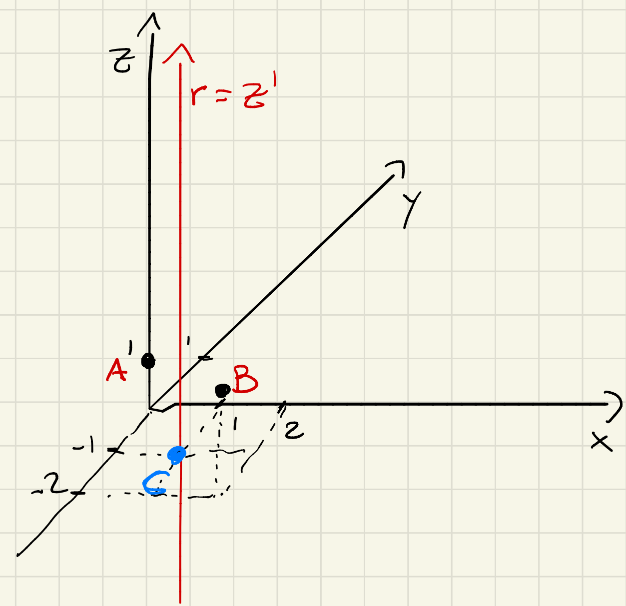
Lezione 18

9.16. Scrivi rototraslazione in $\mathbb{R}^3$

con asse $r = \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ t.c. $f\underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}}_{B}$

$$r = \left\{ \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{C} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$



$$x' = x - 1$$
$$y' = y + 1$$
$$z' = z - 0$$

Nelle nuove coordinate

$$f\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Se non si intuisce,

$$f\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta & 0 \\ \operatorname{sen}\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix}$$

si impone che $f(A) = B$

Ora: $A = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ $B = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta & 0 \\ \operatorname{sen}\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\left.\begin{aligned} -\cos\vartheta - \operatorname{sen}\vartheta &= 1 \\ -\operatorname{sen}\vartheta + \cos\vartheta &= -1 \end{aligned}\right\} \quad \vartheta = \pi$$

$$1 + d = 2 \quad \rightarrow \quad d = 1$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\right) + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

D: L'asse è r?

$$f\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$f(A) = B$?

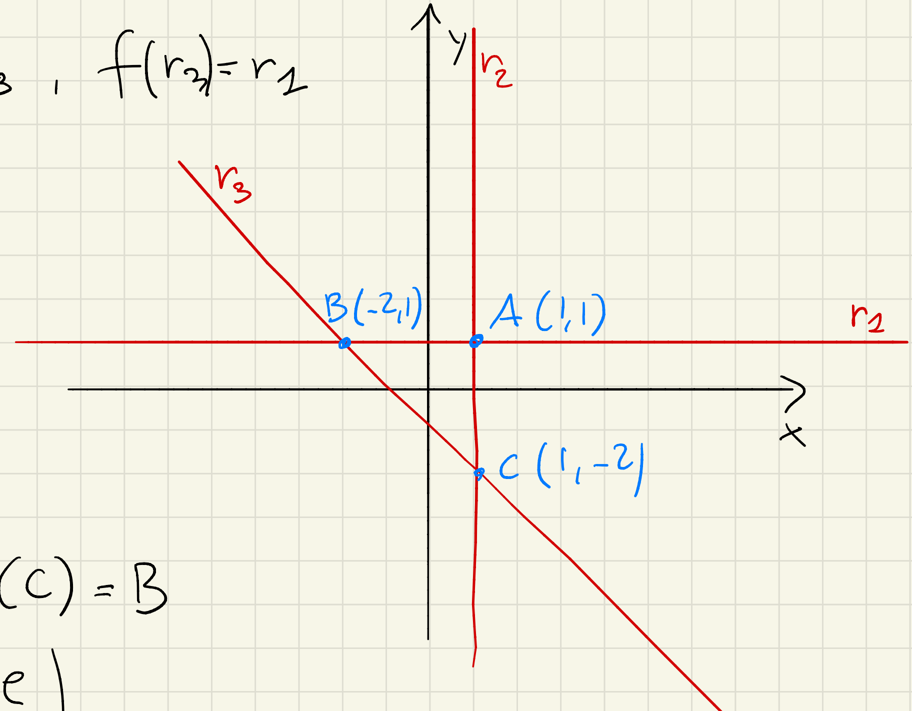
D: $f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$?

