2$ such that Ju is a Radon measure has been introduced by R. Jerrard and H.M. Soner in [17], and called $B2V(\Omega, \mathbb{R}^2)$ by analogy with the space $BV(\Omega)$ of functions with bounded variation. In this context, S^1-valued functions play a rôle similar to finite perimeter sets within BV theory. In particular there holds a rectifiability result, analogous to De Giorgi's theorem for finite perimeter sets: given $u : \Omega \to S^1$ of class $B2V$, Ju is supported on a rectifiable set of codimension two, and more precisely $\star Ju$ is, up to a factor π, an integral current without boundary in Ω—cf. Notes 18 and 19.

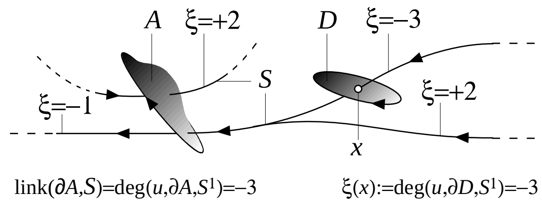

FIGURE 4

REMARK 8. For a better understanding, it is convenient to see what happens in dimension three. In this case $\star Ju$ agrees with the vector product

measure. Within currents, integral ones represent in a sense the closure of regular surfaces (more precisely, of polyhedral chains). If the current T is represented by a measure (valued in k-vectors), its total variation $\|T\|$ is called mass; for integral currents it is just the k-dimensional measure of S, counted with the multiplicity ξ

([19]) As the distributional derivative is defined by the integration-by-parts formula, so the boundary of a k-current T is defined via Stokes theorem, namely by setting $\langle \partial T; \omega \rangle := \langle T; d\omega \rangle$ for every $(k-1)$-form ω of class C_c^∞. In this case $\partial(\star Ju) = 0$ because $d(Ju) = d^2(u^1 \cdot du^2) = 0$, and $d^2\omega = 0$ of every ω.

([20]) More precisely, $\star$ is defined by $\star(dx_i \wedge dx_j) := (-1)^{i+j} e_{\widehat{ij}}$ for every $i < j$, where $e_{\widehat{ij}}$ stands for the $(n-2)$-vector given by the wedge product of all $e_1, \ldots, e_n$ except e_i and e_j.

$\nabla u^1 \times \nabla u^2$ for every u sufficiently regular (e.g. of class $W^{1,2}$); if u is an S^1-valued map which is smooth out of a 1-manifold S without boundary in Ω (namely, the union of finitely many rectifiable curves in Ω with boundary included in $\partial\Omega$), then $\star Ju$ is a singular vector measure supported on S of the form

$$\star Ju = \pi\xi \cdot \tau \cdot \mathscr{H}^1 \llcorner S, \tag{13}$$

where $\mathscr{H}^1 \llcorner S$ denotes the restriction of (Hausdorff) 1-dimensional measure to S, $\tau(x)$ is the tangent vector to S at x, and the multiplicity $\xi(x)$ si given by $\deg(u, \partial D, S^1)$ where D is any embedded disk such that $D \cap S$ consist of the point x only [21].

The functional $\|Ju\|$, that is, $\| \star Ju\|$, is weakly lower semicontinuous on $W^{1,1}$ (cf. Note 17) and agrees with the $(n-2)$-dimensional measure of the singular set of u when u is sufficiently regular (cf. (13)). Hence it is a good candidate to Γ-limit of the functionals in (8) in every dimension (up to a suitable factor).

THE COMPACTNESS PROBLEM. – We face now an essential difficulty: it is possible to prove that the functionals F_ε in (8) Γ-converge to a functional of type $F(u) := \sigma\| \star Ju\|$ with respect to the weak topology of $W^{1,1}$ (or some more or less equivalent metric), but not to prove the coerciveness condition in Definition 1.

