Kähler Finsler manifolds of constant holomorphic curvature

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June 1995

0. Introduction

The classification of simply connected Kähler manifolds of constant holomorphic curvature is a classical result. According to the classification, up to biholomorphic isometry there are only three possibilities: $\mathbb{C}^n$ endowed with the euclidean metric, $\mathbb{P}^n(\mathbb{C})$ endowed with (a suitable costant multiple of) the Fubini-Study metric, and the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ endowed with (a suitable costant multiple of) the hyperbolic metric. In recent years questions coming from geometric function theory, and in particular the study of invariant metrics of complex manifolds, suggested to investigate the geometry of complex Finsler (rather than Hermitian) metrics with constant holomorphic curvature, satisfying some natural Kähler condition (agreeing with the usual one in the case of Hermitian metrics) and whose curvature has symmetries enjoyed by the function theoretic examples. In [AP1], and then in [AP2], among other results it was shown that these hypotheses are equivalent to the existence of geodesic complex curves. Since complex Finsler metrics have been considered for quite some time (we recall among other contributions [Ri] who possibly introduced them, [Ru], [K] who indicated the right setting for their study, [Ro], [F], [P]) it is natural to ask whether, at least under natural geometric assumptions, it is possible to obtain a satisfactory classification. Examples show that one should not expect a short list of models. In fact the strongly convex domains in $\mathbb{C}^n$ with their Kobayashi metric provide an infinite dimensional family of not equivalent (neither holomorphically nor isometrically) complex (weakly) Kähler Finsler manifolds of constant negative holomorphic curvature. Furthermore it is easy to endow $\mathbb{C}^n$ with infinite non isometric flat complex Kähler Finsler metrics (the strongly pseudoconvex Minkowski metrics). On the other hand, no example is known of non Hermitian complex Kähler Finsler manifold of positive constant holomorphic curvature.

The difference of availability of examples seems to hint that there is a different situation according to the sign of the curvature, in striking contrast with the Hermitian situation. Indeed there are difficulties which do not allow one to extend easily the techniques of the Hermitian case — and even in the real case the classification of constant curvature Finsler manifolds is not clearly established. Finally, the relationship between complex and real geometry is not as effective as in the Hermitian situation.

In this paper, using heavily the results of [AP2] and the previous work on the subject by the authors (in particular [AP1]), we address the classification problem and we are able to clarify the situation completely in the non-negative case and make some substantial progress in the negative one. Our work shows that the examples gave the right feeling about the problem. Namely, up to biholomorphic isometries, if some natural symmetry of the curvature is assumed the only complex Kähler Finsler manifold of positive constant holomorphic curvature is $\mathbb{P}^n(\mathbb{C})$ endowed with (a suitable constant multiple of) the Fubini-Study metric, and the only simply connected flat ones are $\mathbb{C}^n$ endowed with strongly pseudoconvex Minkowski metrics. For the negative case we are able to give sufficient conditions ensuring the existence of a Monge-Ampère exhaustion as in the case of strongly convex domains in $\mathbb{C}^n$ and to show that the metric is (a suitable multiple of) the Kobayashi metric of M. We like to thank J. Bland for some very useful remarks and his interest in our work.

1. Preliminaries and statements of main results

In order to give precise statements we need to introduce some notations and add preliminaries which will be used in the paper. We refer to [AP2] and to the literature quoted there for details.

Let M be a complex manifold. We shall denote by $T^{1,0}M$ its holomorphic tangent bundle, and set $\tilde{M} = T^{1,0}M \setminus \{\text{Zero section}\}$. To local coordinates $(z^1,\ldots,z^n)$ for M are associated local coordinates $(z^1,\ldots,z^n,v^1,\ldots,v^n)$ on $T^{1,0}M$, and a local frame $\{\partial_1,\ldots,\partial_n,\dot{\partial}_1,\ldots,\dot{\partial}_n\}$ for $T^{1,0}\tilde{M}$, where

$$\partial_\alpha = \frac{\partial}{\partial z^\alpha} \quad \text{and} \quad \dot{\partial}_\alpha = \frac{\partial}{\partial v^\alpha}.$$

For ϕ of class C^∞ on an open set of $T^{1,0}M$, we shall denote derivatives with indexes as in these examples:

$$\phi_{\alpha\bar\beta}=\frac{\partial^2\phi}{\partial v^\alpha\partial\bar v^\beta},\qquad \phi_{;\mu\nu}=\frac{\partial^2\phi}{\partial z^\mu\partial z^\nu},\qquad \phi_{\alpha;\bar\nu}=\frac{\partial^2\phi}{\partial\bar z^\nu\partial v^\alpha}.$$

Now we can define Finsler metrics. A (smooth, complex) *strongly pseudoconvex Finsler metric* is an upper semicontinuous function $F\colon T^{1,0}M\to\mathbb{R}^+$ such that

$$G=F^2\in C^\infty(\tilde M), \tag{1.1}$$

$$F(p;v)>0\quad\forall\,(p;v)\in\tilde M, \tag{1.2}$$

$$F(p;\zeta v)=|\zeta|F(p;v)\quad\forall\,(p;v)\in T^{1,0}M,\forall\zeta\in\mathbb{C}, \tag{1.3}$$

$$\big(G_{\alpha\bar\beta}(p;v)\big)>0\quad\forall\,(p;v)\in\tilde M. \tag{1.4}$$

Condition (1.4) holds iff all the *indicatrices* of F, defined by

$$I_F(p)=\{v\in T^{1,0}_pM\mid F(v)<1\},$$

are strongly pseudoconvex. Clearly $G=F^2$ is a Hermitian metric iff $G\in C^\infty\left(T^{1,0}M\right)$.

The simplest examples of such metrics are the so called complex Minkowski metrics which are defined as follows. Let $\tilde g\colon\mathbb{P}^{n-1}(\mathbb{C})\to\mathbb{R}^+$ be a smooth function such that if $g\colon\mathbb{C}^n\to\mathbb{R}^+$ is any lift of $\tilde g$ then the exhaustion of $\mathbb{C}^n$ defined by $\tau(z)=g(z)\|z\|^2$ is strictly plurisubharmonic on $\mathbb{C}^n\setminus\{0\}$. Then a *complex Minkowski metric* $\mu\colon\mathbb{C}^n\times\mathbb{C}^n\cong T^{1,0}(\mathbb{C}^n)\to\mathbb{R}^+$ is given by $\mu(z,v)=\sqrt{\tau(v)}$.

Thanks to the results of [L], as mentioned before, the Kobayashi metric of strongly convex domains in $\mathbb{C}^n$ provide a large class of nontrivial examples. In this case the indicatrices of the metrics are always strongly convex.

The study of the differential geometry of complex Finsler metrics on M is reduced to the analysis of a suitable Hermitian metric on $\tilde M$. If $\pi\colon T^{1,0}M\to M$ is the projection and $d\pi\colon T^{1,0}\tilde M\to T^{1,0}M$ is its differential, the *vertical bundle* $\mathcal{V}$ over $\tilde M$ of rank $n=\dim M$ is defined by restricting $\operatorname{Ker}d\pi$ over $\tilde M$. A local frame for $\mathcal{V}$ is given by $\{\dot\partial_1,\dots,\dot\partial_n\}$ and a natural section $\iota\colon\tilde M\to\mathcal{V}$, the *radial vertical field,* is well defined by

$$\iota\left(v^\alpha\frac{\partial}{\partial z^\alpha}\right)=v^\alpha\dot\partial_\alpha, \tag{1.5}$$

where here and in the rest of the paper we are using the Einstein convention.

The Finsler metric F induces a Hermitian metric on $\mathcal{V}$: if $(p;v)\in\tilde M$ e $W,Z\in\mathcal{V}_{(p;v)}$ then one (well!) defines

$$\langle W,Z\rangle=G_{\alpha\bar\beta}(p;v)W^\alpha\bar Z^\beta. \tag{1.6}$$

Since $G(p;v)=G_{\alpha\bar\beta}(p;v)v^\alpha\bar v^\beta=\langle\iota(v),\iota(v)\rangle$, the radial vertical field $\iota\colon\tilde M\to\mathcal{V}$ is an isometry. If ∇ is the covariant derivative associated to the Chern connection of the metric defined by (1.6), let $\Lambda\colon T^{1,0}\tilde M\to\mathcal{V}$ be defined by $\Lambda(X)=\nabla_X\iota$. The *horizontal bundle* $\mathcal{H}$ over $\tilde M$ is then defined by $\mathcal{H}=\operatorname{Ker}\Lambda$, the subbundle of $T^{1,0}\tilde M$ of vectors with respect to which ι is parallel. Then $T^{1,0}\tilde M=\mathcal{V}\oplus\mathcal{H}$ and a natural local frame $\{\delta_1,\dots,\delta_n\}$ for $\mathcal{H}$ is given by

$$\delta_\mu=\partial_\mu-\Gamma^\alpha_{;\mu}\dot\partial_\alpha, \tag{1.7}$$

where $\Gamma^\alpha_{;\mu}=G^{\bar\tau\alpha}G_{\bar\tau;\mu}$ and $(G^{\bar\tau\alpha})$ is the inverse matrix of $(G_{\alpha\bar\tau})$. The *horizontal map* $\Theta\colon\mathcal{V}\to\mathcal{H}$ given locally by $\Theta(\dot\partial_i)=\delta_i$ is well defined, and so we get a section

$$\chi=\Theta\circ\iota\colon\tilde M\to\mathcal{H}, \tag{1.8}$$

called *radial horizontal field* such that $\chi\left(v^\alpha\frac{\partial}{\partial z^\alpha}\right)=v^\alpha\delta_\alpha$. More generally, for any $v\in T^{1,0}_pM$ we have a map $\chi_v\colon T^{1,0}_pM\to\mathcal{H}_v$ given in local coordinates by

$$\chi_v\left(w^\alpha\left.\frac{\partial}{\partial z^\alpha}\right|_p\right)=w^\alpha\delta_\alpha|_v.$$

This allows us to canonically lift vector fields over M to horizontal vector fields over $\tilde{M}$: if ξ is a vector field over M (i.e., a section of $T^{1,0}M$), then we set

$$\xi^H(v) = \chi_v\Big(\xi\big(\pi(v)\big)\Big),$$

where $\pi\colon T^{1,0}_p M \to M$ is the canonical projection.

