Preface

This book is the story of a success.

Classical Euclidean geometry has always had a weak point (as Greek geometers were well aware): it was not able to satisfactorily study curves and surfaces, unless they were straight lines or planes. The only significant exception were conic sections. But it was an exception confirming the rule: conic sections were seen as intersections of a cone (a set consisting of the lines through a point forming a constant angle with a given line) with a plane, and so they could be dealt with using the linear geometry of lines and planes. The theory of conic sections is rightly deemed one of the highest points of classical geometry; beyond it, darkness lay. Some special curves, a couple of very unusual surfaces; but of a general theory, not even a trace.

The fact is, classical geometers did not have the language needed to speak about curves or surfaces in general. Euclidean geometry was based axiomatically on points, lines, and planes; everything had to be described in those terms, while curves and surfaces in general do not lend themselves to be studied by means of that vocabulary. All that had to be very frustrating; it suffices to take a look around to see that our world is full of curves and surfaces, whereas lines and planes are just a typically human construction.

Exeunt Greeks end Egyptians, centuries go by, Arabs begin looking around, Italian algebraists break the barrier of third and fourth degree equations, enter Descartes and discovers Cartesian coordinates. We are now at the beginning of 1600s, more than a millennium after the last triumphs of Greek geometry; at last, very flexible tools to describe curves and surfaces are available: they can be seen as images or as vanishing loci of functions given in Cartesian coordinates. The menagerie of special curves and surfaces grows enormously, and it becomes clear that in this historical moment the main theoretical problem is to be able to precisely define (and measure) how curves and surfaces differ from lines and planes. What does it mean that a curve is curved (or that a surface is curved, for that matter)? And how can we measure just *how much* a curve or a surface is curved?

Compared to the previous millennium, just a short wait is enough to get satisfactory answers to these questions. In the second half of 17th century, Newton and Leibniz discovered differential and integral calculus, redefining what mathematics is (and what the world will be, opening the way for the coming of the industrial revolution). Newton's and Leibniz's calculus provides effective tools to study, measure, and predict the behavior of moving objects. The path followed by a point moving on a plane or in space is a curve. The point moves along a straight line if and only if its velocity does not change direction; the more the direction of its velocity changes, the more its path is curved. So it is just natural to measure how much curved is the path by measuring how much the direction of the velocity changes; and differential calculus was born exactly to measure variations. *Voilà*, we have an effective and computable definition of the curvature of a curve: the length of the acceleration vector (assuming, as we may always do, that the point moves along the curve at constant scalar speed).

The development of differential geometry of curves and surfaces in the following two centuries is vast and impetuous. In 18th century many gifted mathematicians already applied successfully to geometry the new calculus techniques, culminating in the great accomplishments of 19th century French school, the so-called *fundamental theorems of local theory of curves and surfaces*, which prove that the newly introduced instruments are (necessary and) sufficient to describe *all* local properties of curves and surfaces.

Well, at least locally. Indeed, as in any good success story, this is just the beginning; we need a *coup de théatre*. The theory we have briefly summarized works very well for curves; not so well for surfaces. Or, more precisely, in the case of surfaces we are still at the surface (indeed) of the topic; the local description given by calculus techniques is not enough to account for all global properties of surfaces. Very roughly speaking, curves, even on a large scale, are essentially straight lines and circles somewhat crumpled in plane and space; on the other hand, describing surfaces simply as crumpled portions of plane, while useful to perform calculations, is unduly restrictive and prevents us from understanding and working with the deepest and most significant properties of surfaces, which go far beyond than just measuring curvature in space.

Probably the person who was most aware of this situation was Gauss, one of the greatest (if not *the* greatest) mathematician of first half of 19th century. With two masterly theorems, Gauss succeeded both in showing how much was still to be discovered about surfaces, and in pointing to the right direction to be followed for further research.

With the first theorem, his *Theorema Egregium*, Gauss showed that while, seen from inside, all curves are equal (an intelligent one-dimensional being from within such a one-dimensional world would not be able to decide whether he lives in a straight line or in a curve), this is far from true for surfaces: there is a kind of curvature (the *Gaussian curvature*, in fact) that can be measured from within the surface, and can differ for different surfaces. In other words, it is possible to decide whether Earth is flat or curved without leaving one's backyard: it is sufficient to measure (with sufficiently precise instruments. . .) the Gaussian curvature of our garden.

The second theorem, called *Gauss-Bonnet theorem* since it was completed and extended by Pierre Bonnet, one of Gauss's brightest students, disclosed that by just studying local properties one can fail to realize that deep and significant phenomena take place at a global level, having geometrical implications at a local level too. For instance, one of the consequences of the Gauss-Bonnet theorem is that no matter how much we deform in space a sphere, by stretching it (without breaking) so to *locally* change in an apparently arbitrary way its Gaussian curvature, *the integral of Gaussian curvature on the whole surface is constant*: every local variation of curvature is necessarily compensated by an opposite variation somewhere else.

