Ritt's theorem and the Heins map in hyperbolic complex manifolds

Marco Abate* and Filippo Bracci*

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Abstract. Let X be a Kobayashi hyperbolic complex manifold, and assume that X does not contain compact complex submanifolds of positive dimension (e.g., X Stein). We shall prove the following generalization of Ritt's theorem: every holomorphic self-map $f\colon X \to X$ such that $f(X)$ is relatively compact in X has a unique fixed point $\tau(f) \in X$, which is attracting. Furthermore, we shall prove that $\tau(f)$ depends holomorphically on f in a suitable sense, generalizing results by Heins, Joseph-Kwack and the second author.

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0. Introduction

The classical Wolff-Denjoy theorem (see, e.g., [A2, Theorem 1.3.9]) says that the sequence of iterates of a holomorphic self-map f of the unit disk $\Delta \subset \mathbb{C}$, except when f is an elliptic automorphism of Δ or the identity, converges uniformly on compact subsets to a point $\tau(f) \in \overline{\Delta}$, the *Wolff point* of f. Furthermore, if $\tau(f) \in \Delta$ then it is the unique fixed point of f; and if $\tau(f) \in \partial\Delta$ then it is still morally fixed, in the sense that $f(\zeta)$ tends to $\tau(f)$ when ζ tends to $\tau(f)$ non-tangentially.

In 1941, Heins [H] proved that the map $\tau\colon \mathrm{Hol}(\Delta,\Delta) \setminus \{\mathrm{id}\} \to \overline{\Delta}$, associating to every elliptic automorphism its fixed point and to any other map its Wolff point, is continuous. More than half a century later, using the first author's version (see [A1]) of the Wolff-Denjoy theorem for strongly convex domains in $\mathbb{C}^n$, Joseph and Kwack [JK] extended Heins' result to strongly convex domains.

In 2002, the second author started investigating further regularity properties of the Heins map. If D is a bounded domain in $\mathbb{C}^n$, then $\mathrm{Hol}(D,D)$ is a subset of the complex Banach space $H^\infty(D)^n$ of n-uples of bounded holomorphic functions defined on D; so one may ask whether the Heins map, when defined, is holomorphic on some suitable open subset of $\mathrm{Hol}(D,D)$. And indeed, in [B] the second author proved that, when D is strongly convex, the Heins map is well-defined and holomorphic on $\mathrm{Hol}_c(D,D)$, the open subset of holomorphic self-maps of D whose image is relatively compact in D.

The aim of this paper is to prove a similar result for the space $\mathrm{Hol}_c(X,X)$ of the holomorphic self-maps of a Kobayashi hyperbolic Stein manifold whose image is relatively compact in X. First of all, we shall generalize the classical Ritt's theorem, proving (Theorem 1.1) that every $f \in \mathrm{Hol}_c(X,X)$ admits a unique fixed point $\tau(f) \in X$; therefore the Heins map $f \mapsto \tau(f)$ is well-defined and continuous (Lemma 2.1).

To study further regularity properties of the Heins map, one apparently needs a complex structure on $\mathrm{Hol}_c(X,X)$. Unfortunately, we do not know whether such a structure exists in general; so we shall instead prove (Theorem 2.3) that the Heins map is holomorphic when restricted to any holomorphic family inside $\mathrm{Hol}_c(X,X)$, a fact equivalent to τ being holomorphic with respect to any sensible complex structure on $\mathrm{Hol}_c(X,X)$. For instance, we obtain (Corollary 2.4) that the Heins map is holomorphic on $\mathrm{Hol}_c(D,D)$ for any bounded domain D in $\mathbb{C}^n$.

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1. Ritt's theorem

Let X be a complex manifold. We shall denote by $\mathrm{Hol}_c(X,X)$ the space of holomorphic self-maps $f\colon X \to X$ of X such that $f(X)$ is relatively compact in X.

