Isometrie vettoriali di $\mathbb{R}^n$:

più in generale, se $\mathcal{B}$ è ortonormale
T isometria $\Leftrightarrow$ $[T]_{\mathcal{B}}^{\mathcal{B}}$ è ortogonale

- Sono tutte del tipo $L_A : \mathbb{R}^n \to \mathbb{R}^n$, $x \longmapsto A \cdot x$ per qualche A ortogonale

  cioè ${}^tA \cdot A = I$

- Una matrice di cambiamento di base fra basi ortonormali è sempre ortogonale

$n=2$: $\quad Rot_\vartheta$ e Rif_ϑ

$n=3$: più complicate, però prendendo base ortonormale giusta diventano

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{Rot_ϑ}} \\ 0 & & \end{pmatrix}$$

$\nearrow$ rotazione se $+1$

$\searrow$ antirotazione se -1

In generale, una riflessione lungo $U \subseteq V$ sottospazio ha $\det = (-1)^{\dim V - \dim U}$

$\underline{n=3:}$ $V = \mathbb{R}^3$

$$\begin{cases} U = \text{piano} \Rightarrow \det = -1 \Rightarrow \text{antirotaz. di angolo } 0 \\ U = \text{retta} \Rightarrow \det = 1 \Rightarrow \text{rotaz. di angolo } \pi \\ U = \text{origine} \Rightarrow \det = -1 \Rightarrow \text{antirotaz. di angolo } \pi \end{cases}$$

$$\begin{pmatrix} -1 & \\ & \mathrm{Rot}_\vartheta \end{pmatrix}_{\vartheta = \pi} = \underset{U = \{0\}}{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}}, \quad \underset{U = \text{retta}}{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}, \quad \underset{U = \text{piano}}{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

$\underline{8.16:}$

$r_U : \mathbb{R}^3 \to \mathbb{R}^3$ $\quad U = \{x - y + 3z = 0\}$

$$U^\perp = \mathrm{Span}\underset{= v}{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}}$$

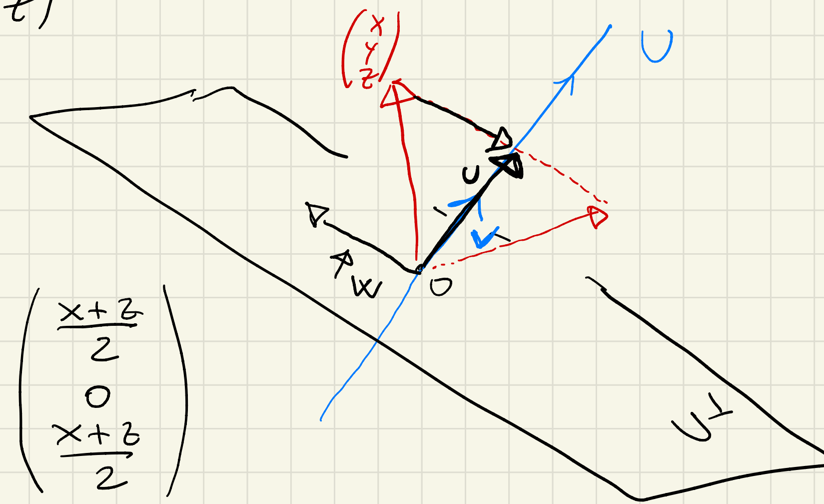
$$r_U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2\,\frac{\left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, v \right\rangle}{\langle v, v \rangle}\, v$$

$$= \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2\,\frac{x-y+3z}{11}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{9}{11}x + \frac{2}{11}y - \frac{6}{11}z \\ \frac{2}{11}x + \frac{9}{11}y + \frac{6}{11}z \\ -\frac{6}{11}x + \frac{6}{11}y - \frac{7}{11}z \end{pmatrix}$$

$$= \frac{1}{11}\begin{pmatrix} 9 & 2 & -6 \\ 2 & 9 & 6 \\ -6 & 6 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$81+4+36 = 121$

