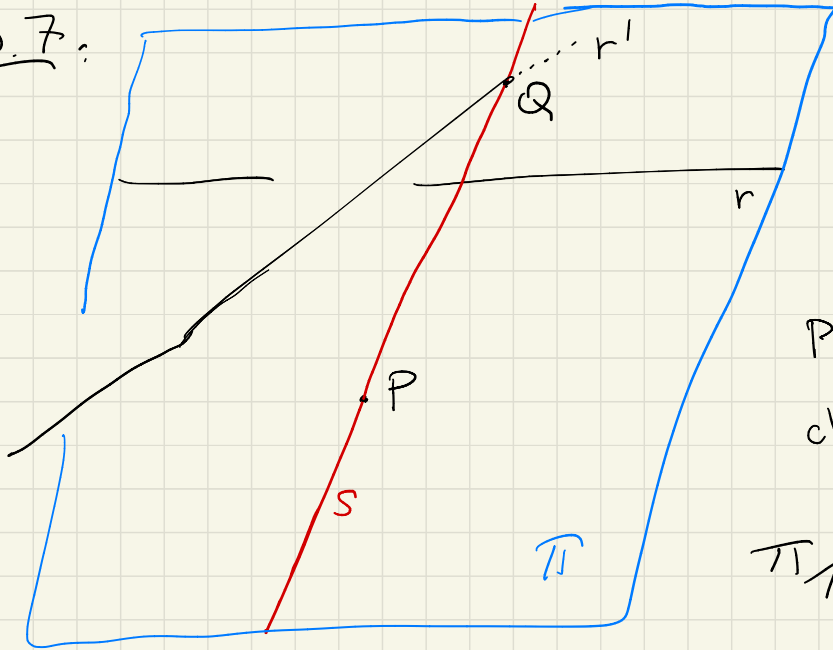
9.7:



Tesi: $\exists!\, s$ che interseca P, r, r'.

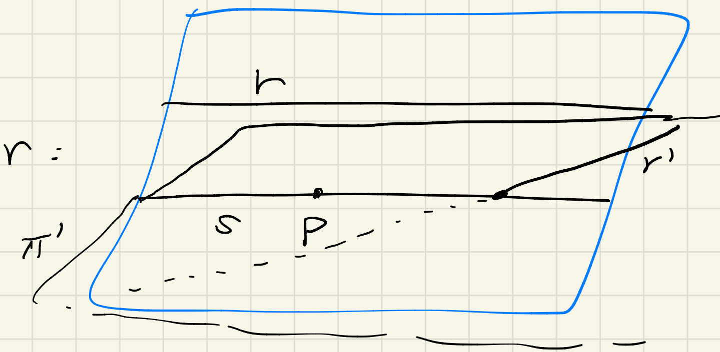
P r $\quad \exists!\, \pi$ piano che contiene P e r

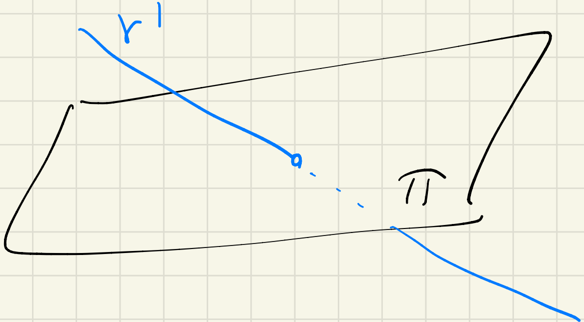
$\pi \not\parallel r' \Rightarrow \pi \cap r' = \{Q\}$

$s :=$ retta che passa per P e Q.

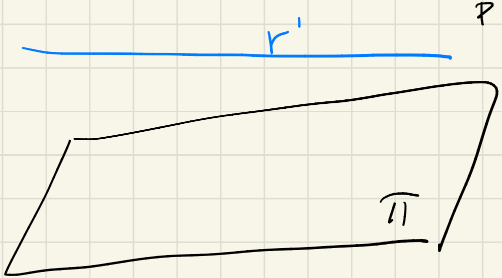
Devo mostrare che s non è parallela a r:

questo è vero perché π' non è parallelo a r





$r' \cap \pi$ è un punto

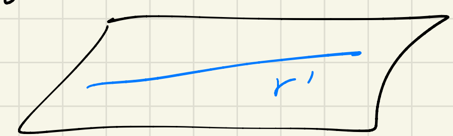


PIANO E RETTA:

