Convex-like properties of the Teichmüller metric

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November 1997

0. Introduction

Bers' realization of a finite dimensional Teichmüller space $T(\Gamma)$ is a bounded, topologically trivial pseudoconvex domain in $\mathbb{C}^N$. It was Royden who clearly realized that many properties of $T(\Gamma)$ may be interpreted and studied in the setting of the theory of functions of several complex variables. In particular the cornerstone of this point of view is Royden's discovery that the Kobayashi metric of $T(\Gamma)$ is the Teichmüller metric, which is a complex Finsler metric naturally defined on $T(\Gamma)$ representing a fundamental tool of investigation in Teichmüller theory.

Following Royden's inspiration we would like to raise the interest of complex analysts in this subject presenting some results about intrinsic metrics of Teichmüller spaces. Namely, while it is known that Bers' realization of a Teichmüller space is very far from being convex, especially in the finite dimensional case the Kobayashi-Teichmüller metric enjoys an amazing number of properties of the Kobayashi metric of (strictly) convex domains. In fact, although in the literature ([Kru1]) it is claimed that the Kobayashi-Teichmüller metric differs from the Carathéodory metric, many results seem to hint that they are very closely related and equal at least in many directions. We shall illustrate how much the Kobayashi-Teichmüller metric resembles the Kobayashi metric of a convex domain, and we shall also describe some applications and open problems.

1. Teichmüller spaces and their intrinsic metrics

We start by outlining the basic notions of differential geometry for the Teichmüller metric following closely [EE] and [G2]. For the necessary ideas of complex Finsler geometry we refer to [AP1].

If $\mathbb{H}^+$ is the upper half plane in $\mathbb{C}$ and M denotes the unit ball in $L^\infty(\mathbb{H}^+,\mathbb{C})$, the *Teichmüller metric* $\sigma\colon T^{1,0}M \cong M\times L^\infty(\mathbb{H}^+,\mathbb{C}) \to \mathbb{R}^+$ is the complex Finsler metric on M defined by

$$\sigma(\mu;\nu) = \left\| \frac{|\nu|}{1-|\mu|^2} \right\|_\infty , \tag{1.1}$$

where $\|\cdot\|_\infty$ denotes the L^∞ norm; note that $|\nu(z)|/(1-|\mu(z)|^2)$ is the Poincaré length of the tangent vector $\nu(z)$ at the point $\mu(z)\in\Delta$. The *Teichmüller distance* on M is just the

† Both authors have been partially supported by a grant of GNSAGA, Consiglio Nazionale delle Ricerche, Italy.

integrated distance d_σ of the Finsler metric σ; it is known that M with the Teichmüller distance is a complete metric space and that

$$d_\sigma(\mu_1,\mu_2) = \tanh^{-1}\left\|\frac{\mu_1-\mu_2}{1-\overline{\mu_1}\mu_2}\right\|_\infty . \tag{1.2}$$

The group $G = \mathrm{Aut}(\mathbb{H}^+)$ acts naturally on M as a group of linear isometries via the action

$$\forall (A,\mu)\in G\times M \qquad (A,\mu)\mapsto \mu^A = \frac{(\mu\circ A)\bar{A}'}{A'}. \tag{1.3}$$

If Γ is a Fuchsian group, i.e., a subgroup of the automorphism group of $\mathbb{H}^+$ acting properly discontinuously on $\mathbb{H}^+$, then a closed subspace of $L^\infty(\mathbb{H}^+,\mathbb{C})$ is given by

$$L^\infty(\Gamma) = \bigl\{\mu\in L^\infty(\mathbb{H}^+,\mathbb{C}) \bigm| \mu=\mu^A \quad \forall A\in\Gamma\bigr\} \tag{1.4}$$

The unit ball in the Banach space $L^\infty(\Gamma)$

$$M(\Gamma) = M\cap L^\infty(\Gamma) \tag{1.5}$$

is the space of *Beltrami differentials* relative to Γ. The Teichmüller metric and distance on $M(\Gamma)$, which we denote again by σ and d_σ respectively, are obtained by restriction, and $M(\Gamma)$ too is a complete Finsler manifold.