$$U = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

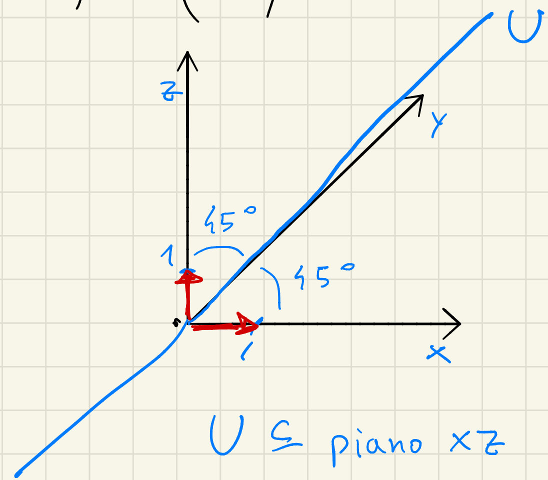


$$u = P_{\underset{V}{\vee}}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{x+z}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{x+z}{2} \\ 0 \\ \frac{x+z}{2} \end{pmatrix}$$

$$w = P_{U^\perp}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - u = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} \frac{x+z}{2} \\ 0 \\ \frac{x+z}{2} \end{pmatrix} = \begin{pmatrix} \frac{x-z}{2} \\ y \\ \frac{z-x}{2} \end{pmatrix}$$

$$r_U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - 2w = \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} x-z \\ 2y \\ z-x \end{pmatrix} = \begin{pmatrix} z \\ -y \\ x \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



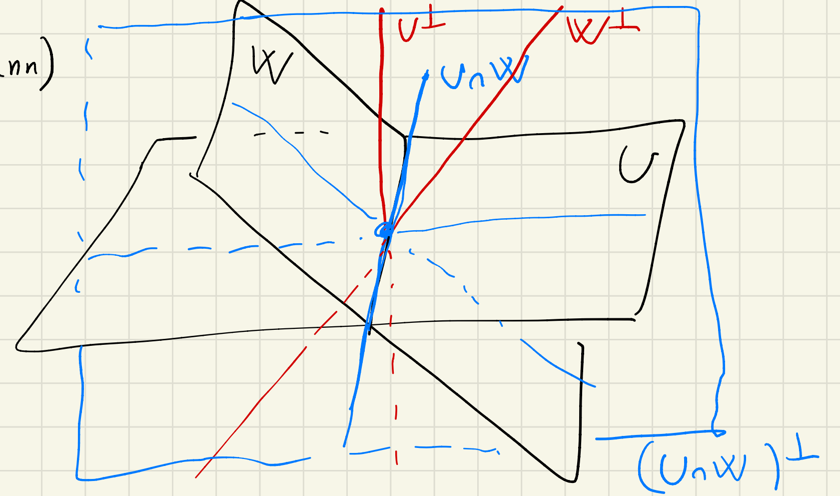
$U \subseteq$ piano xz

3) $U = \{0\}$

$r_U = -I$ sempre.

8.1. $U \cap W$ retta (per Grassmann)

$U^\perp$ $W^\perp$



2) $\underset{\dim = 2}{(U \cap W)^\perp} \overset{?}{=} \underbrace{U^\perp + W^\perp}_{\dim 2}$