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

<u>OK</u>

9.14 : $f : \mathbb{R}^2 \to \mathbb{R}^2$ affinità $f(x) = Ax + b$

$$f(r_1) = r_2 \quad f(r_2) = r_3 \,, \quad f(r_3) = r_1$$



$r_1 : \{y = 1\}$

$r_2 : \{x = 1\}$

$r_3 : \{x + y = -1\}$

$$f(A) = C \quad f(B) = A \quad f(C) = B$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a + b + e \\ c + d + f \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$f\begin{pmatrix}-2\\1\end{pmatrix}=\begin{pmatrix}-2a+b+e\\-2c+d+f\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$f\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}a-2b+e\\c-2d+f\end{pmatrix}=\begin{pmatrix}-2\\1\end{pmatrix}$$

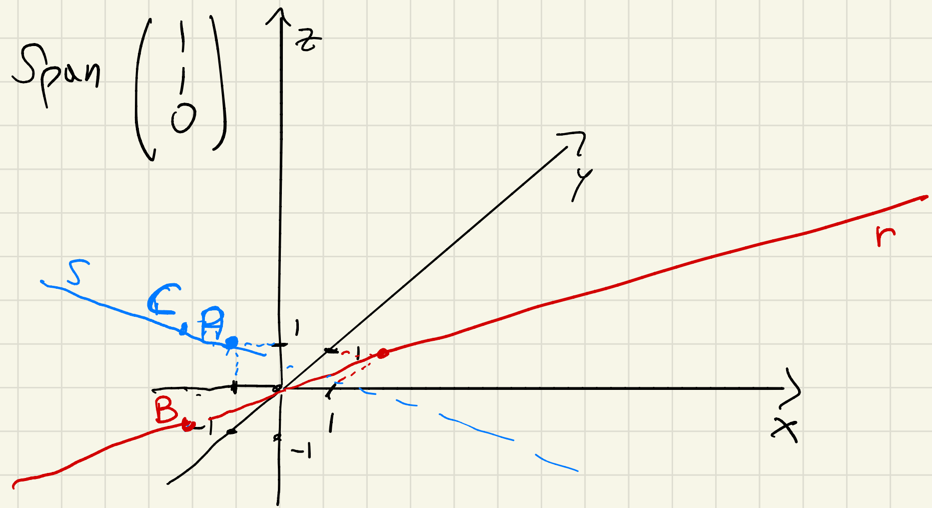
$$\begin{cases}a+b+e=1\\c+d+f=-2\\-2a+b+e=1\\-2c+d+f=1\\a-2b+e=-2\\c-2d+f=1\end{cases}$$

<u>9.20</u> r retta affine

$$r=\mathrm{Span}(e_1+e_2)=\mathrm{Span}\begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$s=\begin{cases}y+z=1\\x+2y+2z=1\end{cases}$$

$$s=\begin{cases}y+z=1\\x=-1\end{cases}$$



$$s = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

A

giac s: $\begin{cases} y + z = 0 \\ x = 0 \end{cases} = \text{Span} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$B = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

$t = -1$: $\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ C

1) Scrivi isometria affine che porti r in s
cioè t.c. $\boxed{f(r) = s}$

Cerco isometria vettoriale che sposti $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ in $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

In questo caso: rotazione intorno a asse y:

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \underline{\text{ok}}.$$

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad \det A = -1$$

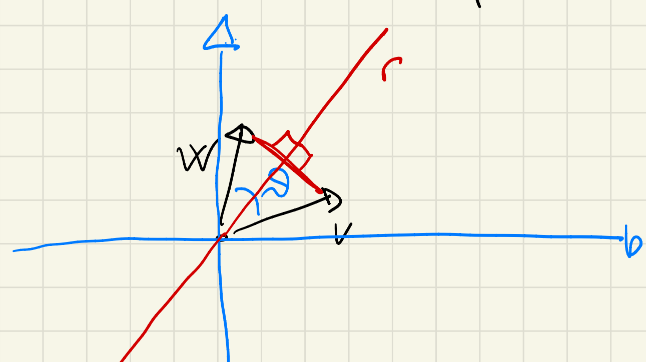
$\underline{\text{riflessione}}$

$$\begin{pmatrix} 1 & \\ & \mathrm{Rot}_\vartheta \end{pmatrix} \qquad \begin{pmatrix} \mathrm{Rot}_\vartheta & \\ & 1 \end{pmatrix}$$

In generale, ho $v, w \in \mathbb{R}^3$ della stessa norma.

