Index theorems for meromorphic self-maps of the projective space

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ABSTRACT. In this short note we would like to show how it is possible to use techniques introduced in the theory of local dynamics of holomorphic germs tangent to the identity to study global meromorphic self-maps of the complex projective space. More precisely, we shall prove three index theorems, relating suitably defined local residues at the fixed and indeterminacy points of a meromorphic map $f: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ with Chern classes of $\mathbb{P}^n$.

In this short note we would like to show how the techniques introduced in [ABT1] (see also [ABT2, 3], [Br], [BT] and [AT2]) for studying the local dynamics of holomorphic germs tangent to the identity can be used to study global meromorphic self-maps of the complex projective space $\mathbb{P}^n$. More precisely, we shall prove the following index theorem:

Theorem 1: *Let $f: \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a meromorphic self-map of degree $\nu + 1 \geq 2$ of the complex n-dimensional projective space. Let $\Sigma(f) = \mathrm{Fix}(f) \cup I(f)$ be the union of the indeterminacy set $I(f)$ of f and the fixed points set $\mathrm{Fix}(f)$ of f. Let $\Sigma(f) = \sqcup_\alpha \Sigma_\alpha$ be the decompositon of Σ in connected components, and denote by N the tautological line bundle of $\mathbb{P}^n$. Then:*

(i) *we can associate to each Σ_α a complex number $\mathrm{Res}^1(f, \Sigma_\alpha) \in \mathbb{C}$, depending only on the local behavior of f nearby Σ_α, so that*

$$\sum_\alpha \mathrm{Res}^1(f, \Sigma_\alpha) = \int_{\mathbb{P}^n} c_1(N)^n = (-1)^n \; ;$$

(ii) *given a homogeneous symmetric polynomial $\varphi \in \mathbb{C}[z_1, \ldots, z_n]$ of degree n we can associate to each Σ_α a complex number $\mathrm{Res}^2_\varphi(f, \Sigma_\alpha) \in \mathbb{C}$, depending only on the local behavior of f nearby Σ_α, such that*

$$\sum_\alpha \mathrm{Res}^2_\varphi(f, \Sigma_\alpha) = \int_{\mathbb{P}^n} \varphi(T\mathbb{P}^n - N^{\otimes \nu}) \; ;$$

(iii) *if $\nu > 1$, given a homogeneous symmetric polynomial $\psi \in \mathbb{C}[z_0, \ldots, z_n]$ of degree n we can associate to each Σ_α a complex number $\mathrm{Res}^3_\psi(f, \Sigma_\alpha) \in \mathbb{C}$, depending only on the local behavior of f nearby Σ_α, such that*

$$\sum_\alpha \mathrm{Res}^3_\psi(f, \Sigma_\alpha) = \int_{P^n} \psi\big((T\mathbb{P}^n \oplus N) - N^{\otimes \nu}\big) \; .$$

In this statement, if $\varphi \in \mathbb{C}[z_1, \ldots, z_n]$ is a homogeneous symmetric polynomial and E and F are vector bundles over $\mathbb{P}^n$, we put

$$\varphi(E - F) = \tilde{\varphi}\big(c_1(E - F), \ldots, c_r(E - F)\big) \; ,$$

where the $c_j(E - F)$ are the Chern classes of the virtual bundle $E - F$, and $\tilde{\varphi} \in \mathbb{C}[z_1, \ldots, z_r]$ is the unique polynomial such that

$$\varphi = \tilde{\varphi}(\sigma_1, \ldots, \sigma_r) \; ,$$

where $\sigma_1, \ldots, \sigma_r \in \mathbb{C}[z_1, \ldots, z_n]$ are the elementary symmetric functions on n variables.

Remark 1: If $\Sigma(f)$ is finite, then the number of points in $\Sigma(f)$, counted with respect to a suitable multiplicity, is

$$\frac{1}{\nu}[(\nu+1)^{n+1} - 1] = \sum_{k=0}^{n} \binom{n+1}{k+1} \nu^k \ ;$$

see, e.g., [AT1].