- paralleli se $giac(r') \subseteq \pi$
- incidenti se $giac(r') \nsubseteq \pi$

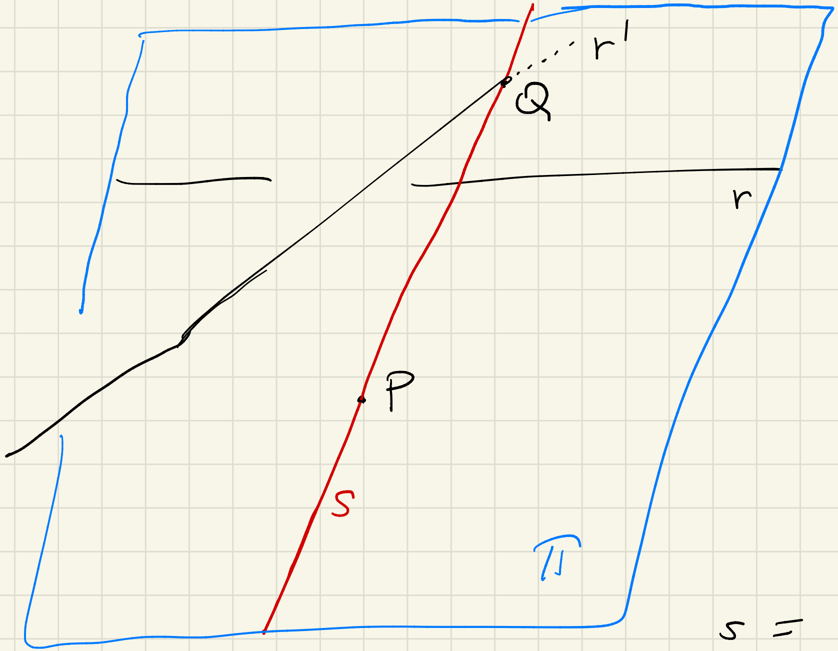
$\rightarrow$ $r' \cap \pi = \emptyset$

$\rightarrow$ $r' \subseteq \pi$



Caso concreto: $P = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$$r = \left\{ \frac{x+3}{2} = 1 - y = -2z \right\} \qquad r' = \begin{cases} 2x - y - 1 = 0 \\ 3x + 3y - z = 0 \end{cases}$$



$$r = \begin{cases} \frac{x+3}{2} + y - 1 = 0 \\ -y + 2z + 1 = 0 \end{cases}$$

$$\pi = \{-y + 2z + 1 = 0\}$$

$$Q = \begin{pmatrix} -10/9 \\ -1/9 \\ -5/9 \end{pmatrix}$$

$$s = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -28/9 \\ -10/9 \\ -5/9 \end{pmatrix} \right\}$$

$$\pi = \left\{ t\left(\frac{x+3}{2} + y - 1\right) + u\left(-y + 2z + 1\right) = 0 \right\}$$

$$\frac{t}{2}x + \frac{3}{2}t + yt - t - uy + 2uz + u = 0$$

$$\frac{t}{2}x + (t-u)y + 2uz + \frac{1}{2}t + u = 0$$

$$t \quad + t - u \quad + \frac{1}{2}t + u = 0$$

$$\frac{5}{2}t = 0 \qquad t = 0 \qquad u = 1$$

$$\begin{cases} 2x - y - 1 = 0 \\ 3x + 3y - z = 0 \\ -y + 2z + 1 = 0 \end{cases} \qquad \begin{pmatrix} 2 & -1 & 0 & -1 \\ 3 & 3 & -1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 4 & -1 & 1 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & -9 & 0 & -1 \\ 1 & 4 & -1 & 1 \\ 0 & -1 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 4 & -1 & 1 \\ 0 & 0 & -18 & -10 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 7 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 9 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 7 & 5 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 5/9 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & 10/9 \\ 0 & 1 & 0 & 1/9 \\ 0 & 0 & 1 & 5/9 \end{pmatrix}$$

$$5 - 7 \cdot 5/9 = 5 - \frac{35}{9} = \frac{10}{9}$$

$$\begin{cases} x + 10/9 = 0 \\ y + 1/9 = 0 \\ z + 5/9 = 0 \end{cases} \qquad Q = \begin{pmatrix} -10/9 \\ -1/9 \\ -5/9 \end{pmatrix}$$

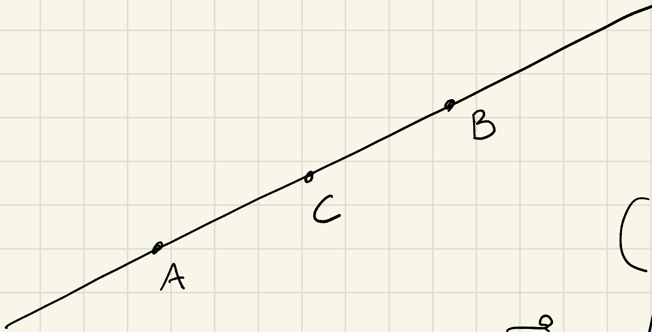
9.1

$$A = \begin{pmatrix} x_A \\ y_A \end{pmatrix} \qquad B = \begin{pmatrix} x_B \\ y_B \end{pmatrix} \qquad C = \begin{pmatrix} x_C \\ y_C \end{pmatrix}$$

sono allineati $\Leftrightarrow \det \begin{pmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{pmatrix} = 0$