Come trovare isometria vettoriale f t.c. $f(v) = w$?

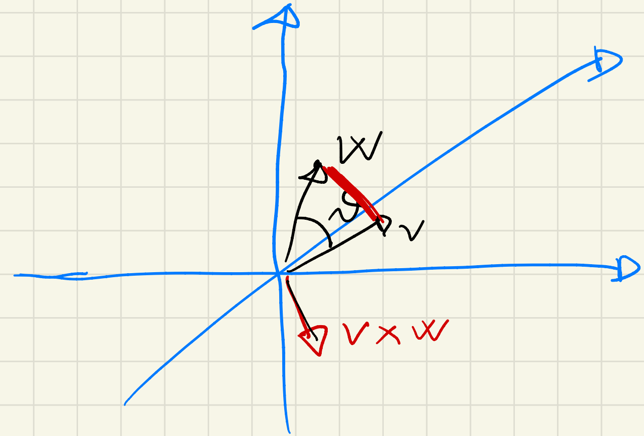
Riflessione lungo r



Rot_ϑ

- Rotazione di asse $v \times w$ e angolo
- Riflessione lungo piano π bisettore

$$\pi = \operatorname{Span}(v-w)^{\perp}$$



$$v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad v \times w = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$u = v - w = \begin{pmatrix} 1 \\ 0 \\ +1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \cdot u = -u$$

$$V_{-1} = \operatorname{Ker}(A + I) = \operatorname{Ker} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \operatorname{Span}(u)$$

$$V_1 = \ker(A - I) = \ker \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} = \text{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right)$$

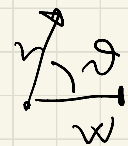
$$\parallel$$

$$\pi$$

Riflessione lungo π

<u>Altra soluzione</u> : Rotaz. intorno a retta $\text{Span}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

con angolo ϑ



$$\cos\vartheta = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|} = \frac{1}{2}$$

$$\vartheta = 60^\circ$$

$$v_1 = \frac{\sqrt{3}}{3} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \qquad v_2 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \frac{\sqrt{6}}{6} \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

$$A = M \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \cdot M^{-1} \qquad M^{-1} = {}^tM$$

$$M = \left(v_1 \,\middle|\, v_2 \,\middle|\, v_3 \right) \qquad M^{-1} = \begin{pmatrix} {}^tv_1 \\ \hline {}^tv_2 \\ \hline {}^tv_3 \end{pmatrix}$$

Cerco b:

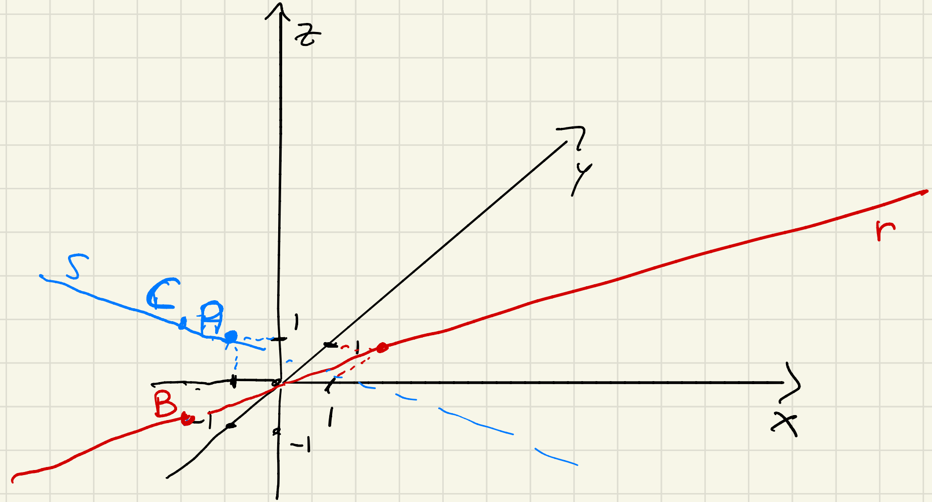
$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Basta $b \in S$