$\subseteq$

$\boxed{\supseteq}$

Dimostro che $U^\perp + W^\perp \subseteq (U \cap W)^\perp$

$v \in U^\perp + W^\perp \iff v = u + w$ con $u \in U^\perp$, $w \in W^\perp$

Tesi: $v \in (U \cap W)^\perp$

$\boxed{u \in U^\perp \Rightarrow u \in (U \cap W)^\perp}$

$w \in W^\perp \Rightarrow w \in (U \cap W)^\perp$

$\Rightarrow v = u + w \in (U \cap W)^\perp$

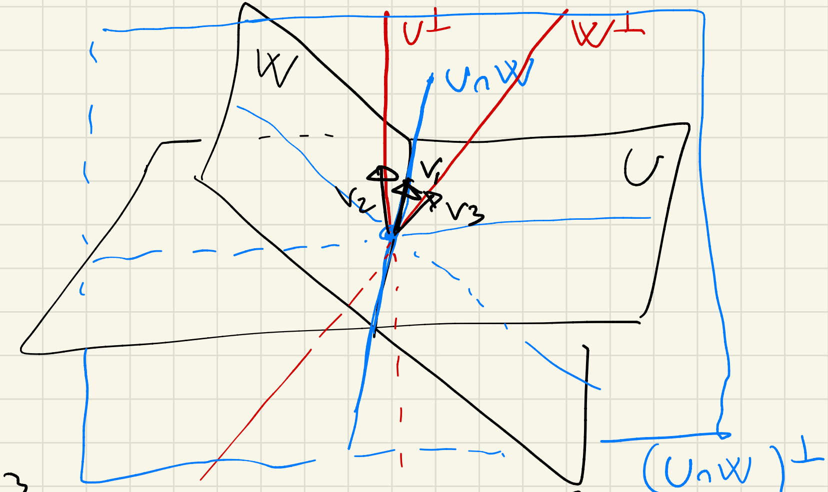
3) $v_1 \in U \cap W$

$v_2 \in U^\perp \qquad v_3 \in W^\perp$

non nulli $\Rightarrow$ sono base

$\{v_2, v_3\}$ base di $U^\perp + W^\perp = (U \cap W)^\perp$

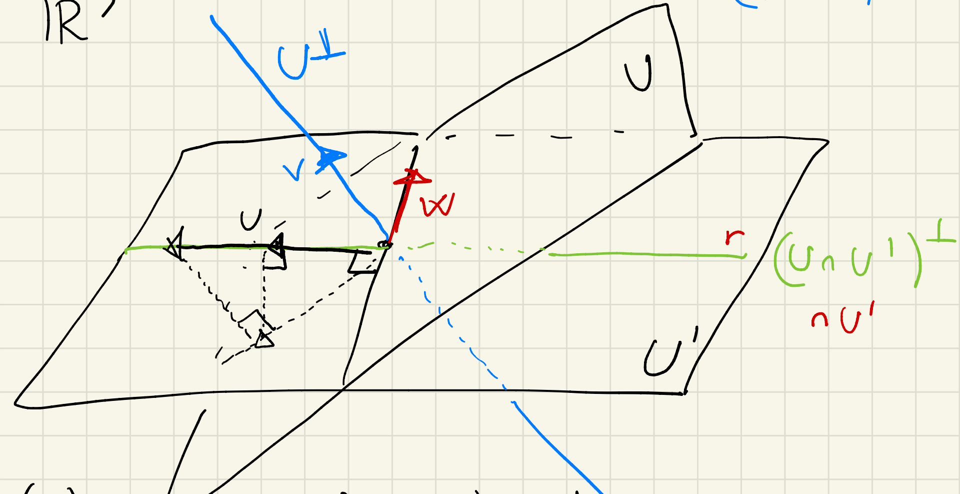
$\Rightarrow \{v_1, v_2, v_3\}$ base di $\mathbb{R}^3$



8.11

$f = P_{U'} \circ P_U$

$\lambda = 0, 1, \lambda_0 \in [0,1]$



$f(v) = P_{U'}(P_U(v)) = P_{U'}(0) = 0$ $\quad \lambda = 0$ trovato

$$f(w) = P_{U'}(P_U(w)) = P_{U'}(w) = w \qquad \lambda = 1 \text{ trovato}$$

$$u \in r \Rightarrow f(u) = \lambda_0 u \qquad 0 \le \lambda_0 \le 1$$

<u>8.7</u>: $U = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) \subseteq \mathbb{R}^4$