Remark 2: Since the total Chern class of $T\mathbb{P}^n$ in terms of the first Chern class of N is given by $c(T\mathbb{P}^n) = \left(1 - c_1(N)\right)^{n+1}$, the total Chern class of $T\mathbb{P}^n - N^{\otimes\nu}$ is given by

$$c(T\mathbb{P}^n - N^{\otimes\nu}) = \frac{\left(1 - c_1(N)\right)^{n+1}}{1 + \nu c_1(N)} = \left(1 - c_1(N)\right)^{n+1} \sum_{j=0}^{n} (-1)^j \nu^j c_1(N)^j$$

and thus the Chern classes of $T\mathbb{P}^n - N^{\otimes\nu}$ are given by

$$c_j(T\mathbb{P}^n - N^{\otimes\nu}) = (-1)^j \sum_{k=0}^{j} \binom{n+1}{k} \nu^{j-k} c_1^j(N) \ .$$

In particular,

$$\int_{\mathbb{P}^n} c_n(T\mathbb{P}^n - N^{\otimes\nu}) = \sum_{k=0}^{n} \binom{n+1}{k} \nu^{n-k} \quad \text{and} \quad \int_{\mathbb{P}^n} c_1^n(T\mathbb{P}^n - N^{\otimes\nu}) = (\nu + 1 + n)^n \ . \tag{1}$$

Arguing in a similar way we get that the Chern classes of $(T\mathbb{P}^n \oplus N) - N^{\otimes\nu}$ are given by

$$c_j\left((T\mathbb{P}^n \oplus N) - N^{\otimes\nu}\right) = (-1)^j \sum_{k=0}^{j} \left[\binom{n+1}{k} - \binom{n+1}{k-1} \right] \nu^{j-k} c_1^j(N) \ ,$$

with the convention $\binom{n+1}{-1} = 0$.

Remark 3: As the proof of Theorem 1 presented at the end of this note will make clear, this index theorem is a direct consequence of the index theorems proved in [ABT1] on holomorphic self-maps pointwise fixing an hypersurface S of a complex manifold M. In turn, as shown in a very general setting in [ABT2], these index theorems follow from the existence of a holomorphic action (or partial connection) of a suitable tensor power $N_S^{\otimes\nu}$ of the normal bundle N_S on a (possibly virtual) vector bundle E on S. The embedding of S into M yields three natural vector bundles to consider: N_S, TS and $TM|_S$; accordingly, we get three index theorems. Indeed, case (i) of Theorem 1 corresponds to the existence of a holomorphic action on N_S (a *Camacho-Sad action*); case (ii) corresponds to the existence of a holomorphic action on $TS - N_S^{\otimes\nu}$ (a *Baum-Bott action*); and case (iii) corresponds to the existence of a holomorphic action on $TM|_S - N_S^{\otimes\nu}$ (a *Lehmann-Suwa action*).

Remark 4: As a personal aside, in my opinion index theorems, giving quantitative and explicit links between local and global objects, rank among the most beautiful results in mathematics, and as such are an end in themselves. For this reason since I was a budding young mathematician I hoped to be able to discover at least one new index theorem during my career. So, as you can imagine, I was pretty excited when in [A2] I realized that I was proving such a theorem; and now that thanks to the work done in [ABT1] and [ABT2] (and in this note) I can say that I actually proved not one but several new index theorems I consider myself (mathematically) realized and content (and thus ready to explore new avenues of research in the future, of course). But index theorems, besides being beautiful, are useful too; for instance they can be of help in classifying objects or behaviors, imposing constraints that should be satisfied. We shall not discuss applications of Theorem 1 in this note; however similar (though less general) theorems already present in the literature have been for instance applied to show that generic holomorphic self-maps of $\mathbb{P}^n(\mathbb{C})$ have infinitely many non-attracting periodic points (see [U]); to classify complex homogenous vector fields

(see [G]); to classify particular classes of Liouville integrable complex Hamiltonian systems (see [P1, 2]); and to study the (non-)existence of compact positive dimensional fixed points set of holomorphic self-maps of complex surfaces (see [ABT1]).