It turns out that the Teichmüller metric and distance on $M(\Gamma)$ agree with the Kobayashi and Carathéodory metric and distance:

Proposition 1.1: *Let Γ be a Fuchsian group. Then the Teichmüller, Carathéodory and Kobayashi metrics (respectively, distances) of $M(\Gamma)$ coincide.*

A simple proof of this fact, partly observed in [EKK, Proposition 1], is given in [AP3, Theorem 2.1] as direct consequence of results due to Harris [H] and Vesentini [V]. In fact it is also a simple corollary of a theorem of Dineen-Timoney-Vigué ([DTV]) which says that the Kobayashi metric (distance) agrees with the Carathéodory metric (distance) on any convex set in a complex Banach space.

Again following [EE] and [G2], let us turn now to Teichmüller spaces . Let $\mathbb{H}^-$ denote the lower half plane in $\mathbb{C}$ and consider the Banach space of holomorphic functions on $\mathbb{H}^-$ with norm

$$||\phi||_B = \sup\bigl\{|z-\bar{z}|^2|\phi(z)| \bigm| z\in\mathbb{H}^-\bigr\} < \infty. \tag{1.6}$$

Then $G = \mathrm{Aut}(\mathbb{H}^+) = \mathrm{Aut}(\mathbb{H}^-)$ acts on B as a group of linear isometries via the action $G\times B\to B$ defined by

$$(A,\phi)\mapsto \phi^A = (\phi\circ A)(A')^2. \tag{1.7}$$

If $\Gamma\subset G$ is a Fuchsian group,

$$B(\Gamma) = \{\phi\in B \mid \phi=\phi^A\}, \tag{1.8}$$

is the subspace of Γ-invariant functions of B. If $\mu \in M$ there exists a unique quasiconformal homeomorphism w_μ of $\mathbb{H}^+$ fixing the points 0, 1, ∞ and satisfying the Beltrami equation $w_{\bar z} = \mu w_z$. It is well known that for $\mu \in M$ there exists a unique homeomorphism w^μ of the Riemann sphere in itself which leaves 0, 1, ∞ fixed and such that w^μ is holomorphic on $\mathbb{H}^-$ while $w^\mu \circ (w_\mu)^{-1}$ is holomorphic on $\mathbb{H}^+$. A theorem of Nehari ensures that, if $[\,\cdot\,]$ denotes the Schwarzian derivative, then the map $\Phi\colon M \to B$ given by $\Phi(\mu) = [w^\mu]$ is well-defined. The image $T = \Phi(M)$ of Φ is called the *universal Teichmüller space*, and

$$T(\Gamma) = \Phi\big(M(\Gamma)\big) \subset B(\Gamma)$$

is called the *Teichmüller space* of the Fuchsian group Γ. This presentation of Teichmüller spaces is equivalent to the presentation as moduli spaces of Riemann surfaces and it is particularly suitable to study the Teichmüller metric. A fundamental result of Bers states that the map Φ is continuous and holomorphic and that the holomorphic and topological structures of $T(\Gamma)$ are just the quotient structure induced by $\Phi\colon M(\Gamma) \to T(\Gamma)$.

The *Teichmüller metric* $\tau_\Gamma\colon T(\Gamma) \times B(\Gamma) \to \mathbb{R}$ on $T(\Gamma)$ is defined using the quotient map Φ as follows:

$$\tau_\Gamma(t;\psi) = \inf\big\{\sigma(\mu;\nu) \bigm| \mu \in M(\Gamma),\, \nu \in L^\infty(\Gamma) \text{ with } t = \Phi(\mu), d\Phi_\mu(\nu) = \psi\big\}, \tag{1.9}$$

where σ is the Teichmüller metric on $M(\Gamma)$. Notice that (1.9) is well posed as σ is invariant under right translations (see [EE] for details). In an analogous way one defines the *Teichmüller distance* d_{τ_Γ}:

$$d_{\tau_\Gamma}(s,t) = \inf\big\{d_\sigma(\alpha,\beta) \bigm| \alpha,\beta \in M(\Gamma) \text{ with } s = \Phi(\alpha), t = \Phi(\beta)\big\} \tag{1.10}$$

which turns out to be always complete.

Royden ([R1]) has shown that the Teichmüller metric (and distance) is intimately linked with the complex structure of the Teichmüller space proving that it coincides with the Kobayashi metric (and distance) of $T(\Gamma)$. Royden's result, valid in the finite dimensional case, was later extended by Gardiner (see [G2]) to the infinite dimensional case by means of an approximation argument. As a consequence it follows that d_{τ_Γ} is exactly the integrated distance of τ_Γ (see also [O]).