In 1920, Ritt [R] proved that if X is a non-compact Riemann surface then every $f \in \mathrm{Hol}_c(X,X)$ has a unique fixed point $z_0 \in X$. Furthermore, this fixed point is *attractive* in the sense that the sequence $\{f^k\}$ of iterates of f converges, uniformly on compact subsets, to the constant map z_0. This theorem has been generalized to bounded domains in $\mathbb{C}^n$ by Wavre [W]; see also Hervé [He, p. 83]. Arguing as in [A2, Corollary 2.1.32] we shall now prove a far-reaching generalization of Ritt's theorem:

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Theorem 1.1: *Let X be a hyperbolic manifold with no compact complex submanifolds of positive dimension. Then every $f \in \mathrm{Hol}_c(X,X)$ has a unique fixed point $z_0 \in X$. Furthermore, the sequence of iterates of f converges, uniformly on compact subsets, to the constant map z_0.*

Proof: Since X is hyperbolic, by [A3] the space $\mathrm{Hol}(X,X)$ of holomorphic self-maps of X is relatively compact in the space $C^0(X,X^*)$ of continuous maps of X into the one-point compactification $X^* = X \cup \{\infty\}$, endowed with the compact-open topology. If $f \in \mathrm{Hol}_c(X,X)$, this implies that the sequence of iterates of f is relatively compact in $\mathrm{Hol}(X,X)$, because $f(X) \subset\subset X$.

Let then $\{f^{k_\nu}\}$ be a subsequence of $\{f^k\}$ converging to $h_0 \in \mathrm{Hol}(X,X)$. We can also assume that $p_\nu = k_{\nu+1} - k_\nu$ and $q_\nu = p_\nu - k_\nu$ tend to $+\infty$ as $\nu \to +\infty$, and that there are ρ_0, $g_0 \in \mathrm{Hol}(X,X)$ such that $f^{p_\nu} \to \rho_0$ and $f^{q_\nu} \to g_0$ in $\mathrm{Hol}(X,X)$. Then it is easy to see that

$$h_0 \circ \rho_0 = h_0 = \rho_0 \circ h_0 \qquad \text{and} \qquad g_0 \circ h_0 = \rho_0 = h_0 \circ g_0,$$

and so

$$\rho_0^2 = \rho_0 \circ \rho_0 = g_0 \circ h_0 \circ \rho_0 = g_0 \circ h_0 = \rho_0.$$

Thus ρ_0 is a holomorphic retraction, whose image is contained in the closure of $f(X)$, which is compact. This means (see Rossi [Ro] and Cartan [C]) that $\rho_0(X)$ is a compact connected complex submanifold of X, i.e., a point $z_0 \in X$. Therefore $\rho_0 \equiv z_0$ and z_0 is a fixed point of f, since f clearly commutes with ρ_0.

We are left to proving that $f^k \to z_0$, which implies in particular that z_0 is the only fixed point of f. Since $\{f^k\}$ is relatively compact in $\mathrm{Hol}(X,X)$, it suffices to show that z_0 is the unique limit point of any converging subsequence of $\{f^k\}$. So let $\{f^{k_\mu}\}$ be a subsequence converging toward a map $h \in \mathrm{Hol}(X,X)$. Arguing as before we find a holomorphic retraction $\rho \in \mathrm{Hol}(X,X)$ such that $h = \rho \circ h$. Furthermore, ρ must again be constant; but since it is obtained as a limit of a subsequence of iterates of f, it must commute with ρ_0, and this is possible if and only if $\rho \equiv z_0$. But then $h = \rho \circ h \equiv z_0$ too, and we are done. □

In particular this theorem holds for hyperbolic Stein manifolds, because a Stein manifold has no compact complex submanifolds of positive dimension.

Remark 1.1: If $f^k \to z_0$, then the spectral radius of df_{z_0} is strictly less than one. Indeed, if df_{z_0} had an eigenvalue $\lambda \in \mathbb{C}$ with $|\lambda| \ge 1$, then $d(f^k)_{z_0}$ would have λ^k as eigenvalue, and $\lambda^k \not\to 0$ whereas $d(f^k)_{z_0} \to O$.