The usefulness of a theorem like Theorem 1 is of course related to the possibility of computing the residues involved, and indeed we have explicit formulas for the (generic) case of point components, expressed in terms of Grothendieck residues (the residues at components of positive dimension can be expressed by using Lehmann-Suwa theory; see [LS], [S, Chapter IV], [ABT1, (6.1)] and [BS]; see also [B]). To state the formulas we recall three definitions.

Definition 1: Let $\mathcal{O}_{w^o}^n$ be the ring of germs of holomorphic functions at $w^o \in \mathbb{C}^n$. Take $g_1, \ldots, g_n \in \mathcal{O}_{w^o}^n$ such that w^o is an isolated zero of $g = (g_1, \ldots, g_n)$. Then the *Grothendieck residue* of $h \in \mathcal{O}_{w^o}^n$ along $g_1, \ldots, g_n$ is defined (see [H], [L, Section 5], [LS, Section 4], [S, pp.105–107]) by the formula

$$\operatorname{Res}_{w^o} \begin{bmatrix} h(w)\, dw_1 \wedge \cdots \wedge dw_n \\ g_1, \ldots, g_n \end{bmatrix} = \left(\frac{1}{2\pi i} \right)^n \int_\Gamma \frac{h}{g_1 \cdots g_n}\, dw_1 \wedge \cdots \wedge dw_n \ ,$$

where $\Gamma = \{ |g_j(w)| = \varepsilon \mid j = 1, \ldots, n \}$ for $\varepsilon > 0$ small enough, oriented so that $d\arg(g_1) \wedge \cdots \wedge d\arg(g_n) > 0$.

Definition 2: Let $\varphi \in \mathbb{C}[z_1, \ldots, z_n]$ be a homogeneous symmetric polynomial in n variables, and $L\colon V \to V$ an endomorphism of an n-dimensional complex vector space V. Then we set

$$\varphi(L) = \varphi(\lambda_1, \ldots, \lambda_n) \ ,$$

where $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are the eigenvalues of L.

Definition 3: We shall say that a homogeneous polynomial self-map $F = (F_0, \ldots, F_n)\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ of degree $\nu + 1$ *induces* the meromorphic self-map $f\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ if

$$f([z_0 : \cdots : z_n]) = [F_0(z_0, \ldots, z_n) : \cdots : F_n(z_0, \ldots, z_n)]$$

for all $[z_0 : \cdots : z_n] \in \mathbb{P}^n$. It is well known (see, e.g., [FS1]) that every meromorphic self-map of $\mathbb{P}^n$ of degree $\nu + 1$ is induced by a unique homogeneous polynomial self-map of $\mathbb{C}^{n+1}$ of degree $\nu + 1$.

We can now state our next theorem:

Theorem 2: *Let $f\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a meromorphic self-map of degree $\nu + 1 \geq 2$ of the complex n-dimensional projective space. Let $\Sigma(f) = \operatorname{Fix}(f) \cup I(f)$ be the union of the indeterminacy set $I(f)$ of f and the fixed points set $\operatorname{Fix}(f)$ of f, and assume that $p = [1 : w_1^o : \cdots : w_n^o]$ is an isolated point in $\Sigma(f)$. Set $w^o = (w_1^o, \ldots, w_n^o)$, let $F = (F_0, \ldots, F_n)\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be the homogeneous polynomial self-map of degree $\nu + 1$ inducing f, and define $g\colon \mathbb{C}^n \to \mathbb{C}^n$ by setting*

$$g_j(w) = F_j(1, w) - w_j F_0(1, w) \tag{2}$$

for all $w \in \mathbb{C}^n$ and $j = 1, \ldots, n$. Then:

(i) *we have*

$$\operatorname{Res}^1(f, p) = \operatorname{Res}_{w^o} \begin{bmatrix} F_0(1, w)^n\, dw_1 \wedge \cdots \wedge dw_n \\ g_1, \ldots, g_n \end{bmatrix} \ ;$$