Proposition 1.1 and the above construction suggest that it is natural to ask whether the Kobayashi-Teichmüller metric (and distance) agrees with the Carathéodory metric (and distance) on Teichmüller spaces. As a matter of fact they do on many direction through any point (see [Kra]) but in the literature it is claimed that they are in fact different ([Kru1]). As announced in the introduction we shall try to indicate how much they look alike by describing properties enjoyed by the Kobayashi-Teichmüller metric which also hold for the Kobayashi metric of strictly convex domains in $\mathbb{C}^n$, domains where an essential feature of the complex geometrical structure is that the Kobayashi and Carathéodory metrics agree.

We end these preliminaries recalling that in the case of finite dimensional Teichmüller spaces, which are just topologically trivial bounded pseudoconvex domains in $\mathbb{C}^N$ (an appealing kind of object for complex analysts), the Kobayashi-Teichmüller metric and distance have nice regularity properties and uniqueness of geodesics:

Proposition 1.2: *Let $T(\Gamma)$ be a finite dimensional Teichmüller space. Then:*

(i) *the Teichmüller distance $\delta_s(t) = d_{\tau_\Gamma}(s,t)$ from a point $s \in T(\Gamma)$ is of class C^1 on $T(\Gamma) \setminus \{s\}$, and the Kobayashi-Teichmüller metric τ_Γ is of class C^1 outside the zero section in $T(\Gamma) \times B(\Gamma)$;*

(ii) *for every $\psi \in B(\Gamma) \cong T^{1,0}_s(T(\Gamma))$ one has*

$$\tau_\Gamma(\psi) = \lim_{h\to 0} \frac{\delta_s(s+h\psi)}{|h|}; \tag{1.11}$$

(iii) *$T(\Gamma)$ with the Kobayashi-Teichmüller metric is straight in the sense of Busemann, i.e., any two point in $T(\Gamma)$ are joined by a unique geodesic of the Teichmüller metric.*

Proof: The regularity of the Kobayashi-Teichmüller metric is due to Royden ([R1]) and presented in details in [G2]. The derivatives of the Kobayashi-Teichmüller distance at a point are computed explicitly in [G1], and in [P] it is given the proof of (1.11) for any taut domain. Finally (iii) is classical, and the uniqueness follows from Teichmüller uniqueness theorem (see again [G2]). □

2. Holomorphic curvature of the Kobayashi-Teichmüller metric

It is known that Teichmüller spaces have not nonpositive real curvature and that they are not hyperbolic in any reasonable real sense (see [MW] for instance). Nevertheless finite dimensional Teichmüller spaces behave very much like hyperbolic manifolds, as it is illustrated for instance by Proposition 1.2.(iii). We shall try to justify this behavior by looking first of all at the holomorphic curvature of the Kobayashi-Teichmüller metric.

We start recalling the notion of holomorphic curvature for a complex Finsler metric F on a complex (Banach) manifold M (see [AP1, 2] for details). A *complex Finsler metric F* is an upper semicontinuous function $F\colon T^{1,0}M \to \mathbb{R}^+$ satisfying

(i) $F(p;v) > 0$ for all $p \in M$ and $v \in T^{1,0}_pM$ with $v \neq 0$;
(ii) $F(p;\lambda v) = |\lambda| F(p;v)$ for all $p \in M$, $v \in T^{1,0}_pM$ and $\lambda \in \mathbb{C}$.

Let us denote by $G\colon T^{1,0}(M) \to \mathbb{R}^+$ the function $G = F^2$. If $p \in M$ and $v \in T^{1,0}_pM$ is a non-zero tangent vector, the *holomorphic curvature* $K_F(p;v)$ of F at $(p;v)$ is given by

$$K_F(p;v) = \sup\{K(\varphi^*G)(0)\},$$

where the supremum is taken with respect to the family of all holomorphic maps $\varphi\colon \Delta \to M$ with $\varphi(0) = p$ and $\varphi'(0) = \lambda v$ for some $\lambda \in \mathbb{C}^*$, and $K(\varphi^*G)$ is the Gaussian curvature of the pseudohermitian metric φ^*G on the unit disk Δ (cf. [He]).