Indeed a sequence (u_ε) such that $F_\varepsilon(u_\varepsilon) \le C < +\infty$, is at most *weakly* compact in some L^p [22]. On the other hand, the Γ-limit of F_ε with respect to any L^p metric (thus in the *strong* topology) is identically zero on all functions u such that $|u| = 1$ a.e. [23], and is therefore meaningless, in the sense that it yields no selection criteria for S^1-valued functions.

[21] In particular, the following equivalent of formula (11) holds: if A is a two-surface relatively compact in Ω and such that $\partial A \cap S = \emptyset$ (cf. Fig. 4), then $\int_A \eta \cdot \star Ju = \pi \cdot \deg(u, \partial A, S^1) = \pi \cdot \mathrm{link}(\partial A, S)$, where η is the unit normal to A, and thus the first integral is the flux of Ju through A.

[22] For the sequence of minimizers u_ε there actually holds something more (cf. [21]), but the proof is quite complicated, and it is deeply rooted in the fact that u_ε solves a certain elliptic system—exactly the kind of difficulties that the variational approach is supposed to bypass.

[23] In fact, every measurable function $u : \Omega \to S^1$ is a.e. the pointwise of a sequence of smooth functions $u_h : \Omega \to S^1$. In general $\int |\nabla u_h|^2$ diverge as $h \to +\infty$, but if we set $u_\varepsilon := u_{h(\varepsilon)}$, then $F_\varepsilon(u_\varepsilon) = |\log\varepsilon|^{-1} \int |\nabla u_{h(\varepsilon)}|^2$ is infinitesimal provided that $h(\varepsilon)$ tends to $+\infty$ slowly enough as $\varepsilon \to 0$. Everything changes if we consider the weak $W^{1,1}$ convergence: for instance, the function $x/|x|$ cannot be approximated by smooth S^1-valued functions, e.g., because the Jacobian of $x/|x|$ is non-trivial, unlike that of any smooth

THE Γ-CONVERGENCE THEOREM, REVISED VERSION. – If we think a second about it, the lack of coercivity in a sufficiently strong topology is not a surprise: if our conjecture on the Γ-limit of F_ε is correct, a bound like $F_\varepsilon(u_\varepsilon) \le C < +\infty$ can imply at most that the Jacobians Ju_ε are uniformly bounded in L^1, which notoriously induce no compactness whatsoever for the functions. This simply means that one cannot go for the compactness of functions, but rather for that of Jacobians [24], and the Γ-convergence result for the functionals F_ε must be reformulated accordingly. Let us begin for simplicity from the dimension two.

THEOREM 9. *Let F_ε be given in* (8) *with $n = 2$. Then*

○ *for every sequence (u_ε) such that $F_\varepsilon(u_\varepsilon) \le C < +\infty$, the functions Ju_ε converge flat* [25] *to measures μ of the form*

$$\mu = \sum \pi d_i \cdot \delta_{x_i} \tag{14}$$

with $x_i \in \Omega$ and d_i integer;

○ *for every μ of the form* (14) *and every (u_ε) such that $Ju_\varepsilon \xrightarrow{\text{flat}} \mu$, there holds*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge 2\pi \sum |d_i| = 2\|\mu\|; \tag{15}$$

○ *for every μ of the form* (14) *there exists (u_ε) such that $Ju_\varepsilon \xrightarrow{\text{flat}} \mu$ and* (15) *holds with the limit instead of the lower limit, and the equality instead of the inequality.*

S^1-valued function. The point is that convergence almost everywhere (unlike weak $W^{1,1}$ convergence) is too weak to sense any kind of topological obstruction—this is not unrelated to the fact that the Jacobian operator defined in (9) and (12) is not continuous with respect to such a convergence.

[24] Here is the big difference between the vector and the scalar G.-L. functionals: for the latter ones, the Γ-limit $F(u)$ corresponds to the mass of the distributional derivative Du (cf. Remark 4), and indeed a bound like $\varepsilon F_\varepsilon(u_\varepsilon) \le C < +\infty$ (almost) implies that the gradients ∇u_ε are uniformly bounded in L^1, and consequently the functions u_ε are pre-compact in L^1 by Rellich theorem. I insist on "almost": a bound on the energies does not imply exactly a bound on the L^1 norms of the derivatives in the scalar case, nor of the Jacobians in the complex case. Indeed the Jacobians Ju_ε are not pre-compact in the weak topology of measures, but in the flat topology (cf. Theorems 9 and 10), which is a source of many technical difficulties.