2. The Heins map

Let X be a hyperbolic manifold with no compact complex submanifolds of positive dimension. The *Heins map* of X is the map $\tau\colon \mathrm{Hol}_c(X,X) \to X$ that associates to any $f \in \mathrm{Hol}_c(X,X)$ its unique fixed point $\tau(f) \in X$, whose existence is proved in Theorem 1.1.

The first observation is that the Heins map is continuous:

Lemma 2.1: *Let X be a hyperbolic manifold with no compact complex submanifolds of positive dimension. Then the Heins map $\tau\colon \mathrm{Hol}_c(X,X) \to X$ is continuous.*

Proof: Let $\{f_k\} \subset \mathrm{Hol}_c(X,X)$ be a sequence converging toward a map $f \in \mathrm{Hol}_c(X,X)$; we must show that $\tau(f_k) \to \tau(f) \in X$.

First of all, we claim that the set $\{\tau(f_k)\}$ is relatively compact in X. Assume that this is not true; then, up to passing to a subsequence, we can assume that the sequence $\{\tau(f_k)\}$ eventually leaves any compact subset of X. Now, the set $f(X)$ is relatively compact in X; we can then find an open set D in X such that

$$f(X) \subset\subset D \subset\subset X.$$

We have $\tau(f_k) \notin \overline{D}$ eventually; therefore for k large enough we can find $R_k > 0$ such that

$$\overline{B\bigl(\tau(f_k),R_k\bigr)} \cap D = \varnothing \qquad \text{and} \qquad \overline{B\bigl(\tau(f_k),R_k\bigr)} \cap \partial D \ne \varnothing,$$

where $B(z,R)$ is the ball of center $z \in X$ and radius $R > 0$ with respect to the Kobayashi distance of X. Choose $z_k \in \overline{B\bigl(\tau(f_k),R_k\bigr)} \cap \partial D$ for every k large enough; since ∂D is compact, up to a subsequence we can assume that $z_k \to z_0 \in \partial D$. In particular, then, $f_k(z_k) \to f(z_0) \in f(X) \subset D$. But, on the other hand, we have $f_k(z_k) \in \overline{B\big(\tau(f_k), R_k\big)} \subset X \setminus D$ for all k large enough, because $\tau(f_k)$ is fixed by f_k and the Kobayashi distance is contracted by holomorphic maps; therefore $f(z_0) \in X \setminus D$, contradiction.

So $\{\tau(f_k)\}$ is relatively compact in X; to prove that $\tau(f_k) \to \tau(f)$ it suffices to show that $\tau(f)$ is the unique limit point of the sequence $\{\tau(f_k)\}$. But indeed if $\tau(f_{k_\nu}) \to x \in X$ we have

$$f(x) = \lim_{\nu\to+\infty} f_{k_\nu}\big(\tau(f_{k_\nu})\big) = \lim_{\nu\to+\infty} \tau(f_{k_\nu}) = x;$$

but $\tau(f)$ is the only fixed point of f, and we are done. □

As stated in the introduction, our aim is to prove that the Heins map is holomorphic in a suitable sense. Since we do not know how to define a holomorphic structure on $\mathrm{Hol}_c(X,X)$ for general manifolds, we shall prove another result which is equivalent to the holomorphy of τ in any reasonable setting (see for instance Corollary 2.4 below). We shall need the following lemma:

Lemma 2.2: *Let $P \subset \mathbb{C}^n$ be a polydisk centered in $p_0 \in \mathbb{C}^n$, and $h\colon P \to \mathbb{C}^n$ a holomorphic map. Then there is a holomorphic map $A\colon P \to M(n,\mathbb{C})$, where $M(n,\mathbb{C})$ is the space of $n\times n$ complex matrices, satisfying the following properties:*