(ii) *if $\varphi \in \mathbb{C}[z_1, \ldots, z_n]$ is a homogeneous symmetric polynomial of degree n then*

$$\operatorname{Res}^2_\varphi(f, p) = \operatorname{Res}_{w^o} \begin{bmatrix} \varphi(dg_w)\, dw_1 \wedge \cdots \wedge dw_n \\ g_1, \ldots, g_n \end{bmatrix} \ ;$$

(iii) *if $\psi \in \mathbb{C}[z_0, \ldots, z_n]$ is a homogeneous symmetric polynomial of degree n then*

$$\operatorname{Res}^3_\psi(f, p) = \operatorname{Res}_{w^o} \begin{bmatrix} \psi\big(F_0(1, w), \mu_1(w), \ldots, \mu_n(w)\big)\, dw_1 \wedge \cdots \wedge dw_n \\ g_1, \ldots, g_n \end{bmatrix} \ ,$$

where $\mu_1(w), \ldots, \mu_n(w)$ are the eigenvalues of dg_w.

This statement is effective because we can explicitly compute a Grothendieck residue using an algorithm suggested by Hartshorne (see [BB] and [H]). Without loss of generality, we can assume $w^o = O$. Since the origin is an isolated zero of g, there exist minimal positive integers $\alpha_1, \ldots, \alpha_n$ such that $w_1^{\alpha_1}, \ldots, w_n^{\alpha_n}$ belong to the ideal generated by $g_1, \ldots, g_n$ in $\mathcal{O}^n = \mathcal{O}_O^n$. Hence there exist holomorphic functions $b_{ij} \in \mathcal{O}^n$ such that

$$w_i^{\alpha_i} = \sum_{j=1}^n b_{ij} g_j \ .$$

The properties of the Grothendieck residue (see [H]) then imply

$$\mathrm{Res}_O \begin{bmatrix} h\, dw_1 \wedge \cdots \wedge dw_n \\ g_1, \cdots, g_n \end{bmatrix} = \mathrm{Res}_O \begin{bmatrix} h \det(b_{ij})\, dw_1 \wedge \cdots \wedge dw_n \\ w_1^{\alpha_1}, \cdots, w_n^{\alpha_n} \end{bmatrix} \ .$$

The right-hand side is now evaluated by expanding $h \det(b_{ij})$ in a power series in the w_i; the residue is given by the coefficient of $w_1^{\alpha_1 - 1} \cdots w_n^{\alpha_n - 1}$.

The easiest case is when $\det(dg_O) \neq 0$. Indeed, since O is an isolated zero of g we can write

$$g_j = \sum_{i=1}^k c_{ji} w_i \ .$$

Differentiating this and evaluating in O we get

$$\frac{\partial g_j}{\partial w_k}(O) = c_{jk}(O) \ ;$$

therefore if $\det(dg_O) \neq 0$ we can invert the matrix (c_{ji}) in a neighborhood of O and write

$$w_i = \sum_{j=1}^n b_{ij} g_j \ ,$$

where (b_{ij}) is the inverse matrix of (c_{ji}). So we have $\alpha_1 = \cdots = \alpha_n = 1$ and $\det(b_{ij})(O) = 1/\det(dg_O)$.

It thus follows that

$$\det(dg_{w^o}) \neq 0 \quad \Longrightarrow \quad \mathrm{Res}_{w^o} \begin{bmatrix} h(w)\, dw_1 \wedge \cdots \wedge dw_n \\ g_1, \ldots, g_n \end{bmatrix} = \frac{h(w^o)}{\mu_1 \cdots \mu_n} \ , \tag{3}$$

where $\mu_1, \ldots, \mu_n \in \mathbb{C}$ are the eigenvalues of dg_{w^o}.