$\boxed{\Rightarrow}$

$\Leftarrow$



$\vec{AC}$ e $\vec{AB}$ sono multipli

(cioè non sono indipendenti)

$$\vec{AC} = \begin{pmatrix} x_C - x_A \\ y_C - y_A \end{pmatrix} \qquad \vec{AB} = \begin{pmatrix} x_B - x_A \\ y_B - y_A \end{pmatrix}$$

$$\det \begin{pmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{pmatrix} = \det \begin{pmatrix} x_A - x_C & y_A - y_C & 0 \\ x_B - x_C & y_B - y_C & 0 \\ x_C & y_C & 1 \end{pmatrix}$$

$$= 0 \iff \det \begin{pmatrix} x_A - x_C & y_A - y_C \\ x_B - x_C & y_B - y_C \end{pmatrix} = 0 \iff$$ Le righe sono multiple

v, w sono dipendenti se sono multipli

cioè $v = kw$ oppure $w = kv$

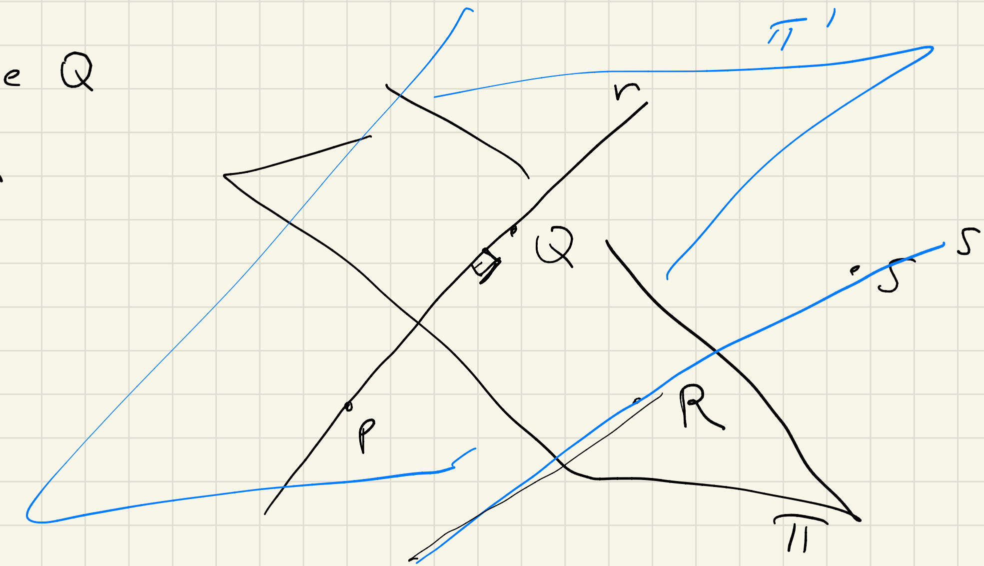
9.4 $\quad P = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad Q = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad R = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \quad S = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

1) r passa per P e Q

2) $\pi \perp r \quad \pi \ni R$

3) $\pi' \supseteq r \quad \pi' \parallel s$

s passa R e S



$$\overrightarrow{PQ} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$r = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix} + t\begin{pmatrix}-2\\-1\\1\end{pmatrix} \right\}$$

$$\pi = \{-2x - y + z = K\}$$

fascio di piani perp. a r

Impongo $R \in \pi$: $-4 = K$

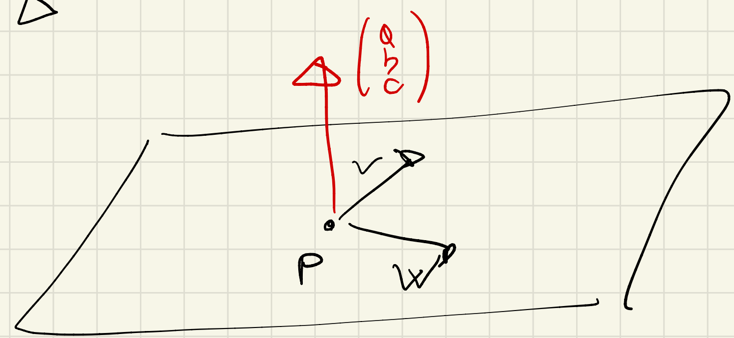
$$\pi = \{-2x - y + z = -4\} = \{2x + y - z = 4\}$$