A Hermitian metric on $T^{1,0}\tilde{M}$ canonically associated to F is defined prescribing $\mathcal{V}\perp\mathcal{H}$ and setting

$$\langle H, K\rangle = \langle \Theta^{-1}(H), \Theta^{-1}(K)\rangle$$

for all $H, K \in \mathcal{H}_v$, so that $\Theta\colon \mathcal{V} \to \mathcal{H}$ and $\chi\colon \tilde{M} \to \mathcal{H}$ are isometries. The Chern connection of this Hermitian structure is referred to as the *Chern-Finsler connection* of F. In local coordinates the corresponding covariant derivative, for $X = X^\mu \delta_\mu + \dot{X}^\gamma \dot{\partial}_\gamma$, is given by

$$\begin{aligned}\nabla_X V &= \Big\{X^\mu\left[\delta_\mu(V^\alpha) + \Gamma^\alpha_{\beta;\mu}V^\beta\right] + \dot{X}^\gamma\left[\dot{\partial}_\gamma(V^\alpha) + \Gamma^\alpha_{\beta\gamma}V^\beta\right]\Big\}\dot{\partial}_\alpha,\\ \nabla_{\overline{X}} V &= \Big\{\overline{X^\mu}\delta_{\bar{\mu}}(V^\alpha) + \overline{X^\gamma}\dot{\partial}_{\bar{\gamma}}(V^\alpha)\Big\}\dot{\partial}_\alpha.\end{aligned}$$

where $\Gamma^\alpha_{\beta;\mu} = \dot{\partial}_\beta(\Gamma^\alpha_{;\mu})$ and $\Gamma^\alpha_{\beta\gamma} = G^{\bar{\tau}\alpha}G_{\beta\bar{\tau}\gamma}$.

The notion of Kähler Finsler metric can now be introduced. The $(2,0)$-torsion θ and the $(1,1)$-torsion τ of F are defined by

$$\nabla_X Y - \nabla_Y X = [X, Y] + \theta(X, Y) \tag{1.9}$$

and

$$\nabla_X \bar{Y} - \nabla_{\bar{Y}} X = \big[X, \bar{Y}\big] + \tau(X, \bar{Y}) + \bar{\tau}(X, \bar{Y}). \tag{1.10}$$

The torsion τ of type $(1,1)$ is $\mathcal{V}$-valued and it is related to curvature. The torsion θ of type $(2,0)$ is $\mathcal{H}$-valued and relates to Kählerianity. The decomposition $T^{1,0}\tilde{M} = \mathcal{V} \oplus \mathcal{H}$ induces decompositions for the bundles of forms. Define the *horizontal part* $p_H\theta$ of θ as the composition of θ with the projection onto the horizontal forms, the *vertical part* $p_V\theta$ of θ as the composition of θ with the projection onto the vertical forms and *mixed part* of θ as $\theta - p_H\theta - p_V\theta$. Then (see [AP2, Proposition 2.3.9]) $p_V\theta \equiv 0$, the mixed part $\theta - p_H\theta$ vanishes identically iff $G = F^2$ is a Hermitian metric and hence $\theta \equiv 0$ iff $G = F^2$ is a Hermitian Kählerian metric. With this in mind, we shall say that F is a *Kähler Finsler* metric iff for all $H \in \mathcal{H}$ one has

$$\theta(H, \chi) = 0. \tag{1.11}$$

In local coordinates, this means that $\Gamma^\alpha_{\beta;\mu}v^\mu = \Gamma^\alpha_{\mu;\beta}v^\mu$.

This notion, which agrees with the usual one for Hermitian metrics, is weaker than the one proposed by Rund [Ru] and it is slightly stronger of the one which is necessary and sufficient for the existence of totally geodesic holomorphic curves through any point and direction and which holds for the Kobayashi metric of strongly convex domains. We singled out this specific definition because it is the correct one to get the second variation formula for complex Finsler metrics (see [AP2, Theorem 2.4.4]).

Finally following Kobayashi [K] let us define the curvature. The usual procedures of Hermitian geometry yield the *curvature operator* $\Omega \in \mathcal{X}(\bigwedge^2(T^*_{\mathbb{C}}\tilde{M}) \otimes \bigwedge^{1,0}\tilde{M} \otimes T^{1,0}\tilde{M})$ associated to the Chern-Finsler connection of F. If $v \in \tilde{M}$, then the *holomorphic curvature* of F *along* v is given by

$$K_F(v) = \frac{2}{[G(v)]^2}\langle \Omega\left(\chi(v), \bar{\chi}(v)\right)\chi(v), \chi(v)\rangle. \tag{1.12}$$

Since locally

$$K_F(v) == -\frac{2}{G^2}G_\alpha \delta_{\bar{\nu}}(\Gamma^\alpha_{;\mu})v^\mu\overline{v^\nu},$$

for Hermitian metrics this is the usual notion. Furthermore, exactly as for Hermitian metrics [W], it can be shown (see [AP1], [AP2], [Ro]) that $K_F(v) = \sup\{K(\phi^*G)(0)\}$ for all $p \in M$ e $v \in \tilde{M}_p$, where the supremum is taken with respect to maps $\phi \in \mathrm{Hol}(\Delta, M)$ with $\phi(0) = p$ and $\phi'(0) = \lambda v$ for some $\lambda \in \mathbb{C}^*$, and $K(\phi^*G)(0)$ is the Gauss curvature in 0 of the metric defined on the unit disk Δ by ϕ^*G.

Finally it is possible to develop a satisfactory theory of geodesics for complex Finsler metrics even under weaker Kähler assumptions (see [AP2]) so that there is a natural notion of completeness of a Finsler metric which in particular is equivalent to the completeness of the manifold as metric space with the distance associated to the metric. With these notions we can state our main results:

Theorem 1.1: *Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex complete Kähler Finsler metric on a complex manifold M with constant positive holomorphic curvature $2c > 0$ and satisfying*

$$\forall H \in \mathcal{H} \qquad \langle \Omega(H, \bar\chi)\chi, \chi\rangle = \langle \Omega(\chi, \bar\chi)H, \chi\rangle. \tag{1.13}$$

Then (M, F) is biholomorphically isometric to the projective space $\mathbb{P}^n(\mathbb{C})$ endowed with a suitable multiple of the Fubini-Study metric.

Theorem 1.2: *Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex complete Kähler Finsler metric on a simply connected complex manifold M with constant vanishing holomorphic curvature and satisfying* (1.13). *Then (M, F) is biholomorphically isometric to $\mathbb{C}^n$ endowed with a complex Minkowski metric.*

We stress that condition (1.13) is a very natural (and mild) assumption on the curvature, because it is necessary for the existence of totally geodesic holomorphic curves (which is something to be expected in constant curvature manifolds).

For the negative curved case we are not yet able to provide a comparable result. As a step toward classification of *tame Finsler metrics*, which are metrics not "too distant" from Hermitian metrics in a sense which is made precise in Section 4, we can prove the following:

Theorem 1.3: *Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex tame Kähler Finsler metric on a simply connected complex manifold M of dimension n with constant negative holomorphic curvature and satisfying* (1.13). *Then $\exp_p: T^{1,0}_pM \to M$ is a Lipschitz homeomorphism diffeomorphic outside the origin for any $p \in M$. Furthermore, M is foliated by isometric totally geodesic holomorphic embeddings of the unit disk Δ endowed with (a suitable multiple of) the Poincaré metric, and F is (a suitable multiple of) the Kobayashi metric of M. If ρ is the distance from p relative to the metric F and $\sigma = (\tanh\rho)^2$, then σ is an exhaustion of M with the following properties:*

(i) $\sigma \in C^0\big(M\big) \cap C^\infty\big(M \setminus \{p\}\big)$;
(ii) *if* $\pi: \check M \to M$ *is the blow-up at* p, *then* $\sigma \circ \pi \in C^\infty\big(\check M\big)$;
(iii) $dd^c\sigma > 0$ *on* $M \setminus \{p\}$;
(iv) $dd^c \log\sigma \ge 0$ *on* $M \setminus \{p\}$;
(v) $(dd^c \log\sigma)^n = 0$ *on* $M \setminus \{p\}$;
(vi) $\log\sigma(z) = \log\|z\|^2 + O(1)$ *with respect to any coordinate system centered in* p.