[25] That is, the integrals with respect to all test functions C^1 converge; this convergence is weaker than the usual weak convergence of measures, and is of the same kind of the $W^{-1,p}$ convergence.

In dimension larger than two we have:

THEOREM 10. *Let F_ε be given in (8) with $n \geq 2$. Then*

∘ *for every sequence (u_ε) such that $F_\varepsilon(u_\varepsilon) \leq C < +\infty$, the currents $\star Ju_\varepsilon$ converge flat, up to a factor π, to codimension-two integral currents T without boundary in Ω (cf. Notes 18 and 19);*

∘ *for every current T as above, and every (u_ε) such that $\star Ju_\varepsilon \xrightarrow{\text{flat}} T$, there holds*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geq 2\pi \|T\|, \tag{16}$$

where $\|T\|$ is the mass of T;

∘ *for every current T as above, there exists (u_ε) such that $\star Ju_\varepsilon \xrightarrow{\text{flat}} \pi T$ and* (16) *holds with the limit instead of the lower limit, and the equality instead of the inequality.*

CONSEQUENCE OF THE Γ-CONVERGENCE THEOREM AND REMARKS. – If the functions u_ε converge weakly in $W^{1,1}$ to some $u : \Omega \to S^1$, then the current T is $\star Ju$. Once this is understood, it is not difficult to recognize the last two statements of Theorem 10 (or 9) as a Γ-liminf and a Γ-limsup inequality, with $\|\star Ju\|$ replaced by $\|T\|$. Even so, a suitable modification of Proposition 2 holds, yielding to conclusions of this kind: if the functions u_ε minimize F_ε under the boundary constraints described above, then the Jacobians $\star Ju_\varepsilon$ converge (up to a factor π) to an integral current T of codimension two which locally minimizes the mass (or area), and the energies $F_\varepsilon(u_\varepsilon)$ converge to 2π times the mass of T (i.e., $\int_S |\xi| d\mathscr{H}^{n-2}$; cf. Note 18). The last statement can be localized, yielding a concentration result for the energies: the rescaled energy densities e_ε associated to the minimizer u_ε, namely the integrand in (8) with u_ε instead of u, converge in the sense of measures to $2\pi|\xi| \cdot \mathscr{H}^{n-2} \llcorner S$.

The relevance of these results in dimension two is very limited. In fact, in that case we simply obtain that the Jacobians of minimizers u_ε converge to a measure μ of the form (14), which minimizes the mass among those with integral equal to d (the degree of the boundary datum). Thus μ minimizes $\sum |d_i|$ under the constraint $\sum d_i = d$, which simply means that all d_i have same sign. But we already know from the analysis in [9] that the degrees d_i, not only have all the same sign, but are also all equal to $+1$ (or -1), and the corresponding singularities x_i are located in order to minimize the energy associated to a certain repulsive potential, which tends to make the distance of a singularity from another, or from the boundary, as large as possible. The point is that our Γ-limit represents only the main term in the expansion of F_ε (the one of order $|\log \varepsilon|$ for the original functionals in (1)). In dimension two this is not sufficient to determine uniquely the behaviour of minimizers u_ε; in particular it does not take into account the repulsion of singularities (which

is due to a lower order term). In higher dimension the situation is different: unless the geometry of the domain Ω is very special, the criterion of minimal area usually gives a finite number of integral currents, if not only one, and provides therefore a good description of the behaviour of minimizers. Let us consider, for instance, the following situation: Ω is a right cylinder in $\mathbb{R}^3$ of the form $x^2 + y^2 < 1$, $0 < z < 1$, and we require that u_ε agrees on the vertical part of the boundary $S^1 \times (0,1)$ with a given smooth function independent of the z variable, namely $g : S^1 \to S^1$ with degree $d > 1$.