$$s = \left\{ \begin{pmatrix}2\\0\\0\end{pmatrix} + t\underbrace{\begin{pmatrix}-2\\0\\1\end{pmatrix}}_{\vec{RS}} \right\}$$

$$\pi' = \left\{ \underbrace{\begin{pmatrix}1\\1\\0\end{pmatrix}}_{P} + t\underbrace{\begin{pmatrix}-2\\-1\\1\end{pmatrix}}_{v} + u\underbrace{\begin{pmatrix}-2\\0\\1\end{pmatrix}}_{w} \right\}$$

$$\pi' = \{ax + by + cz = d\}$$

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-2\\-1\\1\end{pmatrix} \times \begin{pmatrix}-2\\0\\1\end{pmatrix}$$

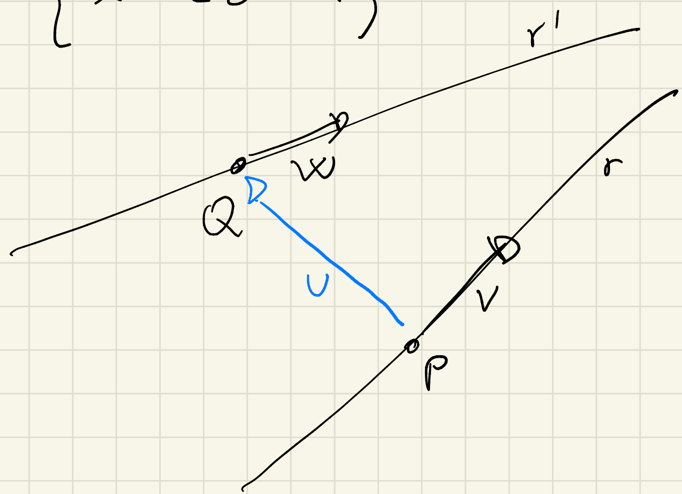


$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} \qquad \pi' = \{-x - 2z = d\}$$

$$P \in \pi' : \ -1 = d \ \Rightarrow \ d = -1$$

$$\pi' = \{x + 2z = 1\}$$

$u = \vec{PQ}$



r e r' sono complanari

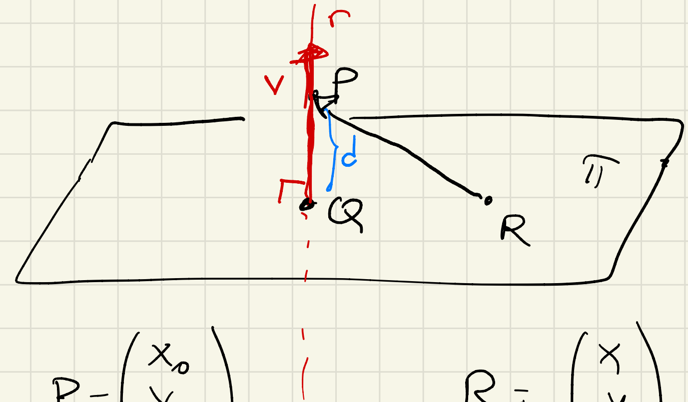
$\iff \det\left(u \ v \ w\right) = 0$

sghembe $\iff \det\left(u\,v\,w\right) \neq 0$

Distanza

Fra PUNTO e PIANO

$d :=$ lunghezza di $\vec{PQ}$



$\pi = \{ax + by + cz - d = 0\}$

$P = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ $\qquad R = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Formula:

$$d = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2+b^2+c^2}}$$

dimostrazione:

$v = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\qquad \vec{RP} = \begin{pmatrix} x_0 - x \\ y_0 - y \\ z_0 - z \end{pmatrix}$

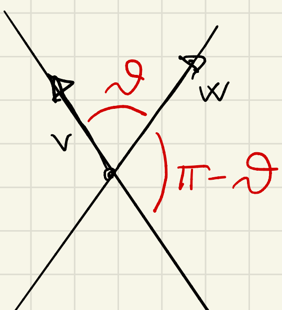
$$d = \| P_v(\vec{RP}) \| = \frac{|\langle \vec{RP}, v \rangle|}{\|v\|} = \frac{|a(x_0-x)+b(y_0-y)+c(z_0-z)|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|a x_0 + b y_0 + c z_0 - (ax + by + cz)|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|a x_0 + b y_0 + c z_0 - d|}{\sqrt{a^2 + b^2 + c^2}}$$

Es: $\pi = \{ 2x - 3y + z = 1 \}$ $\quad P = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$d = \frac{|2x_0 - 3y_0 - z_0 - 1|}{\sqrt{14}} = \frac{|2 - 6 - 3 - 1|}{\sqrt{14}} = \frac{8}{\sqrt{14}}$$