Another important topic pointed to the interrelation between local and global properties: the study of curves on surfaces. One of the fundamental problems in classical differential geometry (even more so because of its practical importance) is to identify the shortest curve (the *geodesic*) between two given points on a surface. Even simply proving that when the points are close to each other there exists a unique geodesic joining them is not completely trivial; if the two points are far from each other, infinitely many geodesics could exist, or none at all. This is another kind of phenomenon of a typically global nature: it cannot be studied with differential calculus techniques only, which are inherently local.

Once more, to go ahead and study global properties a new language and new tools were needed. So we arrive at the twentieth century and at the creation and development (by Poincaré and others) of topology, which turns out to be the perfect environment to study and understand global properties of surfaces. To give just one example, the description given by Hopf and Rinow of the surfaces in which all geodesics can be extended for all values of the time parameter (a property implying, in particular, that each pair of points is connected by a geodesic) is intrinsically topological.

And this is just the beginning of an even more important story. Starting from seminal insights by Riemann (who, in particular, proved that the so-called non-Euclidean geometries were just geometries on surfaces different from the plane and not necessarily embedded in Euclidean space), definitions and ideas introduced to study two-dimensional surfaces have been extended to n-dimensional manifolds, the analogous of surfaces in any number of dimensions. Differential geometry of manifolds has turned out to be one of the most significant areas in contemporary mathematics; its language and its results are used more or less everywhere (and not just in mathematics: Einstein's general theory of relativity, to mention just one example, could not exist without differential geometry). And who knows what the future has in store for us. . .

This book's goal is to tell the main threads of this story from the viewpoint of contemporary mathematics. Chapter 1 describes the local theory of curves, from the definition of what a curve is to the fundamental theorem of the local theory of curves. Chapters 3 and 4 deal with the local theory of surfaces, from the (modern) definition of a surface to Gauss's *Theorema Egregium.* Chapter 5 studies geodesics on a surface, up to the proof of their existence and local uniqueness.

The remaining chapters (and some of the chapters' supplementary material; see below) are devoted instead to global properties. Chapter 2 discusses some fundamental results in the global theory of plane curves, including the *Jordan curve theorem*, which is a good example of the relatively belated development of the interest in global properties of curves: the statement of the Jordan curve theorem seems obvious until you try to actually prove it — and then it turns out to be surprisingly difficult, and far deeper than expected.

Chapter 6 is devoted to the Gauss-Bonnet theorem, with a complete proof is given, and to some of its innumerable applications. Finally, in Chapter 7, we discuss a few important results about the global theory of surfaces (one of them, the Hartman-Nirenberg theorem, proved only in 1959), mainly focusing on the connections between the sign of Gaussian curvature and the global (topological and differential) structure of surfaces. We shall confine ourselves to curves and surfaces in the plane and in space; but the language and the methods we shall introduce are compatible with those used in differential geometry of manifolds of any dimension, and so this book can be useful as a training field before attacking n-dimensional manifolds.

We have attempted to provide several possible paths for using this book as a textbook. A minimal path consists in just dealing with local theory: Chapters 1, 3, and 4 can be read independently of the remaining ones, and can be used to offer, in a two-month course, a complete route from initial definitions to Gauss's Theorema Egregium; if time permits, the first two sections of Chapter 5 can be added, describing the main properties of geodesics. Such a course is suitable and useful both for Mathematics (of course) and Physics students, from the second year on, and for Engineering and Computer Science students (possibly for a Master degree) who need (or are interested in) more advanced mathematical techniques than those learnt in a first-level degree.

In a one-semester course, on the other hand, it is possible to adequately cover the global theory too, introducing Chapter 2 about curves, the final section of Chapter 5 about vector fields, and, at the professor's discretion, Chapter 6 about the Gauss-Bonnet theorem or Chapter 7 about the classification of closed surfaces with constant Gaussian curvature. In our experience, in a one-semester course given to begiing or intermediate undergraduate Mathematics or Physics students, it is difficult to find time to cover both topics, so the two chapters are completely independent; but in the case of an one-year course, or in an one-semester course for advanced students, it might be possible to do so, possibly even touching upon some of the supplementary material (see below).

Each chapter comes with several guided problems, that is, solved exercises, both computational (one of the nice features of local differential geometry of curves and surfaces is that almost everything can be explicitly computed — well, with the exception of geodesics) and of a more theoretical character, whose goal is to teach you how to effectively use the techniques given in the corresponding chapter. You will also be able to test your skill by solving the large number of exercises we propose, subdivided by topic.