2) f ha punto fisso $\iff$ $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} -1 \\ t \\ 1-t \end{pmatrix}$$ ha soluzione

$$b = \begin{pmatrix} -1 \\ t \\ 1-t \end{pmatrix}$$

$$s = \left\{ \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{A} + t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$



$$\begin{cases} x = -z - 1 \\ y = y + t \\ z = -x + 1 - t \end{cases} \quad \longleftarrow \quad \text{Se } t \neq 0 \text{ non ha soluzione}$$

$$\Downarrow$$

NO PUNTI FISSI

Fix	$\emptyset$	$\cdot$		
$\det A = 1$	TRASLAZIONI ROTO-TRASLAZ.		ROTAZIONI	
$\det A = -1$	GLISSO RIFLESSIONI	ANTIROTAZ.		RIFLESSIONE

Se $t=0$?

$$\begin{cases} x = -z-1 \\ y = y \\ z = -x+1 \end{cases}$$

$x = x - 1 - 1$

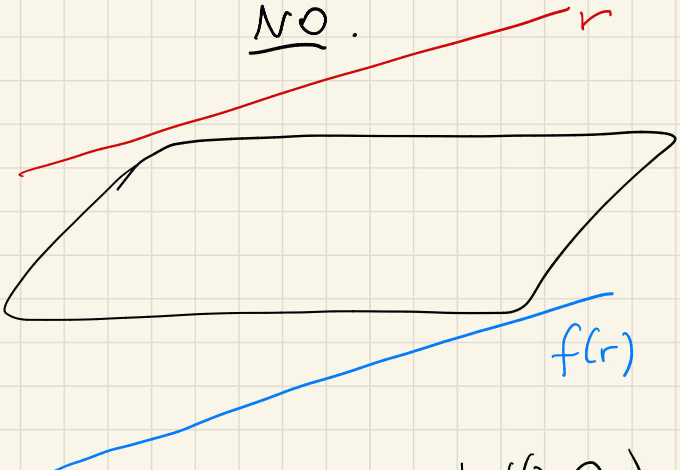
$\nexists$

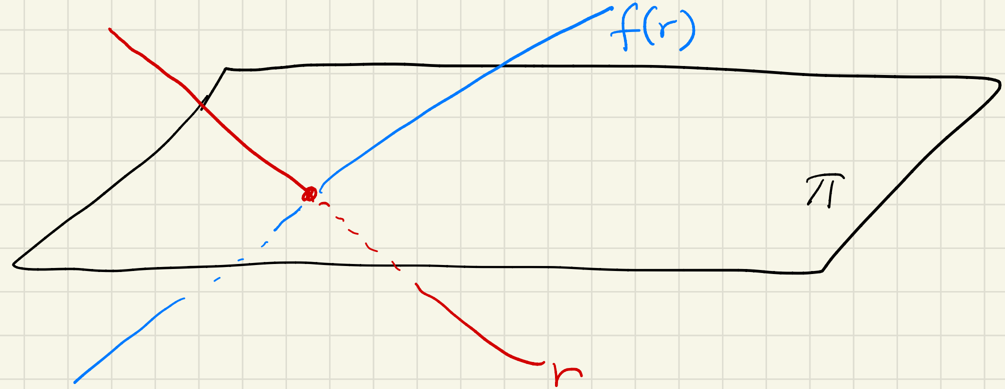
NO SOLUZIONI

Non ho trovato isometrie con punti fissi

3) $\exists$ riflessione f t.c. $f(r) = s$?

<u>NO</u>.





$$\det(\lambda A) = \lambda^n \det A$$

9.19:

$$A = \frac{1}{7}\begin{pmatrix} -6 & 3 & -2 \\ 2 & 6 & 3 \\ 3 & 2 & -6 \end{pmatrix}$$

$$\det A = \left(\frac{1}{7}\right)^3 \left(-6(-42) - 2\cdot(-14) + 3(27)\right) = 1$$

1) Determina il tipo di $L_A : \mathbb{R}^3 \to \mathbb{R}^3$ $\qquad L_A(x) = A\cdot x$

2) Per qual $b \in \mathbb{R}^3$, $\quad f_b(x) = Ax + b$ è rotazione?