ANGOLI

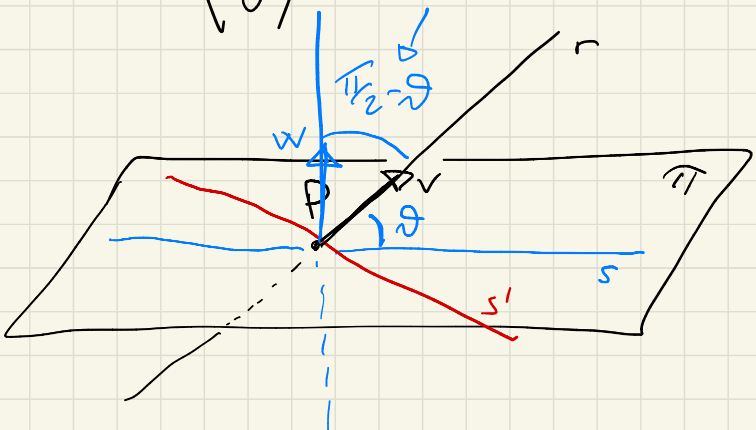


ϑ = angolo fra v e w

$$\cos\vartheta = \frac{\langle v, w \rangle}{\|v\| \cdot \|w\|}$$

$$r = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad s = \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$r \cap s = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \cos\vartheta = \frac{0}{\dots} = 0 \Rightarrow \text{sono ortogonali}$$



L'angolo fra r e π è definito come l'angolo fra r e la sua proiezione s su π.

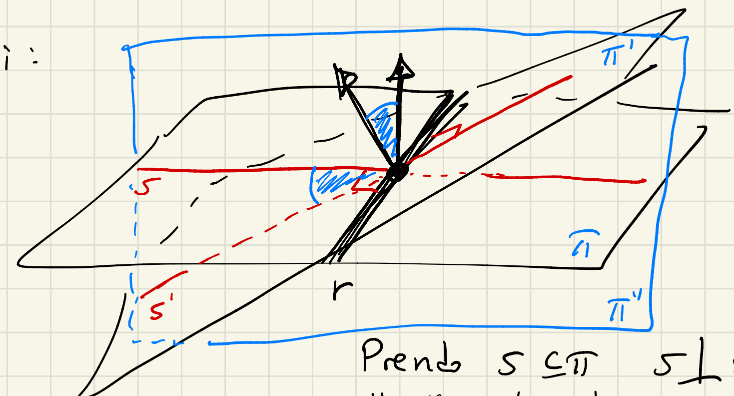
$$r = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \pi = \{ x - y + 2z = 1 \}$$

$$\text{giac}(r) = \text{Span} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{giac}\, \pi = \{ x - y + 2z = 0 \}$$

$$v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \qquad \alpha = \pi/2 - \vartheta$$

$$\cos\alpha = \frac{2}{\sqrt{3} \cdot \sqrt{6}} = \frac{\sqrt{3}\, 2}{9}$$

<u>Tra due piani:</u>



$r = \pi \cap \pi'$

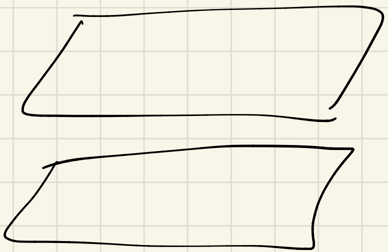
Prendo $s \subseteq \pi$ $\quad s \perp r$

" " $s' \subseteq \pi'$ $\quad s' \perp r$

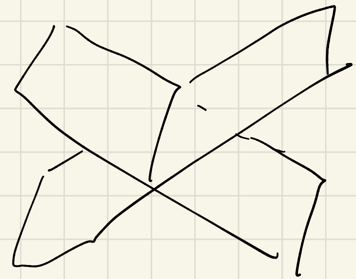
L'angolo fra π e π' è quello fra s e s'

Concretamente:

$\pi = \{2x - y + z = 4\}$ $\quad$ $\pi' = \{x + 2y - 3z = 1\}$



PARALLELI



INCIDENTI

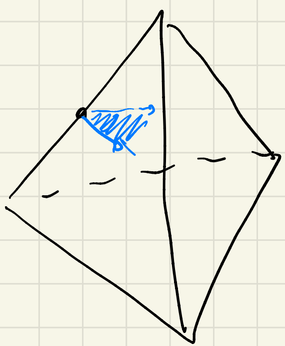
$v = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ $\quad$ $v' = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