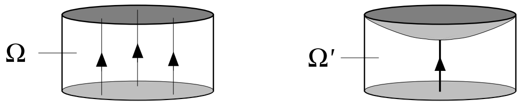

FIGURE 5

From the Γ-convergence result we can only infer that the limit T of the Jacobians $\star Ju_\varepsilon$ is an integral 1-current which minimizes the mass among those whose boundary consists of d Dirac masses on the top face of the cylinder, and $-d$ on the bottom—in other words T is the union of d vertical segments in any possible location (cf. Fig. 5). However, a reduction to the two-dimensional case shows that the locations of the segments are not at all arbitrary, but minimize a specific repulsive potential.

On the other hand, if we replace Ω with a cylinder Ω' with slightly concave top face, e.g., setting $0 < z < 1 - \delta(1 - x^2 - y^2)$ instead of $0 < z < 1$, then the minimality criterion determines a unique T, namely a vertical segment located at the center of the cylinder, thus joining the points of minimal distance between the top and bottom faces, with multiplicity d.

This difference between dimension two and higher is also seen in the asymptotic behaviour of solution of the parabolic equation associated to G.-L. functionals (suitably rescaled in time): in dimension large than two we get, in the limit $\varepsilon \to 0$, an evolution by mean curvature of the singularity, which is the gradient flow of the $(n-2)$-dimensional measure (cf. [5]). On the other hand, in dimension two the gradient flow of the 0-dimensional measures would leave the singular points still, while their evolution is actually determined by the repulsive potential mentioned above, and therefore it is completely different from any kind of curvature-driven evolution (cf. [20]).

To conclude, I shortly comment on the possible generalizations of Theorem 10. If we consider the functionals F_ε defined in (2) for $\mathbb{R}^k$-valued functions, one immediately sees that the problems becomes relevant for $p \geq k$, namely when the functions of class $W^{1,p}$ do not allow for topological singularities of codimension k ([26]). In this case it is reasonable to expect (for a suitable rescaling) a result similar to Theorem 10. Yet there should be a remarkable

([26]) Given $p < k$ and a boundary datum $g : \partial\Omega \to S^{k-1}$, the Γ-limit of difference between the case $p > k$ and the limit case $p = k$. Indeed, for $p > k$ the energy of minimizers concentrates in the transition zone, and not in a neighborhood (cf. Note 12), and the right constant in front of $\|T\|$ in the equivalent of (16) is determined by an optimal profile problem very close to the one which gives the constant σ in Theorem 3 (cf. Note 6). On the other hand, the case $p = k$ should be essentially equivalent to the one of complex G.-L. functionals.

Finally, it is not clear what should happen for $k > 1$ when the integral of the gradient in (2) is replaced by a non-local interaction energy of the form (3); so far it is only known that in the scalar case there holds a Γ-convergence result similar to Theorem 3 (cf. [1]).

Compactness and Γ-liminf inequality. – I briefly sketch now the idea of the proof of the compactness statement in Theorem 9. For the sake of simplicity, I only consider the two-dimensional case, and in addition I assume that the potential W is radially symmetric, i.e., is of the form $W(|u|)$.

Let be given functions u_ε such that $F_\varepsilon(u_\varepsilon) \le C < +\infty$, and denote by e_ε the corresponding energy density, namely $e_\varepsilon := |\log\varepsilon|^{-1}\big[|\nabla u_\varepsilon|^2 + \varepsilon^{-2}W(|u_\varepsilon|)\big]$. For every ε let us fix a positive $\delta = \delta_\varepsilon$, infinitesimal for $\varepsilon \to 0$, and choose a 1-dimensional grid $R = R_\varepsilon$ with size δ (cf. Fig. 6) so that

$$F_\varepsilon(u_\varepsilon) = \int_\Omega e_\varepsilon \ge \delta \int_R e_\varepsilon. \tag{17}$$

The first remark is that if $\delta \gg \varepsilon|\log\varepsilon|$ then $|u_\varepsilon|$ converge to 1 uniformly on R (27). From this point on, we consider only the values of ε such that

F_ε is simply $\int |\nabla u|^p$ for all functions $\in W^{1,p}_g(\Omega, S^{k-1})$, and $+\infty$ elsewhere, and unlike what happens for $p \ge k$, this class is not empty for any datum g sufficiently regular, i.e., in the trace space $W^{1-1/p,p}$.