The holomorphic curvature clearly depends only on the complex line spanned by v in $T^{1,0}_pM$, and not on v itself. Furthermore, the holomorphic curvature defined in this way is invariant under holomorphic isometries, and when F is a honest smooth hermitian metric on M it coincides with the usual holomorphic sectional curvature of F at $(p;v)$ (see [Wu]). Finally we stress that such a notion of holomorphic curvature has a built in Ahlfors' Lemma, even in the infinite dimensional case.

Before moving on we point out the following useful application of the holomorphic curvature which is a simple consequence of Heins-Ahlfors' Lemma in the case of equality ([He]):

Proposition 2.1: *Let F be a complex Finsler metric on a manifold M with holomorphic curvature bounded above by -4. Let $\varphi\colon\Delta\to M$ be a holomorphic map. Then the following are equivalent:*

(i) φ *is* infinitesimally extremal *at one point $\zeta_0\in\Delta$, i.e., it is an isometry at ζ_0 between the Poincaré metric on Δ and F:*

$$F\bigl(\varphi(\zeta_0);\varphi'(\zeta_0)\bigr)=\varphi^*F(\zeta_0;1)=\frac{1}{1-|\zeta_0|^2};$$

(ii) φ *is an* infinitesimal complex geodesic, *i.e., it is infinitesimally extremal at every point.*

We recall that for convex domains in $\mathbb{C}^n$, as a consequence of the fact that it agrees with the Carathéodory metric, the Kobayashi metric has constant negative holomorphic curvature. The same happens, even in the infinite dimensional case, for the Kobayashi-Teichmüller metric:

Theorem 2.2: *Let Γ be any Fuchsian group. Then the holomorphic curvature of the Kobayashi-Teichmüller metric of $T(\Gamma)$ is identically equal to -4. As a consequence, $T(\Gamma)$ is Kobayashi complete hyperbolic.*

Proof: If $T(\Gamma)$ is finite dimensional, then Royden ([R1]; see also [G2, Lemma 7.8]) observed that the holomorphic curvature of the Kobayashi-Teichmüller metric is bounded above by -4. Hence, as a consequence of Ahlfors' Lemma, it follows that $T(\Gamma)$ is complete hyperbolic (see for instance [AP2]), so that the holomorphic curvature of the Kobayashi-Teichmüller metric must also be bounded below by -4 because of results of B. Wong and Suzuki ([W], [S]) and the claim follows. If $T(\Gamma)$ is not finite dimensional, then, as it often happens, it is possible to apply an approximation procedure due to Gardiner (see [G2]). There exists a sequence $\{T_j\}$ of finite dimensional Teichmüller spaces and a sequence of holomorphic maps $\pi_j\colon T(\Gamma)\to T_j$ such that the pull-back metrics $\pi_j^*\tau_{\Gamma_j}$ monotonically converge to τ_Γ. Since the holomorphic curvature of finite dimensional Teichmüller spaces is bounded above by -4, the monotone convergence theorem implies that the holomorphic curvature of τ_Γ is bounded above by -4. In particular, again as a consequence of Ahlfors' Lemma, $T(\Gamma)$ is complete hyperbolic.

To prove that the holomorphic curvature is bounded below by -4 we shall use Proposition 2.1. Let $[\mu]\in T(\Gamma)$ and $\psi\in B(\Gamma)\cong T^{1,0}_{[\mu]}T(\Gamma)$. By [EE, Theorem 3.(c)], up to replacing Γ by an isomorphic Fuchsian group we can assume that $[\mu]=\Phi(0)$. Choose $\nu\in L^\infty(\Gamma)$ such that $\psi=d\Phi_0(\nu)$ and $\tau_\Gamma([\mu];\psi)=\sigma(0;\nu)$ and set $\varphi(\zeta)=\Phi(\zeta\nu/\|\nu\|_\infty)$. Then $\varphi(0)=[\mu]$ and $\tau_\Gamma\bigl(\varphi(0);\varphi'(0)\bigr)=1$. Hence by Proposition 2.1 $\varphi^*\tau_\Gamma$ is the Poincaré metric of Δ, which has Gaussian curvature identically -4. It follows that the holomorphic curvature of τ_Γ at $([\mu];\psi)$ is at least -4. □