(i) *$h(z) - h(p_0) = A(z)\cdot(z-p_0)$ for all $z \in P$;*
(ii) *$A(p_0) = dh_{p_0}$;*
(iii) *for every polydisk $P_1 \subset\subset P$ centered at p_0 there is a constant $C(P_1) > 0$ such that $\|A\|_{P_1} \le C(P_1)\|h\|_{P_1}$.*

Proof: We can write

$$h(z) - h(p_0) = \int_0^1 \frac{\partial}{\partial t} h\big(z_0 + t(z-p_0)\big)\,dt = \sum_{j=1}^n (z^j - p_0^j)\int_0^1 \frac{\partial h}{\partial z^j}\big(z_0 + t(z-p_0)\big)\,dt.$$

Therefore taking

$$A^i_j(z) = \int_0^1 \frac{\partial h^i}{\partial z^j}\big(z_0 + t(z-p_0)\big)\,dt$$

the matrix $A = (A^i_j)$ clearly satisfies (i) and (ii), and (iii) follows from the Cauchy estimates. □

Theorem 2.3: *Let X be a hyperbolic manifold with no compact complex submanifolds of positive dimension, Y another complex manifold, and $F\colon Y\times X \to X$ a holomorphic map so that $f_y = F(y,\cdot) \in \mathrm{Hol}_c(X,X)$ for every $y \in Y$. Then the map $\tau_F\colon Y \to X$ given by $\tau_F(y) = \tau(f_y)$ is holomorphic. Furthermore, for every $y_0 \in Y$ the differential of τ_F at y_0 is given by*

$$d(\tau_F)_{y_0} = \big(\mathrm{id} - d(f_{y_0})_{\tau(f_{y_0})}\big)^{-1} \circ dF_{(y_0,\tau(f_{y_0}))}(\cdot, O).$$

Notice that, by Remark 1.1, $\mathrm{id} - d(f_{y_0})_{\tau(f_{y_0})}$ is invertible.

Proof: Without loss of generality, we can assume that Y is a ball $B^m \subset \mathbb{C}^m$ centered at y_0. Set $p_0 = \tau(f_{y_0})$, and let $P_0 \subset X$ be the domain of a polydisk chart centered at p_0. Since $f_{y_0}(p_0) = p_0$, we can find a polydisk $P_1 \subset\subset P_0$ centered at p_0 such that $f_{y_0}(P_1) \subset\subset P_0$. Furthermore, by Lemma 2.1 there is also a $\delta > 0$ such that $\|y - y_0\| < \delta$ implies $\tau(f_y) \in P_1$ and $f_y(P_1) \subset\subset P_0$. This means that as soon as y is close enough to y_0 we can work inside P_0 and assume, without loss of generality, that X is contained in some $\mathbb{C}^n$.

Write $p_y = \tau(f_y) \in P_1$, and define $h_y\colon \overline{P_1} \to \mathbb{C}^n$ by $h_y = f_y - f_{y_0}$. We have

$$p_y - p_0 = f_{y_0}(p_y) - f_{y_0}(p_0) + h_y(p_y);$$

therefore Lemma 2.2 applied to f_{y_0} yields a matrix $A(y)$, depending continuously on y by Lemma 2.1, such that

$$p_y - p_0 = A(y)\cdot(p_y - p_0) + h_y(p_y).$$

Since $A(y) \to d(f_{y_0})_{p_0}$ as $y \to y_0$, for y close to y_0 $\mathrm{id} - A(y)$ is invertible, and so we can write

$$p_y - p_0 = \big(\mathrm{id} - A(y)\big)^{-1}\cdot h_y(p_y). \tag{2.1}$$

Now, we have

$$dF_{(y_0,\tau(f_{y_0}))}(\cdot,O)=\mathrm{Jac}_y\bigl(f_y(p_0)\bigr)(y_0),$$

where Jac_y is the Jacobian matrix computed with respect to the y variables; in particular,