(27) The bound on the energies and (17) imply that $\int_R W(|u_\varepsilon|) \to 0$, and since W vanishes only in 1, the functions $|u_\varepsilon|$ converge to 1 in measure on R; to obtain the uniform convergence it suffices to prove that the oscillation of $|u_\varepsilon|$ on R tends to 0. We set $v_\varepsilon := |u_\varepsilon|$, and denote by v'_ε the derivative of v_ε in the direction tangent to the grid R. Then (17) yields

$$C \ge \delta \int_R e_\varepsilon \ge \frac{\delta}{|\log\varepsilon|}\int_R |v'_\varepsilon|^2 + \frac{1}{\varepsilon^2}W(v_\varepsilon) \ge \frac{\delta}{\varepsilon|\log\varepsilon|}\int_R 2W^{1/2}(v_\varepsilon)\,|v'_\varepsilon|$$

where the last step follows by applying the inequality $a^2 + b^2 \ge 2ab$ with $a := |v'_\varepsilon|$ and $b := \varepsilon^{-1}W^{1/2}(v_\varepsilon)$. Denoting by H a primitive of $2W^{1/2}$, we get

$$C \ge \frac{\delta}{\varepsilon|\log\varepsilon|}\int_R |(H(v_\varepsilon))'| \ge \frac{\delta}{\varepsilon|\log\varepsilon|}\operatorname{osc}(H(v_\varepsilon), R).$$

$|u_\varepsilon| \ge 1/2$ on R. Then we can talk about the degree d_i of u_ε on the boundary of any cell D_i of the grid, meaning $d_i := \deg(u_\varepsilon/|u_\varepsilon|, \partial D_i, S^1)$.

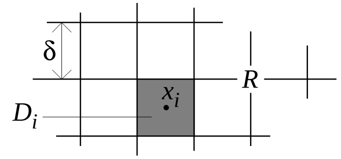

Figure 6

The second remark, which follows by a very nice estimate of R. Jerrard [16], is that for $\delta \gg \varepsilon$, the energy inside each cell D_i must be not less than $2\pi|d_i|$ in the limit $\varepsilon \to 0$ (28), and then

$$C \ge F_\varepsilon(u_\varepsilon) \ge 2\pi\sum_i |d_i| - o(1). \tag{18}$$

Thus we set $\mu_\varepsilon := \sum_i \pi d_i \cdot \delta_{x_i}$, where x_i is the center of the cell D_i (and therefore depends on ε, like d_i and D_i), and obtain a bounded sequence of measures whose limit points (in the weak topology) are measures μ of the form (14).

Notice that the integral of Ju_ε on each D_i is equal to $\pi d_i + o(1)$ because $|u_\varepsilon|$ converge uniformly to 1 on R. Therefore the integral of Ju_ε and μ on every cell of the grid R is (almost) the same. Since the grid gets finer and finer as $\varepsilon \to 0$, one can prove that these two sequences of measures are asymptotically equivalent in the flat metric, and then (Ju_ε) has the same limit points as (μ_ε). Finally, (18) implies the Γ-liminf inequality (15), while the extension to higher dimension can be obtained by slicing.