3. Complex geodesics for the Kobayashi-Teichmüller metric

We need two further definitions. For a complex Finsler metric F on a manifold M we shall say that a holomorphic map $\varphi\colon\Delta\to M$ is *extremal* at $\zeta_0\in\Delta$ if

$$\forall\zeta\in\Delta\qquad\qquad d_F\bigl(\varphi(\zeta_0),\varphi(\zeta)\bigr)=\omega(\zeta_0,\zeta),$$

where d_F is the distance induced by F and ω is the Poincaré distance. We shall say that φ is a *complex geodesic* if it is extremal at all points of Δ, that is if it is a global isometry between the Poincaré distance and d_F. Extremal maps, complex geodesics, infinitesimal extremal maps, infinitesimal complex geodesics are all of great usefulness in conjunction with invariant metrics. It is known that these notions agree for the Carathéodory metric ([V]). Thus the same property holds for the Kobayashi metric of convex domains. As a consequence of Theorem 1.1, the same happens for the Teichmüller metric of the space $M(\Gamma)$ of Beltrami differentials relative to a Fuchsian group Γ. Precisely:

Proposition 3.1: *The following statements are equivalent:*

(i) *φ is infinitesimally extremal at one point for the Teichmüller metric of $M(\Gamma)$;*
(ii) *φ is an infinitesimal complex geodesic for the Teichmüller metric of $M(\Gamma)$;*
(iii) *φ is extremal at one point for the Teichmüller metric of $M(\Gamma)$;*
(iv) *φ is a complex geodesic for the Teichmüller metric of $M(\Gamma)$.*

It was conjectured by Royden that the same happens for Teichmüller spaces, where it is known that extremal maps do exist and are given by the Teichmüller disks which we used above in the proof of Theorem 2.2. The fact, proved by Royden in special cases ([R2]), that infinitesimal extremal maps at one point are infinitesimal complex geodesics follows for any Teichmüller space from Theorem 2.2 (see [AP3]). The available differential geometric techniques do not seem strong enough to provide a proof that infinitesimal complex geodesics are complex geodesics. The full equivalence of the four notions has been proved by Earle, Kra and Krushkal ([EKK]) by means of a suitable lifting technique of holomorphic disks on Teichmüller spaces. The definitive results on the subject may be summarized as follows:

Theorem 3.2: ([EKK, AP3]) *Let Γ be a Fuchsian group. Then:*

(i) *for any point $[\mu] \in T(\Gamma)$ and tangent vector $\psi \in B(\Gamma)$ there exists a infinitesimal complex geodesic $\varphi\colon \Delta \to T(\Gamma)$ such that $\varphi(0) = [\mu]$ and $\varphi'(0)$ is a non-zero multiple of ψ. Furthermore, if $T(\Gamma)$ is finite dimensional then φ is uniquely determined.*
(ii) *for any couple of distinct points $[\mu_1]$, $[\mu_2] \in T(\Gamma)$ there exists a complex geodesic $\varphi\colon \Delta \to T(\Gamma)$ such that $\varphi(0) = [\mu_1]$ and $\varphi(r) = [\mu_2]$ for some $r > 0$. Furthermore, if $T(\Gamma)$ is finite dimensional then φ is uniquely determined.*
(iii) *the four statements in Proposition 3.1 remain equivalent replacing $M(\Gamma)$ by $T(\Gamma)$.*

It should be observed that if $T(\Gamma)$ is finite dimensional the Kobayashi-Teichmüller metric has exactly the same properties concerning existence and uniqueness of (infinitesimal) complex geodesics as the Kobayashi (Carathéodory) metric of smooth strictly convex domains in $\mathbb{C}^n$ ([L]). As an applications, following an idea of Graham ([Gr], see also [Pa]) in [AP3] it was given the following characterization of finite dimensional Teichmüller spaces:

Theorem 3.3: *Let Γ be a Fuchsian group so that $T(\Gamma)$ is finite dimensional. A taut connected complex manifold N is biholomorphic to $T(\Gamma)$ if and only if there exists a holomorphic map $F\colon N \to T(\Gamma)$ which is an isometry for the Kobayashi metric at one point.*

In [AP3] it is given a result in the same spirit for the infinite dimensional case but, because of the lack of uniqueness of complex geodesics, we got a much weaker statement.