You might have noticed another feature peculiar to this book: we address you directly, dear reader. There is a precise reason for this: we want you to feel actively involved while reading the book. A mathematics textbook, no matter at what level, is a sequence of arguments, exposed one after another with (hopefully) impeccable logic. While reading, we are led by the argument, up to a point where we have no idea why the author followed that path instead of another, and — even worse —we are not able to reconstruct that path on our own. In order to learn mathematics, it is not enough to read it; you must *do* mathematics. The style we adopted in this book is chosen to urge you in this direction; in addition to motivations for each notion we shall introduce, you will often find direct questions trying to stimulate you to an active reading, without accepting anything just because you trust us (and, with any luck, they will also help you to stay awake, should you happen to study in the wee hours of the morning, the same hours we used to write this book. . .).

This book also has the (vain?) ambition not to be just a textbook, but something more. This is the goal of the supplementary material. There is a wealth of extremely interesting and important results that does not usually find its place in courses (mostly due to lack of time), and that is sometimes hard to find in books. The supplementary material appended to each chapter display a choice (suggested, as is natural, by our taste) of these results. We go from complete proofs of the Jordan curve theorem, also for curves that are no more than continuous, and of the existence of triangulations on surfaces (theorems often invoked and used but very rarely proved), to a detailed exposition of the Hopf-Rinow theorem about geodesics or Bonnet's and Hadamard's theorems about surfaces with Gaussian curvature having a well-defined sign. We shall also prove that all closed surfaces are orientable, and the fundamental theorem of the local theory of surfaces (which, for reasons made clear at the end of Chapgter 4, is not usually included in a standard curriculum), and much more. We hope that this extra material will answer questions you might have asked yourself while studying, arouse further you curiosity, and provide motivation and examples for moving towards the study of differential geometry in any dimension. A warning: with just some exceptions, the supplementary material is significantly more complicated than the rest of the book, and to be understood it requires a good deal of participation on your part. A reassurance: nothing of what is presented in the supplementary material is used in the main part of the book. On a first reading, the supplementary results may be safely ignored with no detriment to the comprehension of the rest of the book. Lastly, mainly due to space considerations, the supplementary material includes only a limited number of exercises.

Two words about the necessary prerequisites. As you can imagine, we shall use techniques and ideas from differential and integral calculus in several real variables, and from general topology. The necessary notions from topology are really basic: open sets, continuous functions, connectedness and compactness, and even knowing those in metric spaces is enough. Since these topics are usually covered in any basic Geometry course, and often in Calculus courses too, we did not feel the need to provide specific references for the few topology theorems we shall use. If necessary, you will find all of the above (and much more) in the first four chapters of [11].

Much more care has been put in giving the results in calculus we shall need, both because they typically are much deeper than those in general topology, and to give you statements consistent with what we need. All of them are standard results, covered in any Advanced Calculus course (with the possible exception of Theorem 4.9.1); a good reference text is [5].

On the other hand, we do not require you to have been exposed to algebraic topology (in particular, if you do not have any idea of what we are talking about, you do not have to worry). For this reason, Section 2.1 contains a complete introduction to the theory of the degree for continuous maps of the circle to itself, and Section *7.5* discusses everything we shall need about the theory of covering maps.

Needless to say, this is not the first book about this topic, and it could not have been written ignoring the earlier books. We found especially useful the classical texts by do Carmo [4] and Spivak [22], and the less classical but not less useful ones by Lipschutz [13] and Montiel and Ros [16]; if you will like this book, you might want to have a look at those books as well. On the other hand, good starting points for studying differential geometry in any dimension are the already cited [22], and [10] and [12].

Lastly, the pleasant duty of thanks. This book would never have seen the light of the day, and would certainly have been much worse, without the help, assistance, understanding, and patience of (in alphabetical order) Luigi Ambrosio, Francesca Bonadei, Piermarco Cannarsa, Cinzia Casagrande, Ciro Ciliberto, Michele Grassi, Adele Manzella, and Jasmin Raissy. We are particularly grateful to Daniele A. Gewurz, who flawlessly completed the daunting task of translating our book from Italian to English. A special thanks goes to our students of all these years, who put up with several versions of our lecture notes and have relentlessly pointed out even the smallest error. Finally, a very special thanks to Leonardo, Jacopo, Niccolò, Daniele, Maria Cristina, and Raffaele, who have fearlessly suffered the transformation of their parents in an appendix to the computer keyboard, and that with their smiles remind us that the world still deserved to be lived.

Pisa and Rome, July 2011 *Marco Abate, Francesca Tovena*