In particular M is a Stein manifold.

In [AP2] Theorem 3.2.10 gives the same conclusion assuming a symmetry property of the curvature stronger than (1.13) in order to simplify a technical point of the proof. Unfortunately, as it was pointed out to us by J. Bland, the stronger curvature assumption, which is not verified in the nontrivial examples, implies that the metric is Hermitian and hence that the result should have a much simpler proof! With a bit more work we have been able to remove the technical assumption getting in this way a more interesting result.

Besides the notation introduced in this section, we shall freely use the expressions in local coordinates of curvature, covariant derivatives and torsions proved in [AP2] which we refer to for details.

2. Computation of the curvature

The starting point of our work is a careful estimate of the second variation of a geodesic. We recall that if F is a Kähler Finsler metric (in fact weakly Kähler is enough [AP2]) on a complex manifold M then a regular curve $\sigma_0: [a, b] \to M$ with $F(\dot\sigma) \equiv c_0 > 0$ is a geodesic for F iff

$$\nabla_{T^H + \overline{T^H}} T^H \equiv 0, \tag{2.1}$$

where $T^H(v) = \chi_v\big(\dot\sigma(t)\big) \in \mathcal{H}_v$ for all $v \in \tilde M_{\sigma(t)}$. The equation (2.1) is obtained by taking the first variation of the length of curves. In order to study the global behavior of geodesics one computes the second variation. We recall here Theorem 2.4.4 of [AP2]. If the *horizontal* $(1,1)$-*torsion* $\tau^{\mathcal{H}}$ is defined by $\tau^{\mathcal{H}}(X, \overline{Y}) = \Theta\big(\tau(X, \overline{Y})\big)$ then one has $\tau^{\mathcal{H}}(X, \overline{Y}) = \Omega(X, \overline{Y})\chi$. A *symmetric product* $\langle\!\langle\, , \rangle\!\rangle: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ locally given by

$$\forall H, K \in \mathcal{H}_v \qquad \langle\!\langle H, K\rangle\!\rangle_v = G_{\alpha\beta}(v)\, H^\alpha K^\beta,$$

is clearly globally well-defined, and for all $H \in \mathcal{H}$ it satisfies $\langle\langle H, \chi \rangle\rangle = 0$. Let $\sigma_0 \colon [a,b] \to M$ with $F(\dot\sigma_0) \equiv 1$ be a geodesic for a Kähler Finsler metric on a complex manifold M and $\Sigma \colon (-\varepsilon, \varepsilon) \times [a,b] \to M$ be a regular variation of σ_0 with fixed extremes. If

$$T = \frac{\partial \Sigma^\alpha}{\partial t} \frac{\partial}{\partial z^\alpha} \qquad \text{and} \qquad U = \frac{\partial \Sigma^\alpha}{\partial s} \frac{\partial}{\partial z^\alpha}$$

are respectively the vector field tangent to the geodesics and the transversal vector field of the variation, assume further that $\mathrm{Re}\langle U^H, T^H \rangle_{\dot\sigma_0} \equiv 0$, where T^H and U^H are the horizontal lifts of T and U respectively. Then the second variation formula for σ_0 is given by

$$\begin{aligned} \frac{d^2 \ell_\Sigma}{ds^2}(0) = \int_a^b \Big\{ &\| \nabla_{T^H + \overline{T^H}} U^H \|^2_{\dot\sigma_0} \\ &- \mathrm{Re}\Big[\langle \Omega(T^H, \overline{U^H}) U^H, T^H \rangle_{\dot\sigma_0} - \langle \Omega(U^H, \overline{T^H}) U^H, T^H \rangle_{\dot\sigma_0} \\ &+ \langle\langle \tau^{\mathcal{H}}(U^H, \overline{T^H}), U^H \rangle\rangle_{\dot\sigma_0} - \langle\langle \tau^{\mathcal{H}}(T^H, \overline{U^H}), U^H \rangle\rangle_{\dot\sigma_0} \Big] \Big\}\, dt; \end{aligned} \tag{2.2}$$

note that $T^H(\dot\sigma_0) = \chi(\dot\sigma_0)$, because σ_0 is a geodesic.

We shall compute the curvature term in (2.2) under the weakest possible hypotheses. We have the following

Theorem 2.1: *Let $F \colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Kähler Finsler metric on a complex manifold M with constant holomorphic curvature $2c \in \mathbb{R}$, that is*

$$\langle \Omega(\chi, \bar\chi)\chi, \chi \rangle = c\, G^2, \tag{2.3}$$

and satisfying the symmetry condition

$$\forall H \in \mathcal{H} \qquad \langle \Omega(H, \bar\chi)\chi, \chi \rangle = \langle \Omega(\chi, \bar\chi) H, \chi \rangle. \tag{2.4}$$

Then for all H, $K \in \mathcal{H}$

$$\begin{aligned} \mathrm{Re}\Big[&\langle \Omega(\chi, \overline{K}) H, \chi \rangle - \langle \Omega(H, \bar\chi) K, \chi \rangle + \langle\langle \tau^{\mathcal{H}}(H, \bar\chi), K \rangle\rangle - \langle\langle \tau^{\mathcal{H}}(\chi, \overline{K}), H \rangle\rangle \Big] \\ &= \frac{c}{2} \mathrm{Re}\Big\{ G\big[\langle H, K \rangle - \langle\langle H, K \rangle\rangle\big] + \langle H, \chi \rangle \big[\langle \chi, K \rangle - 2\langle K, \chi \rangle\big] - G \langle \hat\theta^{\mathcal{V}}(\chi, \overline{K}), \hat\theta^{\mathcal{V}}(\chi, \overline{H}) \rangle \Big\}, \end{aligned}$$

where $\hat\theta^{\mathcal{V}}$ is the dual $(1,1)$-torsion to be defined in (2.9).

We recall [AP2, Proposition 3.1.7] that (2.3) and (2.4) together are equivalent to

$$\tau^{\mathcal{H}}(\chi, \bar\chi) = c\, G \chi, \tag{2.5}$$

the integrability condition for existence and uniqueness of complex geodesic curves.

The first step in evaluating the curvature term under the assumptions of the theorem was already carried out in [AP2, Proposition 3.2.3]:

Proposition 2.2: *If $F \colon T^{1,0}M \to \mathbb{R}^+$ is a strongly pseudoconvex Finsler metric on a complex manifold M with constant holomorphic curvature $2c \in \mathbb{R}$ satisfying* (2.4), *then*

$$\langle \Omega(H, \bar\chi) K, \chi \rangle = c\big\{ \langle H, \chi \rangle \langle K, \chi \rangle + G \langle\langle H, K \rangle\rangle \big\}.$$

The computation of the other $(2,0)$-addend requires some preliminary results.

Lemma 2.3: *Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold M. Then for all $V \in \mathcal{V}$ and H, $K \in \mathcal{H}$ we have*

$$(\nabla_V \Omega)(H, \overline{K})\chi = \tau^{\mathcal{H}}\left(H, \overline{\tau(K, \overline{V})}\right) - \tau^{\mathcal{H}}\big(\theta(V,H), \overline{K}\big).$$

Proof: Clearly it suffices to consider $\Omega^{\mathcal{H}} = \Omega^\alpha_\beta \otimes dz^\beta \otimes \delta_\alpha$; in particular,

$$\nabla_V \Omega^{\mathcal{H}} = (\nabla_V \Omega^\alpha_\beta) \otimes dz^\beta \otimes \delta_\alpha - \Omega^\alpha_\gamma \otimes \omega^\gamma_\beta(V) dz^\beta \otimes \delta_\alpha + \Omega^\gamma_\beta \otimes dz^\beta \otimes \omega^\alpha_\gamma(V)\delta_\alpha.$$