In our situation, $g_j(w) = F_j(1, w) - w_j F_0(1, w)$; therefore we have

$$\frac{\partial g_j}{\partial w_k}(w) = \frac{\partial F_j}{\partial w_k}(1, w) - w_j \frac{\partial F_0}{\partial w_k}(1, w) - \delta_k^j F_0(1, w) \ ,$$

where δ_k^j is Kronecker's delta.

So if $p = [1 : w_1^o : \cdots : w_n^o]$ is an indeterminacy point we have $F_0(1, w^o) = 0$ and

$$\mathrm{Jac}(g)(w^o) = \left(\frac{\partial F_j}{\partial w_k}(1, w^o) - w_j^o \frac{\partial F_0}{\partial w_k}(1, w^o) \right)_{j,k=1,\ldots,n} \ .$$

If instead p is a fixed point, we can consider the differential of f at p. Let χ be the usual chart of $\mathbb{P}^n$ centered at $[1 : 0 : \cdots : 0]$. Then

$$\tilde{f}(w) = \chi \circ f \circ \chi^{-1}(w) = \left(\frac{F_1(1, w)}{F_0(1, w)}, \ldots, \frac{F_n(1, w)}{F_0(1, w)} \right) \ ,$$

and so

$$\frac{\partial \tilde{f}_j}{\partial w_k}(w) = \frac{1}{F_0(1,w)}\left[\frac{\partial F_j}{\partial w_k}(1,w) - \frac{F_j(1,w)}{F_0(1,w)}\frac{\partial F_0}{\partial w_k}(1,w)\right] .$$

In particular,

$$\frac{\partial \tilde{f}_h}{\partial w_k}(w^o) = \frac{1}{F_0(1,w^o)}\left[\frac{\partial F_j}{\partial w_k}(1,w^o) - w_j^o\frac{\partial F_0}{\partial w_k}(1,w^o)\right] ; \tag{4}$$

therefore

$$\mathrm{Jac}(g)(w^o) = F_0(1,w^o)\big[\mathrm{Jac}(\tilde{f})(w^o) - I\big] .$$

It follows that the eigenvalues $\mu_1,\ldots,\mu_n$ of dg_{w^o} are related to the eigenvalues $\lambda_1,\ldots,\lambda_n$ of df_p by the formula

$$\forall j = 1,\ldots,n \qquad\qquad \mu_j = F_0(1,w^o)(\lambda_j - 1) .$$

In particular, $\det(dg_{w^o}) \neq 0$ if and only if 1 is not an eigenvalue of df_p, that is if and only if p is a simple fixed point of f.

Summing up, these computations give the following particular case of Theorem 2:

Corollary 3: *Let* $f\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *be a meromorphic self-map of degree* $\nu + 1 \geq 2$ *of the complex n-dimensional projective space.*

(a) *Assume that* $p \in \mathbb{P}^n$ *is a simple (necessarily isolated) fixed point of* f, *and let* $\lambda_1,\ldots,\lambda_n \neq 1$ *be the eigenvalues of* df_p. *Then:*

    (i) *we have*

$$\mathrm{Res}^1(f,p) = \frac{(-1)^n}{(1-\lambda_1)\cdots(1-\lambda_n)} ;$$

    (ii) *if* $\varphi \in \mathbb{C}[z_1,\ldots,z_n]$ *is a homogeneous symmetric polynomial of degree* n *then*

$$\mathrm{Res}^2_\varphi(f,p) = \frac{\varphi(1-\lambda_1,\ldots,1-\lambda_n)}{(1-\lambda_1)\cdots(1-\lambda_n)} ;$$

    (iii) *if* $\psi \in \mathbb{C}[z_0,\ldots,z_n]$ *is a homogeneous symmetric polynomial of degree* n *then*

$$\mathrm{Res}^3_\psi(f,p) = (-1)^n \frac{\psi(1,\lambda_1-1,\ldots,\lambda_n-1)}{(1-\lambda_1)\cdots(1-\lambda_n)} .$$