$$h_y(p_0)-dF_{(y_0,\tau(f_{y_0}))}(y-y_0,O)=o(\|y-y_0\|).$$

This means that to show that τ_F is holomorphic and $d\tau_F$ has the claimed expression it suffices to show that

$$\lim_{y\to y_0}\frac{\bigl\|\tau_F(y)-\tau_F(y_0)-\bigl(\mathrm{id}-d(f_{y_0})_{p_0}\bigr)^{-1}\cdot h_y(p_0)\bigr\|}{\|y-y_0\|}=0,$$

which is equivalent to proving that

$$\lim_{y\to y_0}\frac{\bigl\|\bigl(\mathrm{id}-d(f_{y_0})_{p_0}\bigr)\cdot(p_y-p_0)-h_y(p_0)\bigr\|}{\|y-y_0\|}=0. \tag{2.2}$$

Now, (2.1) yields

$$\begin{aligned}\frac{\bigl\|\bigl(\mathrm{id}-d(f_{y_0})_{p_0}\bigr)\cdot(p_y-p_0)-h_y(p_0)\bigr\|}{\|y-y_0\|}&=\frac{\bigl\|\bigl(\mathrm{id}-A(y)\bigr)\cdot(p_y-p_0)-h_y(p_0)+\bigl(A(y)-d(f_{y_0})_{p_0}\bigr)\cdot(p_y-p_0)\bigr\|}{\|y-y_0\|}\\&\le\frac{\|h_y(p_y)-h_y(p_0)\|}{\|y-y_0\|}+\bigl\|A(y)-d(f_{y_0})_{p_0}\bigr\|\,\frac{\|p_y-p_0\|}{\|y-y_0\|}\,.\end{aligned} \tag{2.3}$$

Since $h_y(z)$ is holomorphic both in y and in z, we have

$$h_y(z)-h_{y_1}(z_1)=O\bigl(\|y-y_1\|,\|z-z_1\|\bigr);$$

in particular,

$$h_y(z)=h_y(z)-h_{y_0}(z)=O(\|y-y_0\|) \tag{2.4}$$

uniformly on P_1. So (2.1) implies that $p_y-p_0=O(\|y-y_0\|)$, and thus the second summand in (2.3) tends to zero as $y\to y_0$.

Finally, if we apply Lemma 2.2 to h_y we get a matrix $B(y)$ and a constant $C>0$ such that

$$\|h_y(p_y)-h_y(p_0)\|\le\|B(y)\|\cdot\|p_y-p_0\|\le C\|h_y\|_{P_2}\|p_y-p_0\|$$

when y is close enough to y_0, where $P_2\subset\subset P_1$ is a fixed polydisk centered at p_0. But then (2.4) yields

$$\|h_y(p_y)-h_y(p_0)\|=O\bigl(\|y-y_0\|^2\bigr),$$

and so (2.2) is proved. □

If X is a bounded domain in $\mathbb{C}^n$, then $\mathrm{Hol}_c(X,X)$ is an open subset of $H^\infty(X)^n$, the complex Banach space of n-uples of bounded holomorphic functions defined on X. Therefore in this case $\mathrm{Hol}_c(X,X)$ has a natural complex structure, and we obtain the following generalization of the main result in [B]:

Corollary 2.4: *Let $D\subset\subset\mathbb{C}^n$ be a bounded domain. Then the Heins map $\tau\colon\mathrm{Hol}_c(D,D)\to D$ is holomorphic.*

Proof: It follows from Theorem 2.3 and [FV, Theorem II.3.10]. □

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Marco Abate
Dipartimento di Matematica, Università di Pisa
Largo Pontecorvo 5, 56127 Pisa, Italy
E-mail: abate@dm.unipi.it

Filippo Bracci
Dipartimento di Matematica, Università di Roma "Tor Vergata"
Via della Ricerca Scientifica, 00133 Roma, Italy
E-mail: fbracci@mat.uniroma2.it