Let us compute. Using [AP2, Lemma 2.3.3], $\Gamma^\alpha_{\beta\sigma} v^\beta = 0$ and $\Gamma^\alpha_{\beta;\mu} v^\beta = \Gamma^\alpha_{;\mu}$, we have:

$$\begin{aligned}
(\nabla_V \Omega^\alpha_\beta)(H, \overline{K}) v^\beta &= \big[V(R^\alpha_{\beta;\mu\bar\nu}) - R^\alpha_{\beta;\rho\bar\nu}\omega^\rho_\mu(V)\big] H^\mu \overline{K^\nu} v^\beta \\
&= -\big[\dot\partial_\lambda \delta_{\bar\nu}(\Gamma^\alpha_{\beta;\mu}) + \dot\partial_\lambda\big(\Gamma^\alpha_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu})\big) - \delta_{\bar\nu}(\Gamma^\alpha_{\beta;\rho})\Gamma^\rho_{\mu\lambda} - \Gamma^\alpha_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\rho})\Gamma^\rho_{\mu\lambda}\big] V^\lambda H^\mu \overline{K^\nu} v^\beta \\
&= -\big[\delta_{\bar\nu}\dot\partial_\lambda(\Gamma^\alpha_{\beta;\mu}) - \Gamma^{\bar\tau}_{\lambda;\bar\nu}\dot\partial_{\bar\tau}(\Gamma^\alpha_{\beta;\mu}) + \dot\partial_\lambda(\Gamma^\alpha_{\beta\sigma})\delta_{\bar\nu}(\Gamma^\sigma_{;\mu}) + \Gamma^\alpha_{\beta\sigma}\dot\partial_\lambda\delta_{\bar\nu}(\Gamma^\sigma_{;\mu}) - \delta_{\bar\nu}(\Gamma^\alpha_{\beta;\rho})\Gamma^\rho_{\mu\lambda}\big] V^\lambda H^\mu \overline{K^\nu} v^\beta \\
&= -\big[\delta_{\bar\nu}\big(v^\beta\dot\partial_\lambda(\Gamma^\alpha_{\beta;\mu})\big) - \Gamma^{\bar\tau}_{\lambda;\bar\nu}\Gamma^\alpha_{\bar\tau;\mu} + v^\beta\dot\partial_\lambda(\Gamma^\alpha_{\beta\sigma})\delta_{\bar\nu}(\Gamma^\sigma_{;\mu}) - \delta_{\bar\nu}(\Gamma^\alpha_{;\rho})\Gamma^\rho_{\mu\lambda}\big] V^\lambda H^\mu \overline{K^\nu} \\
&= -\big[-\Gamma^{\bar\tau}_{\lambda;\bar\nu}\Gamma^\alpha_{\bar\tau;\mu} - \Gamma^\alpha_{\lambda\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu}) - \delta_{\bar\nu}(\Gamma^\alpha_{;\rho})\Gamma^\rho_{\mu\lambda}\big] V^\lambda H^\mu \overline{K^\nu}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\Omega^\alpha_\gamma(H, \overline{K})\omega^\gamma_\beta(V) v^\beta &= -\big[\delta_{\bar\nu}(\Gamma^\alpha_{\gamma;\mu}) + \Gamma^\alpha_{\gamma\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu})\big]\Gamma^\gamma_{\beta\lambda} V^\lambda H^\mu \overline{K^\nu} v^\beta = 0, \\
\omega^\alpha_\gamma(V)\Omega^\gamma_\beta(H, \overline{K}) v^\beta &= -\Gamma^\alpha_{\gamma\lambda}\big[\delta_{\bar\nu}(\Gamma^\gamma_{\beta;\mu}) + \Gamma^\gamma_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu})\big] V^\lambda H^\mu \overline{K^\nu} v^\beta \\
&= -\Gamma^\alpha_{\gamma\lambda}\delta_{\bar\nu}(\Gamma^\gamma_{;\mu}) V^\lambda H^\mu \overline{K^\nu};
\end{aligned}$$

hence

$$\begin{aligned}
(\nabla_V \Omega)(H, \overline{K})\chi &= \big[\Gamma^\alpha_{\bar\tau;\mu}\Gamma^{\bar\tau}_{\lambda;\bar\nu} + \delta_{\bar\nu}(\Gamma^\alpha_{;\rho})\Gamma^\rho_{\mu\lambda}\big] V^\lambda H^\mu \overline{K^\nu}\delta_\alpha \\
&= \tau^{\mathcal{H}}\left(H, \overline{\tau(K, \overline{V})}\right) - \tau^{\mathcal{H}}\big(\theta(V,H), \overline{K}\big).
\end{aligned}$$

□

Lemma 2.4: *Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold M with constant holomorphic curvature $2c$ and satisfying* (2.4). *Then for all $H \in \mathcal{H}$*

$$\tau^{\mathcal{H}}(H, \bar\chi) = c\big[\langle H, \chi\rangle\chi + GH\big] - \Omega(\chi, \bar\chi)H - \tau^{\mathcal{H}}\left(\chi, \overline{\tau\big(\chi, \overline{\Theta^{-1}(H)}\big)}\right).$$

Proof: Let $V = \Theta^{-1}(H)$. Then

$$\begin{aligned}
\nabla_V(c\,G\chi) &= c\big[\langle H, \chi\rangle\chi + GH\big], \\
\nabla_V\big(\tau^{\mathcal{H}}(\chi, \bar\chi)\big) = \nabla_V\big(\Omega(\chi, \bar\chi)\chi\big) &= (\nabla_V\Omega)(\chi, \bar\chi)\chi + \Omega(H, \bar\chi)\chi + \Omega(\chi, \bar\chi)H \\
&= \tau^{\mathcal{H}}\left(\chi, \overline{\tau\big(\chi, \overline{V}\big)}\right) + \tau^{\mathcal{H}}(H, \bar\chi) + \Omega(\chi, \bar\chi)H,
\end{aligned}$$

by [AP2, Lemma 2.3.8] and Lemma 2.3, because $\theta(V, \chi) = 0$. Hence the assertion follows from (2.5). □

Lemma 2.5: *Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Kähler Finsler metric on a complex manifold M with constant holomorphic curvature $2c$ and satisfying* (2.4). *Then for all $H \in \mathcal{H}$*

$$\Omega(\chi, \bar\chi)H = \tau^{\mathcal{H}}(H, \bar\chi) = \Omega(H, \bar\chi)\chi.$$

Proof: Indeed

$$\begin{aligned}
\Omega(\chi, \bar\chi)H &= -\big[\delta_{\bar\nu}(\Gamma^\alpha_{\beta;\mu}) + \Gamma^\alpha_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu})\big] v^\mu \overline{v^\nu} H^\beta \delta_\alpha \\
&= -\big[\delta_{\bar\nu}(\Gamma^\alpha_{\beta;\mu} v^\mu) + \Gamma^\alpha_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu}) v^\mu\big] \overline{v^\nu} H^\beta \delta_\alpha \\
&= -\big[\delta_{\bar\nu}(\Gamma^\alpha_{;\beta})\overline{v^\nu} + \Gamma^\alpha_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu}) v^\mu \overline{v^\nu}\big] H^\beta \delta_\alpha \\
&= -\big[\delta_{\bar\nu}(\Gamma^\alpha_{;\beta})\overline{v^\nu} - c\,G\Gamma^\alpha_{\beta\sigma} v^\sigma\big] H^\beta \delta_\alpha \\
&= -\delta_{\bar\nu}(\Gamma^\alpha_{;\beta})\overline{v^\nu} H^\beta \delta_\alpha = \tau^{\mathcal{H}}(H, \bar\chi),
\end{aligned}$$

where we used the Kähler condition, $\Gamma^\alpha_{\beta\sigma} v^\sigma = 0$ and (2.5). □

Proposition 2.6: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Kähler Finsler metric on a complex manifold M with constant holomorphic curvature $2c$ and satisfying* (2.4). *Then for any H, $K \in \mathcal{H}$*

$$\tau^{\mathcal{H}}(H,\bar\chi) = \frac{c}{2}\big[\langle H,\chi\rangle\chi + GH\big] - \frac{1}{2}\,\tau^{\mathcal{H}}\left(\chi, \overline{\tau\big(\chi, \overline{\Theta^{-1}(H)}\big)}\right), \tag{2.6}$$

$$\langle\tau^{\mathcal{H}}(H,\bar\chi),K\rangle = \frac{c}{2}\big[\langle H,\chi\rangle\langle\chi,K\rangle + G\langle H,K\rangle\big] - \frac{1}{2}\langle\tau\big(\chi,\overline{\Theta^{-1}(K)}\big),\tau\big(\chi,\overline{\Theta^{-1}(H)}\big)\rangle \tag{2.7}$$

and

$$\langle\!\langle\tau^{\mathcal{H}}(H,\bar\chi),K\rangle\!\rangle = \frac{c}{2}\,G\langle\!\langle H,K\rangle\!\rangle. \tag{2.8}$$

Proof: (2.6) follows from Lemmas 2.4 and 2.5. (2.7) follows from (2.6) and [AP2, Proposition 2.6.7.(i)]. Furthermore, [AP2, Proposition 2.6.7.(ii)] says that

$$\langle\!\langle\tau^{\mathcal{H}}\left(\chi,\overline{\tau\big(\chi,\overline{\Theta^{-1}(H)}\big)}\right),K\rangle\!\rangle = \langle\theta(K,\chi),\tau^{\mathcal{H}}\big(\chi,\overline{\Theta^{-1}(H)}\big)\rangle = 0,$$

because F is Kähler, and (2.8) follows from $\langle\!\langle\chi,K\rangle\!\rangle = 0$. □