(b) *Assume that* $p = [1 : w_1^o : \cdots : w_n^o]$ *is an isolated indeterminacy point, and that* $\det G \neq 0$, *where*

$$G = \left(\frac{\partial F_j}{\partial w_k}(1,w^o) - w_j^o\frac{\partial F_0}{\partial w_k}(1,w^o)\right)_{j,k=1,\ldots,n} ,$$

and $F = (F_0,\ldots,F_n)\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ *is the homogeneous polynomial self-map of degree* $\nu + 1$ *inducing* f. *Denote by* $\mu_1,\ldots,\mu_n \neq 0$ *the eigenvalues of* G. *Then:*

    (i) *we have*

$$\mathrm{Res}^1(f,p) = 0 ;$$

    (ii) *if* $\varphi \in \mathbb{C}[z_1,\ldots,z_n]$ *is a homogeneous symmetric polynomial of degree* n *then*

$$\mathrm{Res}^2_\varphi(f,p) = \frac{\varphi(\mu_1,\ldots,\mu_n)}{\mu_1\cdots\mu_n} ;$$

    (iii) *if* $\psi \in \mathbb{C}[z_0,\ldots,z_n]$ *is a homogeneous symmetric polynomial of degree* n *then*

$$\mathrm{Res}^3_\psi(f,p) = \frac{\psi(0,\mu_1,\ldots,\mu_n)}{\mu_1\cdots\mu_n} .$$

This corollary shows that our Theorem 1 is related to Ueda's index theorem, which however applies only to holomorphic self-maps having only simple fixed points:

Theorem 4: (Ueda [U]) *Let $f: \mathbb{P}^n \to \mathbb{P}^n$ be holomorphic of degree $\nu + 1 \geq 2$, and assume that all fixed points of f are simple (and thus isolated). For $p \in \mathrm{Fix}(f)$, let $\lambda_1(p), \ldots, \lambda_n(p) \neq 1$ denote the eigenvalues of df_p. Then*

$$\forall k = 0, \ldots, n \qquad \sum_{p \in \mathrm{Fix}(f)} \frac{\sigma_k(df_p)}{\left(1 - \lambda_1(p)\right) \cdots \left(1 - \lambda_n(p)\right)} = (-1)^k (\nu + 1)^k .$$

For instance, the case $k = 0$ is a consequence of Theorem 1.(i), and the cases $k > 0$ can be linked to Theorem 1.(ii) and (iii) by using the formula

$$\sigma_j(I - L) = \sum_{\ell=0}^{j} \binom{n - \ell}{n - j} (-1)^\ell \sigma_\ell(L)$$

valid for every endomorphism L of an n-dimensional complex vector space.

Remark 5: Corollary 3 shows that our Theorem 1 generalizes to the case of non-simple or non-isolated fixed (or indeterminacy) points the classical holomorphic Lefschetz formula (see, e.g., [GH, p. 426]), as well as results obtained by Guillot [G] and, in a different context, by Przybylska [P1, P2].

Example 1: Let us consider the map $f([z]) = [z_0^{\nu+1} : \cdots : z_n^{\nu+1}]$. This map is holomorphic, and all its fixed points are simple. More precisely, for each $\ell = 0, \ldots, n$ the set $\mathrm{Fix}(f)$ contains exactly $\binom{n+1}{\ell+1} \nu^\ell$ fixed points of the form $[\zeta_0 : \cdots : \zeta_n]$, where $\ell + 1$ of the ζ_j are ν-th roots of unity, and the remaining ζ_j's are equal to 0. If $p \in \mathrm{Fix}(f)$ has exactly $\ell + 1$ non-zero homogeneous coordinates, we shall say that p is a fixed point of *level* ℓ.