From Theorem 3.2 it follows that the Kobayashi-Teichmüller metric shares another interesting property with the Kobayashi (Carathéodory) metric of convex domains in $\mathbb{C}^n$. Namely, it is known that the indicatrices for the invariant metrics on convex domains are always convex. The same is true for the Kobayashi-Teichmüller metric even in the infinite dimensional case:

Theorem 3.4: *Let* Γ *be a Fuchsian group. The indicatrices*

$$I_{[\mu]} = \{\psi \in B(\Gamma) \cong T^{1,0}_{[\mu]} T(\Gamma) \mid \tau_\Gamma(\psi) \leq 1\}$$

of the Kobayashi-Teichmüller metric τ_Γ *are convex for every* $[\mu] \in T(\Gamma)$.

Proof: Theorem 3.2 implies that the Kobayashi distance between two points of $T(\Gamma)$ may be achieved using only one holomorphic disk rather than a chain of holomorphic disks. In the terminology of Pang ([P]), $T(\Gamma)$ is Kobayashi simple and hence if $T(\Gamma)$ is finite dimensional, Theorem (4.1) of [P] yields the convexity of $I_{[\mu]}$ for every $[\mu] \in T(\Gamma)$. As Pang uses compactness arguments which do not carry over in the infinite dimensional case, we cannot invoke directly his result if $\dim T(\Gamma) = \infty$. Again we may use instead Gardiner's approximation technique ([G2]). There exists a monotonically increasing sequence τ_n of Finsler metrics converging to τ_Γ such that τ_n is the Kobayashi-Teichmüller metric of a finite dimensional Teichmüller space and hence it has convex indicatrix at every point or, in other words, it has convex restriction on the holomorphic tangent space at each point. Since the sequence τ_n is monotonically increasing, it follows that τ_Γ has convex restriction on the holomorphic tangent space at each point, which is equivalent to our claim. $\square$

4. Pluricomplex Green function for Teichmüller spaces and intrinsic metrics

Let $\Omega \subset\subset \mathbb{C}^n$ be an open connected subset and $z \in \Omega$. The *pluricomplex Green function* at z is a function $u_z: \overline{\Omega} \to [-\infty, 0]$ with $u_z \in C^0(\overline{\Omega} \setminus \{z\}) \cap PSH(\Omega)$ such that

$$(dd^c u_z)^n = 0 \qquad \text{on} \quad \Omega \setminus \{z\}, \tag{4.1}$$

$$u_z|_{\partial\Omega} \equiv 0, \tag{4.2}$$

$$u_z(w) = \log ||z - w|| + O(1) \qquad \text{near } z. \tag{4.3}$$

It is known ([De]) that if Ω is hyperconvex (this is the case for instance if Ω is pseudoconvex and has Lipschitz boundary) then at every point of Ω the pluricomplex Green function exists and it is unique. The pluricomplex Green function at z is obtained by means of a Perron type of argument as the supremum of all plurisubharmonic functions $v: \overline{\Omega} \to [-\infty, 0]$ such that $v(w) = \log ||z - w|| + O(1)$ near z. On the other hand, Poletsky ([Po]) has shown that if, for $\varphi \in \mathrm{Hol}(\Delta, \Omega)$ with $\varphi(0) = w$, we denote

$$v_\varphi(w, z) = \sum_{\zeta_j \in \varphi^{-1}(z)} m_j \log |\zeta_j|$$

where m_j is the multiplicity of φ at ζ_j, then

$$u_z(w) = \inf\{v_\varphi(w, z) \mid \varphi \in \mathrm{Hol}(\Delta, \Omega) \text{ with } \varphi(0) = w\}. \tag{4.4}$$

Using Poletsky ([Po]), Krushkal ([Kru2]) has outlined (in the infinite dimensional case too) the proof of the following important result :

Theorem 4.1: *Let $T(\Gamma)$ be a finite dimensional Teichmüller space. Then for every $[\mu] \in T(\Gamma)$ the pluricomplex Green function at $[\mu]$ is of class C^1 on $T(\Gamma)$ and it is given for any $[\nu] \in T(\Gamma)$ by*

$$u_{[\mu]}([\nu]) = \log\bigl(\tanh\bigl(d_{\tau_\Gamma}([\mu],[\nu])\bigr)\bigr), \tag{4.5}$$

where d_{τ_Γ} is the Kobayashi-Teichmüller distance of $T(\Gamma)$.