For the evaluation of the $(1,1)$-addends we need a new object. Define a $T^{1,0}\tilde M$-valued $(1,1)$-form $\hat\theta \in \mathcal{X}(\bigwedge^{1,1}\tilde M \otimes T^{1,0}\tilde M)$ by the formula

$$\langle\theta(X,Y),Z\rangle = \langle X,\hat\theta(Z,\overline{Y})\rangle \tag{2.9}$$

for all X, Y, $Z \in T^{1,0}\tilde M$; the form $\hat\theta$ is the *dual* $(1,1)$*-torsion*. In local coordinates, $\hat\theta$ is given by

$$\hat\theta = \hat\theta^{\mathcal{H}} + \hat\theta^{\mathcal{V}} = \hat\theta^\sigma\otimes\delta_\sigma + \dot{\hat\theta}^\alpha\otimes\dot\partial_\alpha,$$

where

$$\begin{aligned}\hat\theta^\sigma &= G^{\bar\tau\sigma}G_{\mu\bar\rho}(\Gamma^{\bar\rho}_{\bar\nu;\bar\tau} - \Gamma^{\bar\rho}_{\bar\tau;\bar\nu})\,dz^\mu\wedge d\bar z^\nu - G^{\bar\tau\sigma}G_{\bar\tau\mu\bar\gamma}\,dz^\mu\wedge\overline{\psi^\gamma},\\ \dot{\hat\theta}^\alpha &= G^{\bar\tau\alpha}G_{\bar\tau\mu\bar\nu}\,dz^\mu\wedge d\bar z^\nu.\end{aligned}$$

Since $\hat\theta^{\mathcal{V}}(H,\bar\chi)\equiv 0$, we immediately see that F is Kähler iff $\hat\theta(H,\bar\chi)\equiv 0$. On the other hand,

$$\hat\theta^{\mathcal{V}}(\chi,\overline{K}) = G^{\bar\tau\alpha}G_{\bar\tau\bar\nu}\overline{K^\nu}\dot\partial_\alpha$$

vanishes for all $K\in\mathcal{H}$ iff $G_{\tau\nu}\equiv 0$; recalling [AP2, Proposition 2.3.9.(i)] we have proved the

Lemma 2.7: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold M. Then F comes from a Hermitian metric iff*

$$\forall K\in\mathcal{H} \qquad \hat\theta^{\mathcal{V}}(\chi,\overline{K}) = 0.$$

The form $\hat\theta^{\mathcal{V}}(\chi,\overline{K})$ in a way transforms the Hermitian product into the symmetric product and conversely. In fact it is easy to check that

$$\langle\hat\theta^{\mathcal{V}}(\chi,\overline{K}),\Theta^{-1}(H)\rangle = \overline{\langle\!\langle H,K\rangle\!\rangle} \tag{2.10}$$

and

$$\langle\!\langle\hat\theta^{\mathcal{V}}(\chi,\overline{K}),\Theta^{-1}(H)\rangle\!\rangle = \langle\hat\theta^{\mathcal{V}}(\chi,\overline{K}),\hat\theta^{\mathcal{V}}(\chi,\overline{H})\rangle \tag{2.11}$$

for all H, $K\in\mathcal{H}$.

Proposition 2.8: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Finsler metric on a complex manifold M with constant holomorphic curvature $2c \in \mathbb{R}$ and satisfying* (2.4). *Then*

$$\langle \tau^{\mathcal{H}}(\chi, \overline{K}), H\rangle = c\left[\langle \chi, H\rangle\langle \chi, K\rangle + G\overline{\langle\!\langle H, K\rangle\!\rangle}\right], \tag{2.12}$$

$$\tau(\chi, \overline{K}) = c\left[\langle \chi, K\rangle\iota + G\,\hat\theta^{\mathcal{V}}(\chi, \overline{K})\right] \tag{2.13}$$

and

$$\langle\!\langle \tau^{\mathcal{H}}(\chi, \overline{K}), H\rangle\!\rangle = c\,G\langle \hat\theta^{\mathcal{V}}(\chi, \overline{K}), \hat\theta^{\mathcal{V}}(\chi, \overline{H})\rangle \tag{2.14}$$

for all H, $K \in \mathcal{H}$.

Proof: By [AP2, (2.5.2) and Lemma 2.3.8] we know that

$$\langle \tau^{\mathcal{H}}(\chi, \overline{K}), H\rangle = \langle \Omega(\chi, \overline{K})\chi, H\rangle = \overline{\langle \Omega(K, \bar\chi)H, \chi\rangle},$$

and (2.12) follows from [AP2, Proposition 3.2.3]. (2.10) then yields (2.13), and (2.14) follows from $\langle\!\langle \chi, H\rangle\!\rangle = 0$ and (2.11). □

Proposition 2.9: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex weakly Kähler (i.e., such that*

$$\langle \theta(H, \chi), \chi\rangle = 0$$

for all $H \in \mathcal{H}$) Finsler metric on a complex manifold M with constant holomorphic curvature $2c \in \mathbb{R}$ and satisfying (2.4). *Then*

$$\langle \Omega(\chi, \overline{K})H, \chi\rangle = \frac{c}{2}\left[\langle H, \chi\rangle\langle \chi, K\rangle + G\langle H, K\rangle + G\langle \hat\theta^{\mathcal{V}}(\chi, \overline{K}), \hat\theta^{\mathcal{V}}(\chi, \overline{H})\rangle\right]$$

for all H, $K \in \mathcal{H}$.

Proof: [AP2, Proposition 3.2.2] yields

$$\langle \Omega(\chi, \overline{K})H, \chi\rangle + \langle \Omega(H, \overline{K})\chi, \chi\rangle = c\big[\langle H, \chi\rangle\langle \chi, K\rangle + G\langle H, K\rangle\big].$$

Furthermore, the proof of [AP2, Proposition 3.2.5] yields

$$\langle \Omega(\chi, \overline{K})H, \chi\rangle - \langle \Omega(H, \overline{K})\chi, \chi\rangle = \langle\!\langle \tau^{\mathcal{H}}(\chi, \overline{K}), H\rangle\!\rangle \tag{2.15}$$

for all H, $K \in \mathcal{H}$. The assertion then follows from Proposition 2.8. □

Summing up, Theorem 2.1 follows from Propositions 2.2, 2.6, 2.8 and 2.9.

3. The positive curvature case

We are almost ready to characterize the constant positive curvature complex Finsler manifolds. We still need one lemma.

Lemma 3.1: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Kähler Finsler metric on a complex manifold M with constant holomorphic curvature $2c \in \mathbb{R}$ and satisfying* (2.4). *Then*

$$\langle \Omega(\chi, \overline{K})H, \chi\rangle - \langle \Omega(H, \overline{K})\chi, \chi\rangle = -\langle \tau\big(\chi, \overline{\Theta^{-1}(K)}\big), \tau\big(\chi, \overline{\Theta^{-1}(H)}\big)\rangle$$

for all H, $K \in \mathcal{H}$.

Proof: Let $W = \Theta^{-1}(K)$. Then

$$\begin{aligned}
\overline{W}\langle \Omega(H, \bar\chi)\chi, \chi\rangle &= \langle (\nabla_{\overline{W}}\Omega)(H, \bar\chi)\chi, \chi\rangle + \langle \Omega(\nabla_{\overline{W}}H, \bar\chi)\chi, \chi\rangle \\
&\qquad + \langle \Omega(H, \overline{K})\chi, \chi\rangle + \langle \Omega(H, \bar\chi)\chi, K\rangle,\\
\overline{W}\langle \Omega(\chi, \bar\chi)H, \chi\rangle &= \langle (\nabla_{\overline{W}}\Omega)(\chi, \bar\chi)H, \chi\rangle + \langle \Omega(\chi, \overline{K})H, \chi\rangle \\
&\qquad + \langle \Omega(\chi, \bar\chi)(\nabla_{\overline{W}}H), \chi\rangle + \langle \Omega(\chi, \bar\chi)H, K\rangle.
\end{aligned}$$

Recalling (2.4), Lemma 2.5 and [AP2, Lemma 3.1.5], we then obtain

$$\langle \Omega(\chi, \overline{K})H, \chi \rangle - \langle \Omega(H, \overline{K})\chi, \chi \rangle = -\langle (\nabla_{\overline{W}}\Omega)(\chi, \bar{\chi})H, \chi \rangle. \tag{3.1}$$

The proof of [AP2, Lemma 3.1.5] shows that

$$\begin{aligned}
&\langle (\nabla_{\overline{W}}\Omega)(\chi, \bar{\chi})H, \chi \rangle \\
&\quad = -\left[-\delta_{\bar{\nu}}(G_{\beta\sigma}\Gamma^{\sigma}_{\bar{\gamma};\mu}) + G_{\beta\sigma}\delta_{\bar{\nu}}(\Gamma^{\sigma}_{\bar{\gamma};\mu}) - \delta_{\bar{\rho}}(G_{\alpha}\Gamma^{\alpha}_{\beta;\mu})\Gamma^{\bar{\rho}}_{\bar{\nu}\bar{\gamma}} - G_{\beta\sigma}\delta_{\bar{\rho}}(\Gamma^{\sigma}_{;\mu})\Gamma^{\bar{\rho}}_{\bar{\nu}\bar{\gamma}}\right] v^{\mu}\overline{v^{\nu}}H^{\beta}\overline{W^{\gamma}} \\
&\quad = -\left[-\Gamma^{\sigma}_{\bar{\gamma};\mu}\delta_{\bar{\nu}}(G_{\beta\sigma})\right] v^{\mu}\overline{v^{\nu}}H^{\beta}\overline{W^{\gamma}} = G_{\beta\bar{\tau}}\Gamma^{\bar{\tau}}_{\sigma;\bar{\nu}}\Gamma^{\sigma}_{\bar{\gamma};\mu}v^{\mu}\overline{v^{\nu}}H^{\beta}\overline{W^{\gamma}} \\
&\quad = \langle H, \tau^{\mathcal{H}}\big(\chi, \overline{\tau(\chi, \overline{W})}\big)\rangle = \langle \tau(\chi, \overline{W}), \tau\big(\chi, \overline{\Theta^{-1}(H)}\big)\rangle,
\end{aligned}$$

where we used respectively [AP2, (2.6.18)] and [AP2, Proposition 2.6.7.(i)], and we are done. □