Using (4) it is easy to see that if $p \in \mathrm{Fix}(f)$ is a fixed point of level ℓ then the eigenvalues of df_p are $\nu + 1$ with multiplicity ℓ and 0 with multiplicity $n - \ell$. In particular, $\left(1 - \lambda_1(p)\right) \cdots \left(1 - \lambda_n(p)\right) = (-1)^\ell \nu^\ell$, and so Theorem 1.(i) becomes

$$(-1)^n \sum_{\ell=0}^{n} \binom{n + 1}{\ell + 1} (-1)^\ell = (-1)^n ,$$

while Theorem 1.(ii) becomes

$$(-1)^n \sum_{\ell=0}^{n} \binom{n + 1}{\ell + 1} (-1)^\ell \varphi(\underbrace{\nu, \ldots, \nu}_{\ell \text{ times}}, -1, \ldots, -1) = \int_{\mathbb{P}^n} \varphi(T\mathbb{P}^n - N^{\otimes \nu}) ,$$

and Theorem 1.(iii) becomes

$$(-1)^n \sum_{\ell=0}^{n} \binom{n + 1}{\ell + 1} (-1)^\ell \psi(1, \underbrace{\nu, \ldots, \nu}_{\ell \text{ times}}, -1, \ldots, -1) = \int_{\mathbb{P}^n} \psi\big((T\mathbb{P}^n \oplus N) - N^{\otimes \nu}\big) .$$

For instance, taking $\varphi = \sigma_1^n$ and recalling (1) we get

$$(\nu + 1 + n)^n = \int_{\mathbb{P}^n} c_1^n(T\mathbb{P}^n - N^{\otimes \nu}) = (-1)^n \sum_{\ell=0}^{n} \binom{n + 1}{\ell + 1} (-1)^\ell \big((\nu + 1)\ell - n\big)^n .$$

The (non-trivial) equality of the left and right-hand sides in this formula can also be proved directly by using Abel's formula

$$(x + y)^r = \sum_{k=0}^{r} \binom{r}{k} x(x - kz)^{k-1} (y + kz)^{r-k} \tag{5}$$

valid for all $r \in \mathbb{N}$ and $x, y, z \in \mathbb{C}$ with $x \neq 0$ (see, e.g., [K, p. 56]). Indeed we have

$$
\sum_{\ell=0}^{n} \binom{n+1}{\ell+1} (-1)^\ell \big((\nu+1)\ell - n\big)^n
$$

$$
= \sum_{\ell=0}^{n} \binom{n+1}{\ell+1} \big(n - (\nu+1)\ell\big)^\ell \big(-n + (\nu+1)\ell\big)^{n-\ell}
$$

$$
= \sum_{k=1}^{n+1} \binom{n+1}{k} \big(n+\nu+1 - k(\nu+1)\big)^{k-1} \big(-n-\nu-1 + k(\nu+1)\big)^{n+1-k}
$$

$$
= \frac{1}{n+\nu+1} \Bigg[\sum_{k=0}^{n+1} \binom{n+1}{k} (n+\nu+1)\big(n+\nu+1 - k(\nu+1)\big)^{k-1} \big(-n-\nu-1 + k(\nu+1)\big)^{n+1-k}
$$

$$
\qquad\qquad + (-1)^n (n+\nu+1)^{n+1} \Bigg]
$$

$$
= (-1)^n (\nu+n+1)^n \;,
$$

thanks to (5) applied with $r = n+1$, $x = -y = n+\nu+1$ and $z = \nu+1$.

Analogously, taking $\varphi = \sigma_n$ we get

$$
\sum_{k=0}^{n} \binom{n+1}{k} \nu^{n-k} = \int_{\mathbb{P}^n} c_n(T\mathbb{P}^n - N^{\otimes \nu}) = \sum_{\ell=0}^{n} \binom{n+1}{\ell+1} \nu^\ell = \frac{(\nu+1)^{n+1} - 1}{\nu} = \sum_{\ell=0}^{n} (\nu+1)^\ell \;,
$$

in agreement with both (1) and Theorem 4.

Let us finally show how to use the local theory developed in [ABT1] and [AT2] to prove Theorems 1 and 2.