Fix	$\emptyset$	$\cdot$		
$\det A = 1$	TRASLAZIONI ROTO-TRASLAZ.		ROTAZIONI	
$\det A = -1$	GLISSO RIFLESSIONI	ANTIROTAZ.		RIFLESSIONE

$$\operatorname{tr} A = \frac{1}{7}(-6) = -\frac{6}{7} = 1 + 2\cos\theta$$

$$A \text{ è simile a } \begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta & 0 \\ \operatorname{sen}\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad tr = 1 + 2\cos\vartheta$$

$$\cos\vartheta = -\frac{13}{14}$$

Asse: $r = V_1 = \operatorname{Ker}(A - I)$

$$= \operatorname{Ker} \frac{1}{7} \begin{pmatrix} -6-7 & 3 & -2 \\ 2 & 6-7 & 3 \\ 3 & 2 & -6-7 \end{pmatrix} =$$

$$= \operatorname{Ker} \frac{1}{7} \begin{pmatrix} -13 & 3 & -2 \\ 2 & -1 & 3 \\ 3 & 2 & -13 \end{pmatrix} =$$

$$\ker\begin{pmatrix} -13 & 3 & -2 \\ 2 & -1 & 3 \\ 3 & 2 & -13 \end{pmatrix} = \ker\begin{pmatrix} -13 & 3 & -2 \\ 2 & -1 & 3 \\ 1 & 3 & -16 \end{pmatrix} = \ker$$

$$\ker\begin{pmatrix} -10 & 5 & -15 \\ 2 & -1 & 3 \\ 3 & 2 & -13 \end{pmatrix} = \ker\begin{pmatrix} 2 & -1 & 3 \\ 3 & 2 & -13 \end{pmatrix} =$$

$$\ker\begin{pmatrix} 2 & -1 & 3 \\ 1 & 3 & -16 \end{pmatrix} = \ker\begin{pmatrix} 1 & -4 & 19 \\ 1 & 3 & -16 \end{pmatrix} =$$

$$\ker\begin{pmatrix} 1 & -4 & 19 \\ 0 & 7 & -35 \end{pmatrix} = \ker\begin{pmatrix} 1 & -4 & 19 \\ 0 & 1 & -5 \end{pmatrix}$$

$$= \text{Ker}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -5 \end{pmatrix} = \text{Span}\begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix}$$

$$\begin{array}{ccc} x & y & z=1 \end{array}$$

$$\begin{cases} x - z = 0 \\ y - 5z = 0 \end{cases} \qquad \begin{cases} x = 1 \\ y = 5 \end{cases}$$

2) Per quali $b \in \mathbb{R}^3$ $\quad f_b(x) = Ax + b$ è rotazione?

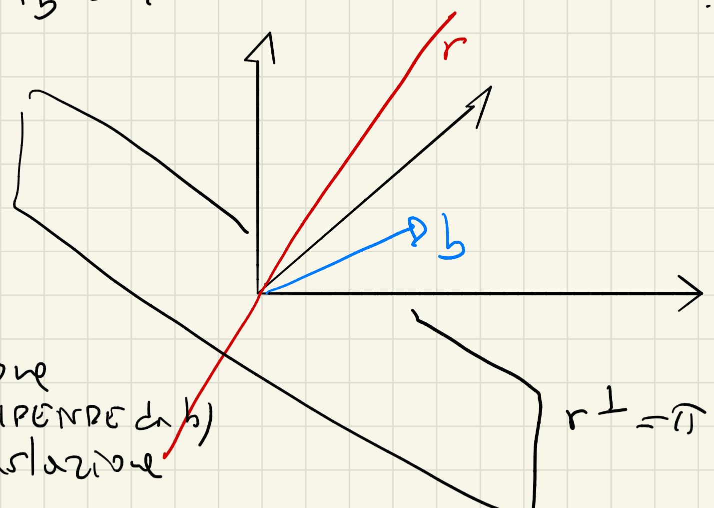
f_b è rototraslazione

o rotazione

$\pi = r^\perp = \{x + 5y + z = 0\}$

a) se $b \in \pi \Rightarrow$ rotazione (L'ASSE DIPENDE da b)

b) Se $b \notin \pi \Rightarrow$ rototraslazione



(L'ASSE DIPENDE da b)

$\pi = \{ z = 0 \}$

$$f\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} Rot_\vartheta & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

<u>ES</u>: f ha punti fissi $\iff b_z = 0$