We are now in condition of proving Theorem 1.1:

Proof of Theorem 1.1 It suffices to prove that under these hypotheses F comes from a Hermitian metric, and then invoke the analogous result for Kähler Hermitian manifolds. But indeed Lemma 3.1, (2.15) and Proposition 2.8 yield

$$c\,G\langle \hat{\theta}^{\mathcal{V}}(\chi, \overline{K}), \hat{\theta}^{\mathcal{V}}(\chi, \overline{K})\rangle = -\langle \tau\big(\chi, \overline{\Theta^{-1}(K)}\big), \tau\big(\chi, \overline{\Theta^{-1}(K)}\big)\rangle$$

for all $K \in \mathcal{H}$. Being $c > 0$ this forces $\hat{\theta}^{\mathcal{V}}(\chi, \overline{K}) \equiv 0$, and the assertion follows from Lemma 2.7. □

4. Existence of Monge-Ampère potentials

While there exists a unique complex manifold and a unique complete Kähler Finsler metric with constant positive holomorphic curvature satisfying (2.4), the situation for manifolds with nonpositive constant holomorphic curvature is quite different. As first step in this case we show that it is possible to construct exhaustions satisfying the complex Monge-Ampère equation. Before stating the main result of this section (which includes Theorem 1.3), we need a further definition.

We say that a strongly pseudoconvex Finsler metric F is *tame* if it satisfies

$$\mathrm{Re}\big[\langle H, H\rangle + \langle\!\langle H, H\rangle\!\rangle\big] \geq \langle \hat{\theta}^{\mathcal{V}}(\chi, \overline{H}), \hat{\theta}^{\mathcal{V}}(\chi, \overline{H})\rangle$$

for all $H \in \mathcal{H}$ such that $\langle H, \chi\rangle = 0$ (and then for all $H \in \mathcal{H}$). Note that this is only a punctual requirement on F (i.e., it depends on the derivatives of F along the v directions only, and not on derivatives along the z directions). We shall discuss briefly the meaning of this notion at the end of this section.

Suppose that F is a strongly pseudoconvex Kähler Finsler with constant positive holomorphic curvature $2c \leq 0$ and satisfying (2.4). Assume furthermore that if $c < 0$ then F is tame. Then Theorem 2.1 implies that the curvature term in the second variation either vanishes or is positive. Therefore, exactly how it is done in [AP2] one may reconstruct the global geometry of M. In particular the possibility of controlling the second variation is the key to get in this setting the analogue of Cartan-Hadamard Theorem ([AP2, Theorem 3.2.7]) under our weaker assumption. Once this result is obtained, as only Kählerianity and constant curvature are needed in the proofs, one may recover also Theorem 3.2.10 of [AP2] with exactly the same proof. We can therefore invoke the proofs of Theorem 3.2.7 and of Theorem 3.2.10 of [AP2] to conclude

Theorem 4.1: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a complete strongly pseudoconvex Kähler Finsler metric on a simply connected complex manifold M with nonpositive constant holomorphic curvature $2c \leq 0$ and satisfying* (2.4). *Assume furthermore that if $c < 0$ then F is tame. Then $\exp_p\colon T^{1,0}_p \to M$ is a Lipschitz homeomorphism diffeomorphic outside the origin for any $p \in M$. The manifold M is foliated by isometric totally geodesic holomorphic embeddings of the unit disk Δ endowed with a suitable multiple of the Poincaré metric if $c < 0$, or by isometric totally geodesic holomorphic embeddings of $\mathbb{C}$ endowed with the euclidean metric if $c = 0$. In particular, if $c = -2$ then F is the Kobayashi metric of M, and if $c = 0$ then the Kobayashi metric of M vanishes identically. Furthermore if ρ is the distance from p relative to the metric F, and setting $\sigma_c = (\tanh\rho)^2$ when $c < 0$ and $\sigma_0 = \rho^2$ otherwise, then*

(i) $\sigma_c \in C^0(M) \cap C^\infty(M \setminus \{p\})$;
(ii) *if* $\pi: \check{M} \to M$ *is the blow-up at* p, *then* $\sigma_c \circ \pi \in C^\infty(\check{M})$;
(iii) $dd^c \sigma_c > 0$ *on* $M \setminus \{p\}$;
(iv) $dd^c \log \sigma_c \geq 0$ *on* $M \setminus \{p\}$;
(v) $(dd^c \log \sigma_c)^n = 0$ *on* $M \setminus \{p\}$;
(vi) $\log \sigma_c(z) = \log \|z\|^2 + O(1)$ *with respect to any coordinate system centered in* p.

In particular M *is a Stein manifold.*

A remark concerning [AP2] is in order. Let $F: T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex complete Kähler Finsler metric on a complex manifold M with constant negative holomorphic curvature $2c < 0$ and satisfying (2.4). Then Lemma 2.7, (2.15) and Proposition 2.8 imply that

$$\langle \Omega(H, \overline{K})\chi, \chi \rangle = \langle \Omega(\chi, \overline{K})H, \chi \rangle \tag{4.1}$$

for all H, $K \in \mathcal{H}$ iff F comes from a Hermitian metric (we thank J. Bland for pointing this out to us). In particular, this means that the hypotheses of [AP2, Theorem 3.2.7] holds for $c < 0$ iff F comes from a Hermitian metric, disposing of most of the interest of the theorem. But a consequence of Theorem 2.1 is that assuming only (2.4) instead of (4.1) we can recover the same results up to assuming that the metric F is tame.

We end this section with a couple of remarks about tame metrics. It is easy to check that the indicatrices of a strongly pseudoconvex Finsler metric F are strongly convex iff

$$\mathrm{Re}\big[\langle H, H \rangle + \langle\langle H, H \rangle\rangle\big] > 0$$

for all $H \in \mathcal{H}$, with $H \neq 0$. So the indicatrices of a tame metric are somewhat more than strongly convex.

Concerning the existence (and not existence) of tame $(1,1)$-homogeneous functions we state the following two results, whose elementary proofs are left to the reader.

Proposition 4.2: *Let* $g: \mathbb{C}^n \to \mathbb{R}^+$ *be a Hermitian norm,* $f: \mathbb{C}^n \to \mathbb{R}^+$ *any* $(1,1)$*-homogeneous function and* $\varepsilon << 1$. *Then* $G = g + \varepsilon f$ *is a tame* $(1,1)$*-homogeneous function.*

Proposition 4.3: *Let* $G: \mathbb{C}^2 \to \mathbb{R}^+$ *be a strongly convex* $(1,1)$*-homogeneous function such that*

$$G(v) = \left(a + b \,\mathrm{Re}\, \frac{v_1}{v_2}\right) \big[|v_1|^2 + |v_2|^2\big] \tag{4.2}$$

(with a, $b \in \mathbb{R}^+$*) in a conical neighborhood of the diagonal* $\{v_1 = v_2\}$. *Assume also that*

$$\frac{1}{5} \leq \frac{b}{a} < \frac{1}{2\sqrt{2}} \tag{4.3}$$

(e.g., $a = 5$ *and* $b = 1$*). Then* G *is not tame.*

Thus we may conclude that although strongly convex complex Finsler metrics need not to be tame, at least an "open" neighborhood of the Hermitian metrics is made of tame complex Finsler metrics. We add that while this assumption is essential for our proof of Theorem 4.1 (proof based on the estimate of the second variation) to work, the same conclusions hold for the Kobayashi metric of strongly convex domains which need not to be tame. As a matter of fact to estimate the second variation it would suffice to have a tame metric on a large enough set. This may very well be the case for the Kobayashi metric.

5. The zero curvature case

We start with a consequence of Theorem 4.1:

Theorem 5.1: *Let* $F: T^{1,0}M \to \mathbb{R}^+$ *be a simply connected strongly pseudoconvex complete Kähler Finsler metric on a complex manifold* M *with vanishing holomorphic curvature and satisfying* (2.4). *Then the exponential map* $\exp_p: T^{1,0}_p M \cong \mathbb{C}^n \to M$ *is a biholomorphism for every* $p \in M$.

Proof: The proof is a very easy corollary of the geometric theory of complex Monge-Ampère equation (see [B] and [Pa] for example). Since by construction the leaves of the Monge-Ampère foliation associated to σ_0 are all parabolic, then the foliation is holomorphic and hence $\exp_p$ must be biholomorphic. □

From now on, we shall assume that M is simply connected and that F is a strongly pseudoconvex complete Kähler Finsler metric of vanishing holomorphic curvature and satisfying (2.4); in particular, M is biholomorphic to $\mathbb{C}^n$, and Proposition 2.8, (2.15), Lemma 3.1, and Proposition 2.6 yield

$$\tau(H,\bar\chi)=\tau(\chi,\overline{K})=0 \tag{5.1}$$

and

$$\tau(\chi,\overline{W})=0 \tag{5.2}$$

for all H, $K\in\mathcal{H}$ and $W\in\mathcal{V}$.