Theorem 4.1 outlines a further important similarity between the complex geometries of strictly convex domains and of Teichmüller spaces: the relationship between the pluricomplex Green function and the Kobayashi distance. This relation may be exploited to describe yet another property of the Kobayashi-Teichmüller metric. We start by recalling a definition. Let $\Omega = \{z \in \mathbb{C}^n \mid \rho < 0\}$ be a bounded, C^1, pseudoconvex domain. A *stationary disk* for Ω is a proper holomorphic holomorphic map $\varphi \in \mathrm{Hol}(\Delta, \Omega) \cap C^{1/2}(\overline{\Delta})$, such that for some function $p\colon \partial\Delta \to \mathbb{R}_+$ of class $C^{1/2}$ the map

$$\zeta \in \partial\Delta \mapsto \zeta\, p(\zeta) \left(\frac{\partial \rho}{\partial z_1}\bigl(\varphi(\zeta)\bigr), \dots, \frac{\partial \rho}{\partial z_n}\bigl(\varphi(\zeta)\bigr) \right)$$

extends to a map $\tilde{\varphi}\colon \overline{\Delta} \to \mathbb{C}^n$ holomorphic on Δ and of class $C^{1/2}$ on $\overline{\Delta}$. Geometrically this means that the restriction to $\varphi(\partial\Delta)$ of the holomorphic tangent bundle of $\partial\Omega$ extends to a holomorphic tangent bundle of rank $n-1$ along the image of φ. In [L] it is shown that for any smooth strictly convex domain D in $\mathbb{C}^n$ and for any point $z \in D$ the leaves of the Monge-Ampère foliation associated to the Green pluripotential u_z with pole at z (which are the complex geodesics for the Kobayashi-Carathéodory metric of D) are stationary maps. In particular it is easy to see that if φ is a complex geodesic with $\varphi(0) = z$ then

$$\tilde{\varphi}(\zeta) = \zeta \left(\frac{\partial u_z}{\partial z_1}\bigl(\varphi(\zeta)\bigr), \dots, \frac{\partial u_z}{\partial z_n}\bigl(\varphi(\zeta)\bigr) \right).$$

Thus the form ∂u_z has holomorphic restriction along the images of the complex geodesics outside the pole z. While it is hard to give an appropriate notion of stationary disk on Teichmüller spaces because of the complicated structure of the boundary, it is reasonable to ask for the existence of such a form, and in fact we have the following

Proposition 4.2: *Let $T(\Gamma)$ be a finite dimensional Teichmüller space and $[\mu] \in T(\Gamma)$. Then, if $u_{[\mu]}$ is the Green pluripotential of $T(\Gamma)$ with pole at $[\mu]$, the form $\omega_{[\mu]} = \partial u_{[\mu]}$ is continuous on $T(\Gamma) \setminus \{[\mu]\}$ and for every complex geodesic $\varphi\colon \Delta \to T(\Gamma)$ with $\varphi(0) = [\mu]$, the form*

$$\zeta \mapsto \zeta \omega_{[\mu]}(\varphi(\zeta))$$

is holomorphic on Δ.

Proof: Because of Theorem 4.1 the restriction of $u_{[\mu]}$ to the image of a complex geodesic is harmonic. Precisely, for a complex geodesic $\varphi\colon \Delta \to T(\Gamma)$ with $\varphi(0) = [\mu]$ we have

$$u_{[\mu]}(\varphi(\zeta)) = \log|\zeta|. \tag{4.6}$$

Now the (weakly defined) form $dd^c u_{[\mu]}$ is semipositive definite as $u_{[\mu]}$ is plurisubharmonic, and using (4.6) we have for $\zeta \neq 0$ that $dd^c u_{[\mu]}(\varphi'(\zeta), \overline{\varphi'(\zeta)}) = 0$, so that it follows that for every $X \in \mathbb{C}^n$

$$dd^c u_{[\mu]}\big(X, \overline{\varphi'(\zeta)}\big) = 0.$$

Thus, as $dd^c u_{[\mu]} = \frac{1}{2\pi i}\partial\overline{\partial} u_{[\mu]}$, we may conclude that if Z is a holomorphic tangent vector field along $\varphi(\Delta \setminus \{0\})$ then $\overline{Z}(\partial u_{[\mu]}) = 0$, and thus the form $\omega_{[\mu]}$ is holomorphic along $\varphi(\Delta \setminus \{0\})$. On the other hand, from (1.11) it follows that the components of $\zeta\,\omega_{[\mu]}(\zeta)$ are bounded near $\zeta = 0$, and the claim follows. □