- Determina $U^\perp$

$$\begin{cases} w + x + y + z = 0 \\ w + y = 0 \end{cases} \xrightarrow{\text{risolvo}} \text{parametrica}$$

- base ortog. per U e per $U^\perp$

$$\underbrace{v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{w_1} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$v_2' = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad w_2$$

Base di $U^\perp$: $\begin{cases} w + x + y + z = 0 \\ w + y = 0 \end{cases}$

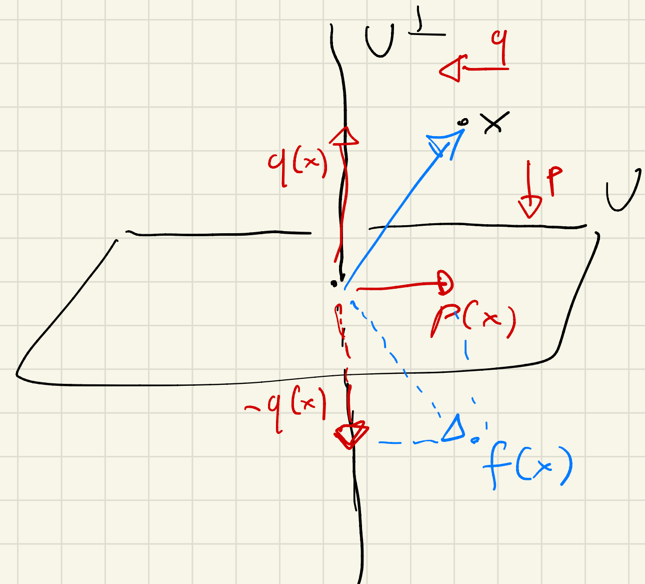
$$\begin{array}{ccc} w_3 & & w_4 \\ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} & , & \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \\ y=1 & & y=0 \\ z=0 & & z=1 \end{array}$$

sono venuti orthogonali

$$e_1 = \frac{\langle e_1, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle e_1, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \frac{\langle e_1, w_3 \rangle}{\langle w_3, w_3 \rangle} w_3 + \frac{\langle e_1, w_4 \rangle}{\langle w_4, w_4 \rangle} w_4$$

$$\frac{1}{4}$$

8.10. $U = \{4x - 3y + z = 0\}$



1) $[p]_{\mathcal{C}}^{\mathcal{C}}$ scrivere (già fatto)

2) $f(x) = p(x) - q(x)$ è diag.bile?

$$x = p(x) + q(x)$$

$f = r_U$ Le riflessioni sono sempre diagonalizzabili

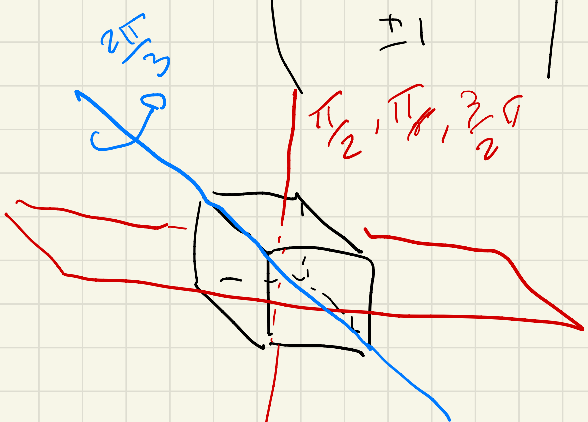
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

8.15: $A = \begin{pmatrix} \quad \end{pmatrix}$ I vettori unitari con coefficienti interi sono solo $\begin{pmatrix} \pm 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix}$

$$\left\| \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\|^2 = a^2 + b^2 + c^2 = 1 \qquad a, b, c \in \mathbb{Z}$$

$$\begin{pmatrix} \pm 1 & & \\ & \pm 1 & \\ & & \pm 1 \end{pmatrix} \quad \begin{pmatrix} \pm 1 & & \\ & & \pm 1 \\ & \pm 1 & \end{pmatrix} \quad \begin{pmatrix} & \pm 1 & \\ \pm 1 & & \\ & & \pm 1 \end{pmatrix}$$

$$\begin{pmatrix} & & \pm 1 \\ \pm 1 & & \\ & \pm 1 & \end{pmatrix} \quad \begin{pmatrix} & & \pm 1 \\ & \pm 1 & \\ \pm 1 & & \end{pmatrix} \quad \begin{pmatrix} & \pm 1 & \\ & & \pm 1 \\ \pm 1 & & \end{pmatrix}$$



$48 = 6 \times 8$ matrici.