To classify the metrics which may occur under these assumptions we need some technical facts.

Lemma 5.2: *Let $F\colon T^{1,0}M\to\mathbb{R}^+$ be a strongly pseudoconvex Kähler Finsler metric on a complex manifold M with constant vanishing holomorphic curvature and satisfying* (2.4)*. Then for all H, $K\in\mathcal{H}$,*

$$\Omega(\chi,\overline{K})H=\tau^{\mathcal{H}}(H,\overline{K})=0,$$

and for all $H\in\mathcal{H}$ and $W\in\mathcal{V}$

$$\Omega(\chi,\overline{W})H=\tau^{\mathcal{H}}(H,\overline{W})=0.$$

Proof: First of all, for H, $K\in\mathcal{H}$

$$\begin{aligned}\Omega(\chi,\overline{K})H&=-\big[\delta_{\bar\nu}(\Gamma^\alpha_{\beta;\mu})+\Gamma^\alpha_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu})\big]v^\mu\overline{K^\nu}H^\beta\delta_\alpha\\&=-\big[\delta_{\bar\nu}(\Gamma^\alpha_{\beta;\mu}v^\mu)+\Gamma^\alpha_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu})v^\mu\big]\overline{K^\nu}H^\beta\delta_\alpha\\&=-\big[\delta_{\bar\nu}(\Gamma^\alpha_{;\beta})\overline{K^\nu}+\Gamma^\alpha_{\beta\sigma}\delta_{\bar\nu}(\Gamma^\sigma_{;\mu})v^\mu\overline{K^\nu}\big]H^\beta\delta_\alpha\\&=-\delta_{\bar\nu}(\Gamma^\alpha_{;\beta})\overline{K^\nu}H^\beta\delta_\alpha=\tau^{\mathcal{H}}(H,\overline{K}),\end{aligned}\tag{5.3}$$

which follows from the Kähler condition and from $\tau(\chi,\overline{K})=0$.

Next, put $V=\Theta^{-1}(H)$ and apply ∇_V to the second equation in (5.1). We get

$$\begin{aligned}0=\nabla_V\big(\Omega(\chi,\overline{K})\chi\big)&=(\nabla_V\Omega)(\chi,\overline{K})\chi+\Omega(H,\overline{K})\chi+\Omega(\chi,\nabla_V\overline{K})\chi+\Omega(\chi,\overline{K})H\\&=\tau^{\mathcal{H}}\big(\chi,\overline{\tau(K,\overline{V})}\big)+\tau^{\mathcal{H}}(H,\overline{K})+\Omega(\chi,\overline{K})H=2\tau^{\mathcal{H}}(H,\overline{K}),\end{aligned}$$

by Lemma 2.3, $\theta(V,\chi)=0$, (5.2) and (5.3).

On the other hand, for $H\in\mathcal{H}$ and $W\in\mathcal{V}$,

$$\Omega(\chi,\overline{W})H=-[\dot\partial_{\bar\gamma}(\Gamma^\alpha_{\beta;\mu})+\Gamma^\alpha_{\beta\sigma}\Gamma^\sigma_{\bar\gamma;\mu}]v^\mu\overline{W^\gamma}H^\beta\delta_\alpha=-\Gamma^\alpha_{\bar\gamma;\beta}\overline{W^\gamma}H^\beta\delta_\alpha=\tau^{\mathcal{H}}(H,\overline{W}),$$

by the Kähler condition and (5.2). Next, if $V\in\mathcal{V}$, again using the Kähler condition, (5.2), $\Gamma^\tau_{\beta\lambda}v^\beta=0$ and $v^\mu\dot\partial_\beta(\Gamma^\alpha_{\bar\gamma;\mu})=v^\mu\dot\partial_{\bar\gamma}(\Gamma^\alpha_{\beta;\mu})=\Gamma^\alpha_{\bar\gamma;\beta}$, we have

$$\begin{aligned}(\nabla_V\Omega)(\chi,\overline{W})\chi=&-\big[\dot\partial_\lambda\dot\partial_{\bar\gamma}\dot\partial_\beta(\Gamma^\alpha_{;\mu})+\dot\partial_\lambda(\Gamma^\alpha_{\beta\sigma}\Gamma^\sigma_{\bar\gamma;\mu})-[\dot\partial_{\bar\gamma}(\Gamma^\alpha_{\beta;\rho})+\Gamma^\alpha_{\beta\sigma}\Gamma^\sigma_{\bar\gamma;\rho}]\Gamma^\rho_{\mu\lambda}\big]v^\mu\overline{W^\gamma}v^\beta V^\lambda\delta_\alpha\\&+[\dot\partial_\lambda\dot\partial_\tau(\Gamma^\alpha_{;\mu})+\Gamma^\alpha_{\tau\sigma}\Gamma^\sigma_{\bar\gamma;\mu}]\Gamma^\tau_{\beta\lambda}v^\mu\overline{W^\gamma}v^\beta V^\lambda\delta_\alpha\\&-[\dot\partial_{\bar\gamma}(\Gamma^\tau_{\beta;\mu})+\Gamma^\tau_{\beta\sigma}\Gamma^\sigma_{\bar\gamma;\mu}]\Gamma^\alpha_{\tau\lambda}v^\mu\overline{W^\gamma}v^\beta V^\lambda\delta_\alpha\\=&-[\dot\partial_\lambda\dot\partial_\beta(\Gamma^\alpha_{\bar\gamma;\mu})+\dot\partial_\lambda(\Gamma^\alpha_{\beta\sigma})\Gamma^\sigma_{\bar\gamma;\mu}]v^\mu\overline{W^\gamma}v^\beta V^\lambda\delta_\alpha\\=&-\dot\partial_\lambda\dot\partial_\beta(\Gamma^\alpha_{\bar\gamma;\mu})v^\mu v^\beta\overline{W^\gamma}V^\lambda\delta_\alpha\\=&-[\dot\partial_\lambda(v^\mu\dot\partial_\beta(\Gamma^\alpha_{\bar\gamma;\mu}))-\dot\partial_\beta(\Gamma^\alpha_{\bar\gamma;\lambda})]v^\beta\overline{W^\gamma}V^\lambda\delta_\alpha\\=&-[\dot\partial_\lambda(\Gamma^\alpha_{\bar\gamma;\beta})-\dot\partial_{\bar\gamma}(\Gamma^\alpha_{\beta;\lambda})]v^\beta\overline{W^\gamma}V^\lambda\delta_\alpha\\=&-[\Gamma^\alpha_{\bar\gamma;\lambda}-\Gamma^\alpha_{\bar\gamma;\lambda}]\overline{W^\gamma}V^\lambda=0\end{aligned}$$

Hence if $V=\Theta^{-1}(H)$ we have

$$0=\nabla_V\big(\Omega(\chi,\overline{W})\chi\big)=(\nabla_V\Omega)(\chi,\overline{W})\chi+\tau^{\mathcal{H}}(H,\overline{W})+\Omega(\chi,\overline{W})H=2\tau^{\mathcal{H}}(H,\overline{W}).$$

□

Proposition 5.3: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Kähler Finsler metric on a complex manifold M with constant vanishing holomorphic curvature and satisfying* (2.4)*. Then:*

(i) $\tau \equiv 0$;
(ii) $\Omega(H, \overline{K})L = 0 = \Omega(H, \overline{W})L$ *for all* H, K, $L \in \mathcal{H}$ *and* $W \in \mathcal{V}$;
(iii) F *is strongly Kähler, i.e.,* $\theta(H, K) = 0$ *for all* H, $K \in \mathcal{H}$

Proof: (i) is the content of Lemma 5.2. It follows from [AP2, Corollary 2.2.1] that the frames $\{\delta_\alpha\}$ are holomorphic; in particular, then, the $\Gamma^\alpha_{;\mu}$ are holomorphic functions of $(z; v)$. Thus also the $\Gamma^\alpha_{\beta;\mu} = \dot\partial_\beta(\Gamma^\alpha_{;\mu})$ are holomorphic; being homogeneous of degree zero in v and using the Kähler condition, we obtain

$$\Gamma^\alpha_{\beta;\mu}(z; v) \equiv a^\alpha_{\beta;\mu}(z),$$

where the $a^\alpha_{\beta;\mu}$ are holomorphic functions of z satisfying $a^\alpha_{\beta;\mu} = a^\alpha_{\mu;\beta}$, and (iii) follows. Finally (ii) is a consequence of the holomorphicity of $\Gamma^\alpha_{\beta;\mu}$ and $\Gamma^\alpha_{;\mu}$. □

Of course, Minkowski metrics on $\mathbb{C}^n$ satisfy such requirements; it turns out that they are the only ones. In fact we have this precise restatement of Theorem 1.2:

Theorem 5.4: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex complete Kähler Finsler metric on a simply connected complex manifold M with constant vanishing holomorphic curvature and satisfying* (2.4)*. Fix $p \in M$, and denote by $\hat F$ the Minkowski metric induced by $F(p; \cdot)$ on $T^{1,0}_p M \cong \mathbb{C}^n$. Then $\exp_p\colon T^{1,0}_p M \to M$ is a biholomorphic isometry from $(T^{1,0}_p M, \hat F)$ to (M, F). In particular, (M, F) is biholomorphically isometric to a Minkowski space.*

Proof: We already know that $\exp_p$ is a biholomorphism (Theorem 5.1); it remains to prove that it is an isometry. The idea is to recast in our terms the similar step for the proof of Cartan-Ambrose-Hicks theorem as provided in Lemma 1.35 of [CE]. Unfortunately to this end it is necessary to formulate the appropriate theory of Jacobi fields which are essential for the proof. Here we shall just outline the basic ideas as it is a simple adaptation of the ideas developed in the real case in [AP2, Section 1.7.1]. As usual a Jacobi field is defined as a vector field J along a geodesic σ tangent to a geodesic variation. Under our hypotheses it is rather easy to recover the equation for Jacobi fields. If J^H is the horizontal lift of J, we have the following

Lemma 5.5: *Let $F\colon T^{1,0}M \to \mathbb{R}^+$ be a strongly pseudoconvex Kähler Finsler metric on a simply connected complex manifold M with constant vanishing holomorphic curvature and satisfying* (2.4)*. If J is a Jacobi field along a geodesic of M with unit tangent field T then*

$$\nabla_{T^H + \overline{T^H}} \nabla_{T^H + \overline{T^H}} (J^H + \overline{J^H}) \equiv 0.$$

Let us postpone the proof of the Lemma and, before proceeding with our argument, remark that as a consequence of the Lemma it follows that a Jacobi field is uniquely determined assigning initial conditions. Now let $v \in (T^{1,0}_p M) \setminus \{0\}$ and let $\sigma\colon [0, t^*] \to T^{1,0}_p M$ be the normalized geodesic from 0 to v in $T^{1,0}_p M$ relative to the metric $\hat F$ (which has as image the segment from 0 to v) and let $\bar\sigma = \exp_p \circ \sigma\colon [0, t^*] \to M$ be the corresponding geodesic on M from p to $\exp_p(v)$. If we denote by A_σ the parallel transport along a geodesic σ, then we set $I_\sigma = A_{\bar\sigma} \circ (d \exp_p)_0 \circ A_{-\sigma}$. For any $W \in T^{1,0}_v(T^{1,0}_p M)$ let J be the Jacobi field along σ such that $J(0) = 0$ and $J(v) = W$. If $\bar J$ is the vector field along $\bar\sigma$ defined by $I_\sigma \circ J$, then, as parallel transport and $d(\exp_p)_0$ are isometries, $\bar J$ is a Jacobi field and $\hat F(J(t)) = F(\bar J(t))$. On the other hand $\bar J = d(\exp_p)_{\sigma(t)}\big(J(t)\big)$ since Jacobi fields on normal geodesic starting from a point p and vanishing at p can be expressed as the push forward via the exponential map at p of Jacobi fields on $(T^{1,0}_p M, \hat F)$ exactly as it is done in the real case (cf. [AP2, Proposition 1.7.2]). But then we have our conclusion because

$$\hat F(W) = \hat F\big(J(t^*)\big) = F\big(\bar J(t^*)\big) = F\big(d(\exp_p)_{\sigma(t^*)}(J(t^*))\big) = F\big(d(\exp_p)_v(W)\big).$$

□

We end by giving the proof of the Lemma:

Proof of Lemma 5.5: Let $\Sigma\colon(-\varepsilon,\varepsilon)\times[0,a]\to M$ be a geodesic variation of a geodesic $\sigma_0\colon[0,a]\to M$, i.e., a regular variation of σ_0 such that $\sigma_s=\Sigma(s,\cdot)$ is a geodesic for any $s\in(-\varepsilon,\varepsilon)$. Let T be the vector field tangent to the geodesics and J be the transversal vector field of the variation Σ. Then we have $\nabla_{T^H+\overline{T^H}}T^H\equiv 0$ and $\nabla_{T^H+\overline{T^H}}\overline{T^H}\equiv 0$. Using the fact that under the assumptions the metric is strongly Kähler, $\tau\equiv 0$ and the horizontal curvature vanishes identically, we can compute:

$$
\begin{aligned}
0 &= \nabla_{J^H+\overline{J^H}}\nabla_{T^H+\overline{T^H}}(T^H+\overline{T^H})\\
&= \nabla_{J^H}\nabla_{T^H}T^H+\nabla_{J^H}\nabla_{\overline{T^H}}T^H+\nabla_{J^H}\nabla_{T^H}\overline{T^H}+\nabla_{J^H}\nabla_{\overline{T^H}}\overline{T^H}\\
&\qquad+\nabla_{\overline{J^H}}\nabla_{T^H}T^H+\nabla_{\overline{J^H}}\nabla_{\overline{T^H}}T^H+\nabla_{\overline{J^H}}\nabla_{T^H}\overline{T^H}+\nabla_{\overline{J^H}}\nabla_{\overline{T^H}}\overline{T^H}\\
&= \nabla_{T^H}\nabla_{J^H}T^H+\nabla_{[J^H,T^H]}T^H\\
&\qquad+\nabla_{\overline{T^H}}\nabla_{J^H}T^H+\nabla_{[J^H,\overline{T^H}]}T^H+\Omega(J^H,\overline{T^H})T^H\\
&\qquad+\nabla_{T^H}\nabla_{J^H}\overline{T^H}+\nabla_{[J^H,T^H]}\overline{T^H}\\
&\qquad+\nabla_{\overline{T^H}}\nabla_{J^H}\overline{T^H}+\nabla_{[J^H,\overline{T^H}]}\overline{T^H}+\Omega(J^H,\overline{T^H})T^H\\
&\qquad+\nabla_{T^H}\nabla_{\overline{J^H}}T^H-\nabla_{[T^H,\overline{J^H}]}T^H-\Omega(T^H,\overline{J^H})T^H\\
&\qquad+\nabla_{\overline{T^H}}\nabla_{\overline{J^H}}T^H+\nabla_{[\overline{T^H},\overline{J^H}]}T^H\\
&\qquad+\nabla_{T^H}\nabla_{\overline{J^H}}\overline{T^H}-\nabla_{[T^H,\overline{J^H}]}\overline{T^H}-\Omega(T^H,\overline{J^H})\overline{T^H}\\
&\qquad+\nabla_{\overline{T^H}}\nabla_{\overline{J^H}}\overline{T^H}+\nabla_{[\overline{T^H},\overline{J^H}]}\overline{T^H}\\
&= \nabla_{T^H+\overline{T^H}}(\nabla_{J^H}T^H+\nabla_{J^H}\overline{T^H}+\nabla_{\overline{J^H}}T^H+\nabla_{\overline{J^H}}\overline{T^H})\\
&\qquad+\nabla_{[J^H,T^H]+[J^H,\overline{T^H}]+[\overline{J^H},T^H]+[\overline{J^H},\overline{T^H}]}(T^H+\overline{T^H}).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\nabla_{J^H}T^H&+\nabla_{J^H}\overline{T^H}+\nabla_{\overline{J^H}}T^H+\nabla_{\overline{J^H}}\overline{T^H}\\
&= \nabla_{T^H}J^H+[J^H,T^H]+\theta(J^H,T^H)\\
&\qquad+\nabla_{\overline{T^H}}J^H+[J^H,\overline{T^H}]+\tau(J^H,\overline{T^H})+\overline{\tau}(J^H,\overline{T^H})\\
&\qquad+\nabla_{T^H}\overline{J^H}+[\overline{J^H},T^H]-\tau(T^H,\overline{J^H})-\overline{\tau}(T^H,\overline{J^H})\\
&\qquad+\nabla_{\overline{T^H}}\overline{J^H}+[\overline{J^H},\overline{T^H}]\\
&= \nabla_{T^H+\overline{T^H}}(J^H+\overline{J^H})\\
&\qquad+[J^H,T^H]+[J^H,\overline{T^H}]+[\overline{J^H},T^H]+[\overline{J^H},\overline{T^H}].
\end{aligned}
$$

Finally as in [AP2, proof of Theorem 2.4.4], we have

$$
\begin{aligned}
&[J^H,T^H]+[J^H,\overline{T^H}]+[\overline{J^H},T^H]+[\overline{J^H},\overline{T^H}]\\
&\qquad\qquad= \tau(T^H,\overline{J^H})-\tau(J^H,\overline{T^H})+\overline{\tau(T^H,\overline{J^H})}-\overline{\tau(J^H,\overline{T^H})}=0,
\end{aligned}
$$

and hence

$$
0=\nabla_{J^H+\overline{J^H}}\nabla_{T^H+\overline{T^H}}(T^H+\overline{T^H})=\nabla_{T^H+\overline{T^H}}\nabla_{T^H+\overline{T^H}}(J^H+\overline{J^H}).
$$

□

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