Given a meromorphic $f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of degree $\nu + 1 \geq 2$, let $F = (F_0, \ldots, F_n) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be the homogeneous polynomial map of degree $\nu + 1$ inducing f. First of all, we associate to F the homogeneous vector field

$$
Q = \sum_{j=0}^{n} F_j \frac{\partial}{\partial z_j} \;.
$$

It is well-known that the time-1 map f_Q of Q is a germ of holomorphic self-map of $\mathbb{C}^{n+1}$ tangent to the identity; furthermore we can write $f_Q(z) = z + F(z) + O(\|z\|^{v+2})$. In particular, by definition a direction $v \in \mathbb{C}^{n+1} \setminus \{O\}$ is a non-degenerate characteristic direction for f_Q if and only if $[v] \in \mathbb{P}^n$ is a fixed point of f; and it is a degenerate characteristic direction for f_Q if and only if $[v]$ is an indeterminacy point of f. Furthermore, since f has degree at least 2 then Q is non-dicritical, that is not all directions are characteristic.

Now let $\pi : M \to \mathbb{C}^{n+1}$ be the blow-up of the origin, and denote by $E = \pi^{-1}(O)$ the exceptional divisor. By construction, E is canonically biholomorphic to $\mathbb{P}^n$; furthermore, the blow-up M can be identified with the total space of the normal bundle N_E of E in M, and N_E is isomorphic to the tautological line bundle N on $\mathbb{P}^n$.

We can now lift f_Q to the blow-up, obtaining (see, e.g., [A1]) a germ $\hat{f}_Q$ about E of holomorphic self-map of M fixing E pointwise, and we may apply all the machinery developed in [ABT1]. First of all, since Q is non-dicritical then $\hat{f}_Q$ is tangential and has order of contact ν with E ([ABT1, Proposition 1.4]). Then we can define ([ABT1, Proposition 3.1 and Corollary 3.2]) the canonical section X_f, which is a global holomorphic section of $TE \otimes (N_E^*)^{\otimes \nu}$. In local coordinates in M centered at $[1 : 0 : \cdots : 0] \in E$ we can write

$$
X_f = \sum_{j=1}^{n} g_j \frac{\partial}{\partial w^j} \otimes (dw_0)^{\otimes \nu} \;, \tag{6}
$$

where $g_1, \ldots, g_n$ are given by (2), and E in these coordinates is given by $\{w_0 = 0\}$.

In particular, we can think of X_f as a bundle morphism $X_f : N_E^{\otimes \nu} \to TE$, vanishing exactly on the characteristic directions of f_Q, and thus we have proved the following (already noted, for instance, in [FS2, p. 409]):

Proposition 5: *Let $f\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a meromorphic self-map of degree $\nu + 1 \geq 2$. Then the canonical section X_f, given locally by (6), defines a global singular holomorphic foliation of $\mathbb{P}^n$ in Riemann surfaces, singular exactly at the fixed and indeterminacy points of f.*

We can now apply the index theorems proved in [ABT1]. Theorem 1.(i) follows immediately from the Camacho-Sad-like index Theorem 6.2 in [ABT1], recalling that $\int_{\mathbb{P}^n} c_1(N)^n = (-1)^n$. Theorem 1.(ii) follows from the Baum-Bott index theorem (see [S, Th. III.7.6] and [ABT1, Theorem 6.4]); and Theorem 1.(iii) follows from the Lehmann-Suwa-like index Theorem 6.3 in [ABT1], that can be applied to this situation because E is automatically comfortably embedded in M ([ABT1, Example 2.4]), and recalling that the exact sequence

$$O \longrightarrow TE \longrightarrow TM|_E \longrightarrow N_E \longrightarrow O$$

implies that $c(TM|_E) = c(T\mathbb{P}^n \oplus N_E)$.

Theorem 2 also follows from [ABT1]. Indeed, Theorem 2.(i) is an immediate consequence of [ABT1, Theorem 6.5.(i)] and the computations in [AT2, Section 5]. The same computations and [ABT1, Theorem 6.6] yield Theorem 2.(iii); and Theorem 2.(ii) follows from [BB] and [S, Th. III.5.5].

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