The Γ-limsup inequality. – To prove the Γ-limsup inequality in dimension two (last statement of Theorem 9) it suffices to take u_ε as follows (29)

$$u_\varepsilon(x) := \phi(\varepsilon^{-1}d_S(x)) \cdot u(x)$$

where $d_S(x)$ is the distance of x from the set $S := \{x_i\}$, $\phi : [0, +\infty] \to [0, 1]$ is any smooth function such that $\phi(0) = 0$ and $\phi(t) = 1$ for $t \ge 1$, and $u : \Omega \to S^1$ is any map asymptotically equivalent to $[(x - x_i)/|x - x_i|]^{d_i}$ at

Then, if $\delta \gg \varepsilon|\log\varepsilon|$, the oscillation of $H(v_\varepsilon)$ on R tends to 0, and the same happens to $v_\varepsilon = |u_\varepsilon|$ because H is strictly increasing.

(28) Only a suitable refinement of this statement is actually true.

(29) This approximating sequence gives the right value of the energy only for those measures μ such that $d_i = \pm 1$ for every i. However, this is enough, because these measures are dense in the class of all μ of type (14).

any x_i, namely a map with Jacobian equal to μ (cf. Remark 6); for instance one can take (cf. Note 16)

$$u(x) := \prod_i \left[\frac{x - x_i}{|x - x_i|}\right]^{d_i}.$$

The first step in the proof of the Γ-limsup inequality in higher dimension (last statement of Theorem 10) consists in finding a function u such that $\star Ju = \pi T$, where $T = \xi \cdot \tau \cdot \mathscr{H}^{n-2} \llcorner S$ is any integral current without boundary (cf. Note 18).

If we assume for simplicity that the support S is an oriented surface of codimension two without boundary, and the multiplicity ξ is identically 1, our task becomes finding an S^1-valued map u, smooth in the complement of S, such that (cf. Note 21)

$$\deg(u, \gamma, S^1) = \mathrm{link}(\gamma, S) \quad \text{for every closed curve } \gamma \text{ s.t. } \gamma \cap S = \emptyset.$$

Such u can be written as $u = e^{i\theta}$, where θ is a smooth function from the complement of S into $\mathbb{R}/2\pi\mathbb{Z}$ the differential of which is a 1-form ω_S satisfying

$$\int_\gamma \omega_S = 2\pi \, \mathrm{link}(\gamma, S) \quad \text{for every closed curve } \gamma \text{ s.t. } \gamma \cap S = \emptyset. \tag{19}$$

Since every 1-form ω_S which fulfills (19) is integrable, in the sense that it agrees with the differential of a map valued into $\mathbb{R}/2\pi\mathbb{Z}$, we just need to exhibit one.

To begin with, we remark that $\mathrm{link}(\gamma, S)$ is (by definition) the degree of the restriction $\gamma \times S$ of the canonic map Φ which takes $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $x \neq y$, into $(x - y)/|x - y| \in S^{n-1}$. On the other hand, the degree of Φ times the $(n-1)$-dimensional measure of the sphere S^{n-1}, is, by the area formula, the integral on $\gamma \times M$ of the pull-back ω of the volume form on S^{n-1} according to Φ. Hence

$$\mathrm{vol}(S^{n-1}) \cdot \mathrm{link}(\gamma, S) = \int_{\gamma \times S} \omega = \int_\gamma \Big[\int_S \omega(x, y)\, dy \Big]\, dx.$$

Thus the 1-form defined by the integral between square brackets satisfies, up to a constant factor, condition (19). Therefore we set

$$\omega_S(x) := \frac{2\pi}{\mathrm{vol}(S^{n-1})} \int_S \omega(x, y)\, dy.$$

Finally one can check that the function u obtained from ω_S has the features required to construct the approximating sequence u_ε.