The importance of Proposition 4.2 resides on the fact that, as in the case of convex domains, the form $\omega_{[\mu]}$ is a tool to investigate when the Kobayashi and Carathéodory metrics agree. In fact the key to prove equality between the two metrics is to show that given a complex geodesic for the Kobayashi metric it is always possible to construct a holomorphic retraction of the domain into the image of the complex geodesic. In this way it is defined a left inverse for the complex geodesic, whose existence implies the equality of Carathéodory and Kobayashi metrics along the chosen complex geodesic. A verbatim repetition of the argument for convex domains as presented for example in Proposition 2.6.22 of [A] gives the following:

Theorem 4.3: *Let $T(\Gamma)$ be a finite dimensional Teichmüller space, $[\mu] \in T(\Gamma)$ and let $\omega = \omega_{[\mu]}$. Let $\varphi\colon \Delta \to T(\Gamma)$ be a complex geodesic for the Kobayashi-Teichmüller metric with $\varphi(0) = [\mu]$. Then φ is a complex geodesic for the Carathéodory metric if for every $[\nu] \in T(\Gamma)$ the winding number of the function*

$$\zeta \mapsto \zeta\omega_{\varphi(\zeta)}([\nu] - \varphi(\zeta))$$

is 1. In this case along the image of φ the Kobayashi-Teichmüller metric and distance agree with the Carathéodory metric and distance.

We end by proposing some questions which we feel deserve to be addressed. Let us start with the following remark:

Proposition 4.4: *Let $T(\Gamma)$ be a finite dimensional Teichmüller space and $[\mu] \in T(\Gamma)$. For $r > 0$, if $\mathbb{B}(r) = \{[\nu] \in T(\Gamma) \mid d_{\mathcal{T}_\Gamma}([\mu],[\nu]) > r\}$ is the Kobayashi-Teichmüller ball of radius r centered at $[\mu]$, then the pluricomplex Green function of $\mathbb{B}(r)$ at $[\mu]$ is of class C^1 on $\overline{\mathbb{B}(r)}$ and is given by*

$$u_{[\mu]}([\nu]) = \log\left(\frac{\tanh\big(d_{\mathcal{T}_\Gamma}([\mu],[\nu])\big)}{\tanh r}\right). \tag{4.7}$$

The proof is a simple application of the uniqueness for the pluricomplex Green function with given pole. As balls for the Kobayashi-Teichmüller distance have C^1 smooth boundary, using similar arguments as above, it follows easily that the leaves of the Monge-Ampère foliation associated to the Green function, which are exactly Teichmüller disks of $T(\Gamma)$ intersected with the ball, are stationary disks. In general, while extremal disks for the Kobayashi distance are necessarily stationary, the converse is not known. In analogy with what happens for convex domains we propose the following

Question 1. If $\varphi\colon \Delta \to T(\Gamma)$ is a complex geodesic of $T(\Gamma)$ such that $\varphi(0) = [\mu]$ then is $\varphi_r\colon \Delta \to \mathbb{B}(r)$ defined by $\varphi_r(\zeta) = \varphi\big((\tanh r)\zeta\big)$ a complex geodesic for $\mathbb{B}(r)$?

Question 2. If $d_{\mathbb{B}(r)}$ is the Kobayashi distance of $\mathbb{B}(r)$, is it true that for all $[\nu] \in \mathbb{B}(r)$

$$d_{\mathbb{B}(r)}([\mu],[\nu]) = \tanh^{-1}\left(\frac{\tanh\big(d_{\tau_\Gamma}([\mu],[\nu])\big)}{\tanh r}\right)?$$

Kra [Kra] has shown that along special complex geodesics (the so-called abelian Teichmüller disks) the Kobayashi and Carathéodory metrics agree. We ask

Question 3. Is there a geometric characterization of the Teichmüller disks which satisfy the assumptions of Theorem 4.3?

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