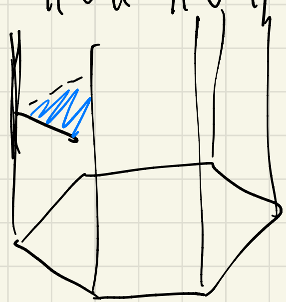
incidenti

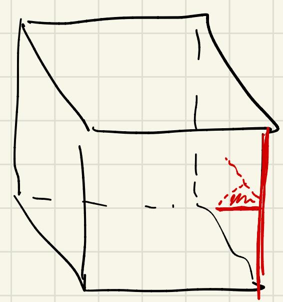
L'angolo fra π e π' è uguale a quello fra v e v'

$$\cos\vartheta = \frac{\langle v, v' \rangle}{\|v\| \cdot \|v'\|}$$

72°







AFFINITÀ

Def: Una AFFINITÀ è una funzione $f: \mathbb{R}^n \to \mathbb{R}^n$ del tipo

$$\boxed{f(x) = Ax + b}$$

A matrice $n \times n$

$b \in \mathbb{R}^n$

Esempi:

Se $b = 0$, $f(x) = Ax$ $f = L_A$

Se $A = I$, $f(x) = x + b$ TRASLAZIONE

A COMPONENTE LINEARE di f

b PARTE TRASLATORIA di f

$n = 1$: $f: \mathbb{R} \to \mathbb{R}$ $f(x) = ax + b = mx + q$ il grafico è una (retta)

Def: f è un ISOMORFISMO AFFINE se è invertibile (cioè se $f: \mathbb{R}^n \to \mathbb{R}^n$ sia corrispondenza biunivoca)

Prop: Una affinità $f(x) = Ax + b$ è un isomorfismo affine $\Leftrightarrow$ A è matrice invertibile. In questo caso l'inversa è

$$\boxed{g(y) = A^{-1}y - A^{-1}b}$$

dim: $\boxed{\Leftarrow}$ $\boxed{f = T_b \circ L_A}$

$L_A(x) = Ax$

$T_b(x) = x + b$

Se A è invertibile allora L_A è invertibile

T_b è invertibile: l'inversa è T_{-b}

$\Rightarrow$ f è composizione di due corr. biunivoche è corr. biuniv.

$\boxed{\Rightarrow}$ Se A non è invertibile

$$\{x \mid Ax + b = 0\} \quad \text{R.C.} \begin{cases} \to \emptyset \\ \to \infty \text{ punti} \end{cases}$$

$$\| $$

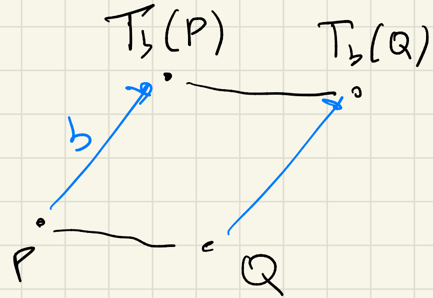
$$\{x \mid f(x) = 0\}$$

$\Rightarrow f$ non è mai iniettiva

AFFINITÀ	ISOMORFISMI AFFINI	ISOMETRIE AFFINI
$f(x) = Ax + b$	A INVERTIBILE	A ORTOGONALE

<u>Def</u>: Un isomorfismo affine $f(x) = Ax + b$ è ISOMETRIA AFFINE se preserva le distanze, cioè $\boxed{d(f(P), f(Q)) = d(P, Q)}$ $\forall P, Q \in \mathbb{R}^n$

<u>Esempio</u>: $T_b : x \longmapsto x + b$ TRASLAZIONE



$$d(f(P), f(Q)) = d(P+b, Q+b)$$

$$\|Q+b-(P+b)\| = \|Q-P\| = d(P,Q)$$

Prop: Un isomorfismo affine $f(x) = Ax + b$ è isometria affine $\Leftrightarrow$ A è matrice ortogonale

dim:

$$f(x) = Ax + b \qquad \boxed{f = T_b \circ L_A}$$

ISOMETRIA AFFINE

$\boxed{\Leftarrow}$ Se A è ortogonale allora $L_A : \mathbb{R}^n \to \mathbb{R}^n$ $\boxed{\text{isometria vettoriale} \Rightarrow \text{isometria affine}}$

(perché mantiene le distanze)