REFERENCES

1. G. Alberti, G. Bellettini: A non-local anisotropic model for phase transitions. Asymptotic behaviour of rescaled energies. *European J. Appl. Math.* **9** (1998), 261–284.
2. G. Alberti: Variational models for phase transitions. An approach via Γ-convergence. *Differential equations and calculus of variations. Topics on geometrical evolution problems and degree theory (Pisa 1996)*, pp. 95–114, edited by G. Buttazzo et al., Springer-Verlag, Berlin-Heidelberg, 2000.
3. G. Alberti, S. Baldo, G. Orlandi: Functions with prescribed singularities. Preprint 2000.
4. G. Alberti, S. Baldo, G. Orlandi: Variational convergence for functionals of Ginzburg-Landau type. Paper in preparation.
5. L. Ambrosio, H.M. Soner: A measure-theoretic approach to higher codimension mean curvature flows. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25** (1997), 27–49.
6. S. Baldo, G. Orlandi: Codimension one minimal cycles with coefficients in $\mathbf{Z}$ or $\mathbf{Z}_p$, and variational functionals on fibered spaces. *J. Geom. Anal.* **9** (1999), 547–568.
7. J.M. Ball: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.* **63** (1977), 337–403.
8. F. Bethuel, X.M. Zheng: Density of smooth functions between two manifolds in Sobolev spaces. *J. Funct. Anal.* **80** (1988), 60–75.
9. F. Bethuel, H. Brezis, F. Hélein: *Ginzburg-Landau vortices.* Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser, Boston, 1994.
10. H. Brezis, J.-M. Coron, E.H. Lieb: Harmonic maps with defects. *Comm. Math. Phys.* **107** (1986), 649–705.
11. J.W. Cahn, J.E. Hilliard: Free energy of a nonuniform system, I. Interfacial free energy. *J. Chem. Phys.* **28** (1958), 258–267.
12. G. Dal Maso: *An Introduction to Γ-convergence.* Progress in Nonlinear Differential Equations and their Applications, 8. Birkhäuser, Boston 1993.
13. V.L. Ginzburg, L.D. Landau: On the theory of superconductivity. *Zh. Eksperim. i Teor. Fiz.* **20** (1950), 1064–1082 (in russo). English translation in *Men of Physics: L.D. Landau, I*, 138–167, edited by D. Ter Haar. Pergamon Press, Oxford 1965.
14. V.L. Ginzburg, L.P. Pitaevskii: On the theory of superfluidity. *Soviet Physics JETP (7) (5)* **34** (1958), 858–861.
15. A. Jaffe, C. Taubes: *Vortices and monopoles. Structure of static gauge theories.* Progress in Physics, 2. Birkhäuser, Boston, 1980.
16. R. Jerrard: Lower bounds for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal.* **30** (1999), 721–746.
17. R. Jerrard, H.M. Soner: Functions of bounded higher variation. Preprint 1999.
18. R. Jerrard, H.M. Soner: The Jacobian and the Ginzburg-Landau energy. Paper in preparation.
19. W. Jin, R.V. Kohn: Singular perturbation and the energy of folds. *J. Nonlinear Sci.* **10** (2000), 355–390.
20. F.-H. Lin: Some dynamical properties of Ginzburg-Landau vortices. *Comm. Pure Appl. Math.* **49** (1996), 323–359.
21. F.-H. Lin, T. Rivière: Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents. *J. Eur. Math. Soc. (JEMS)* **1** (1999), 237–311.
22. L. Modica, S. Mortola: Un esempio di Γ^--convergenza. *Boll. Un. Mat. Ital. B (5)* **14** (1977), 285–299.

23. L. MODICA, S. MORTOLA: Il limite nella Γ-convergenza di una famiglia di funzionali ellittici. *Boll. Un. Mat. Ital. A (5)* **14** (1977), 526–529
24. L. MODICA: The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.* **98** (1987), 123–142.
25. T. RIVIÈRE: Line vortices in the U(1)-Higgs model. *ESAIM Contrôle Optim. Calc. Var.* **1** (1995/96), 77–167. Electronic publication, http://www.emath.fr/cocv/.
26. E. SANDIER: Ginzburg-Landau minimizers from $\mathbf{R}^{n+1}$ into $\mathbf{R}^n$ and minimal connections. Preprint 1999.
27. L. SIMON: *Lectures on geometric measure theory.* Proceedings of the Centre for Mathematical Analysis 3. Australian National University, Canberra, 1983.

Dipartimento di Matematica "L. Tonelli", Università di Pisa
v. Buonarroti 2, 56127 Pisa, Italy
e-mail: alberti@dm.unipi.it