Composizione di due isometrie affini è isometria affine

$\Rightarrow f = T_b \circ L_A$ isometria affine

$\boxed{\Leftarrow}$ Se f isometria, $\quad T_{-b} \circ f = \cancel{T_{-b}} \circ \cancel{T_b} \circ L_A$

allora L_A isometria $\quad L_A = T_{-b} \circ f$

$\Rightarrow A$ ortogonale

$\square$

Calcoliamo l'inversa: $\quad$ <u>A invertibile</u>

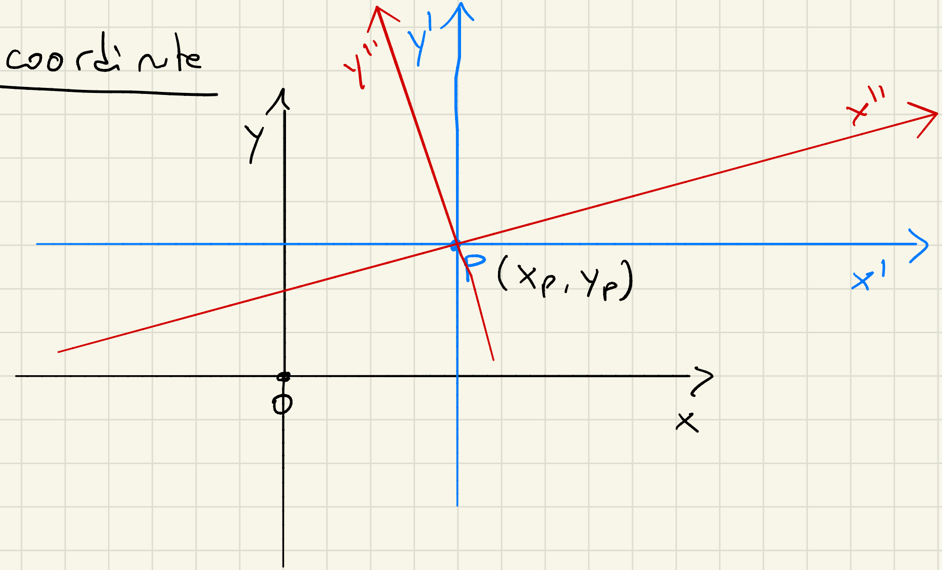
$$y = \boxed{f(x) = Ax + b} \qquad y = Ax + b$$

$$A^{-1}y = \cancel{A^{-1}}\cancel{A}x + A^{-1}b$$

$$\boxed{x = A^{-1}y - A^{-1}b}$$

$$\boxed{g(y) = A^{-1}y - A^{-1}b} \qquad g = T_{-A^{-1}b} \circ L_{A^{-1}}$$

Cambiamento di sistema di coordinate



Traslare l'origine:

$$X = x' + x_P$$

$$Y = y + y_P$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \boxed{X = x' + P}$$

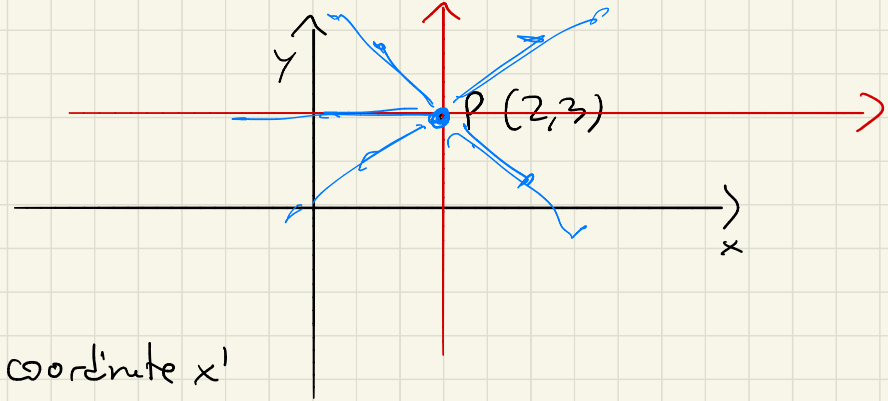
In generale $\boxed{X = Mx' + P}$

M matrice ortogonale

Esempi di isomorfismi affini:

OMOTETIA di fattore $\lambda > 0$ $\quad f(x) = \lambda x$ di centro O

Omotetia di centro $P \in \mathbb{R}^n$
e di fattore λ



$x = x' + P \qquad x' = x - P$

$f(x') = \lambda x'$ nelle coordinate x'

Nelle coordinate x:

In generale: $y' = f(x') = \boxed{Ax' + b}$ $\qquad y' = Ax' + b$

cambio coordinate: $x' = x - P$

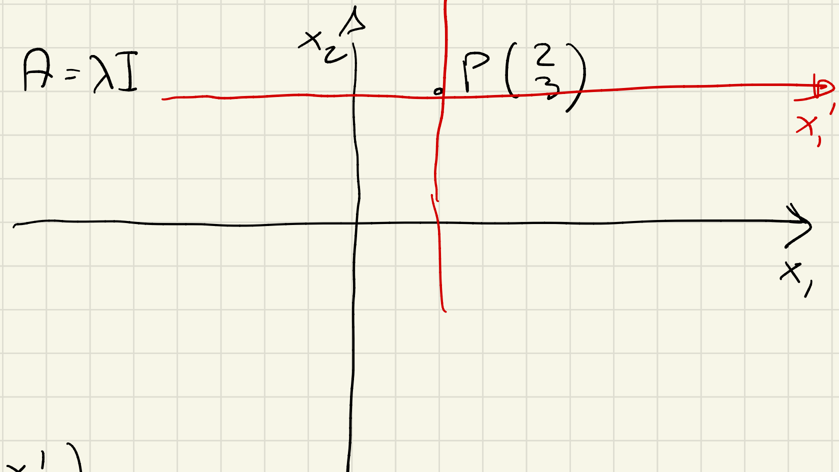
$y' = y - P$

$$y - P = A(x - P) + b$$

$$\boxed{y = Ax \underbrace{- AP + b + P}_{\text{nuovo termine traslatorio}}}$$

$$\boxed{f(x) = Ax - AP + b + P}$$

Omotetia: $f(x') = \lambda x'$ $\quad A = \lambda I$



$$\boxed{f(x) = \lambda x - \lambda P + P}$$

Nel caso $P = \begin{pmatrix}2\\3\end{pmatrix}$ $\quad \lambda = 4$

$$f\begin{pmatrix}x_1'\\x_2'\end{pmatrix} = 4\begin{pmatrix}x_1'\\x_2'\end{pmatrix} = \begin{pmatrix}4x_1'\\4x_2'\end{pmatrix}$$

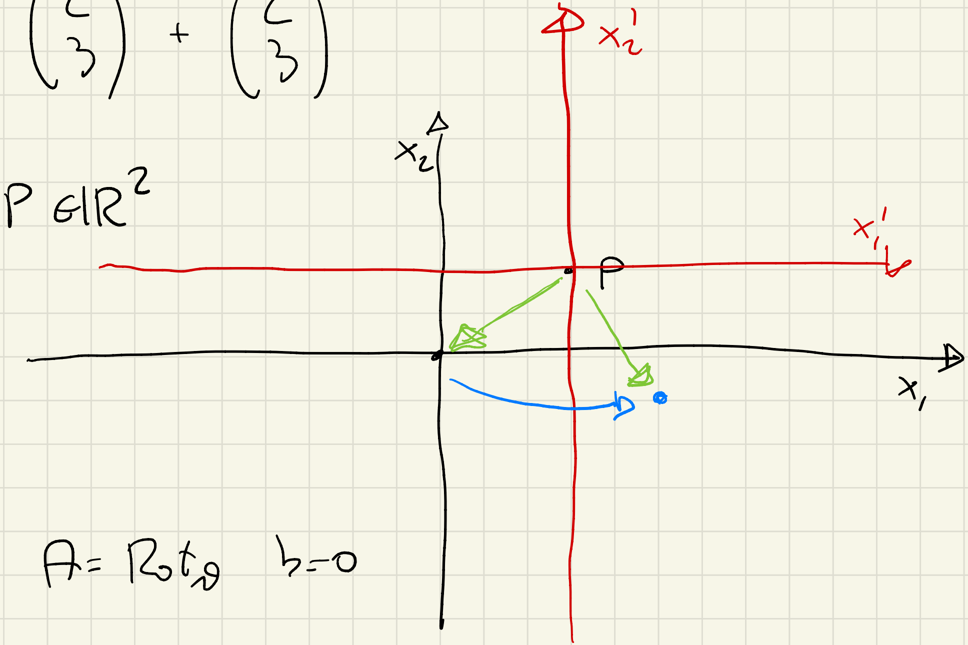
$$\begin{pmatrix}x_1'\\x_2\end{pmatrix} = \begin{pmatrix}x_1\\x_2\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix}$$

$$f\begin{pmatrix}x_1\\x_2\end{pmatrix} = 4\left(\begin{pmatrix}x_1\\x_2\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix}\right) + \begin{pmatrix}2\\3\end{pmatrix}$$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 4\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - 4\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

ISOMETRIA AFFINE

<u>Esempio</u>: Rotazione intorno a $P \in \mathbb{R}^2$ di angolo ϑ



$$f\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = Rot_\vartheta \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

$$f(x') = Rot_\vartheta \cdot x'$$

$A = Rot_\vartheta \quad b = 0$

$$\boxed{f(x) = Rot_\vartheta \cdot x - Rot_\vartheta \cdot P + P}$$

$$\boxed{f(x) = Ax - AP + b + P}$$

Caso $P = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \vartheta = \frac{\pi}{2}$

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 5 \\ -1 \end{pmatrix}}$$

<u>Riprova</u>: $f(x) = Ax + b$ $\qquad$ $y = mx + q$

$x = 0 \qquad f(0) = A \cdot 0 + b = b$