24 rotazioni

8.17: $\mathbb{R}^4$

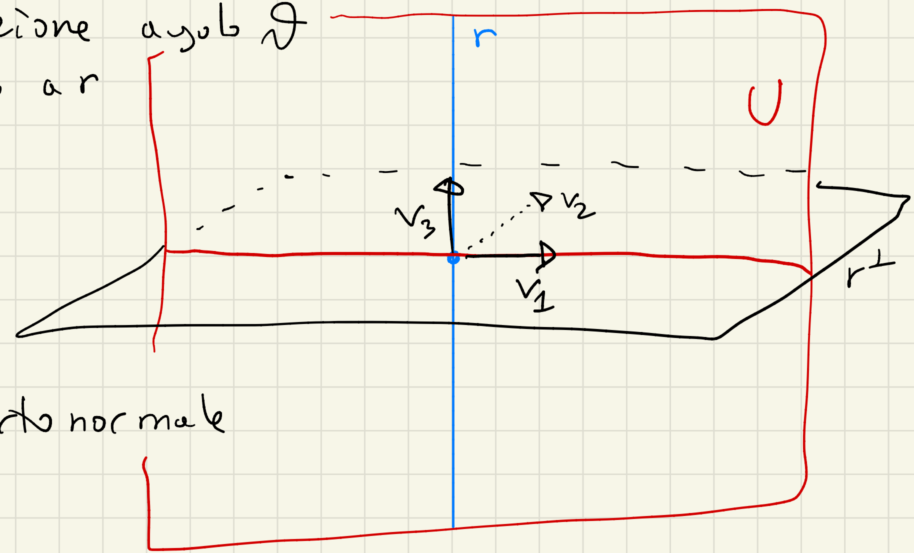
$\mathbb{R}^2$ $\quad Rot_\vartheta$ non ha autovettori se $\vartheta \neq 0, \pi$

$\mathbb{R}^4$ $\quad A = \left(\begin{array}{c|c} Rot_\vartheta & 0 \\ \hline 0 & Rot_\vartheta \end{array}\right)$ è ortogonale (eserciz.)

$A_1, \ldots, A_k$ ortogonali $\Rightarrow$ $\begin{pmatrix} \boxed{A_1} & & & 0 \\ & \boxed{A_2} & & \\ & & \ddots & \\ 0 & & & \boxed{A_k} \end{pmatrix}$ ortog.

$\mathbb{R}^{2n}$ $\quad A = \begin{pmatrix} \boxed{Rot_\vartheta} & & \\ & \ddots & \\ & & \boxed{Rot_\vartheta} \end{pmatrix}$

8.18: $R:\mathbb{R}^3 \to \mathbb{R}^3$ rotazione angolo ϑ intorno a r



$P = P_U$

$\boxed{P \circ R}$ è diag.bile?

Scelgo $\mathcal{B} = \{v_1, v_2, v_3\}$ ortonormale

$$[P]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[R]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta & 0 \\ \operatorname{sen}\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$[P \circ R]^{\mathcal{B}}_{\mathcal{B}} = [P]^{\mathcal{B}}_{\mathcal{B}} \cdot [R]^{\mathcal{B}}_{\mathcal{B}} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta & 0 \\ \operatorname{sen}\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$0, 1, \cos\vartheta$ è diag.bile se $\vartheta \neq 0, \pm\frac{\pi}{2}, \cancel{\pm\pi}$

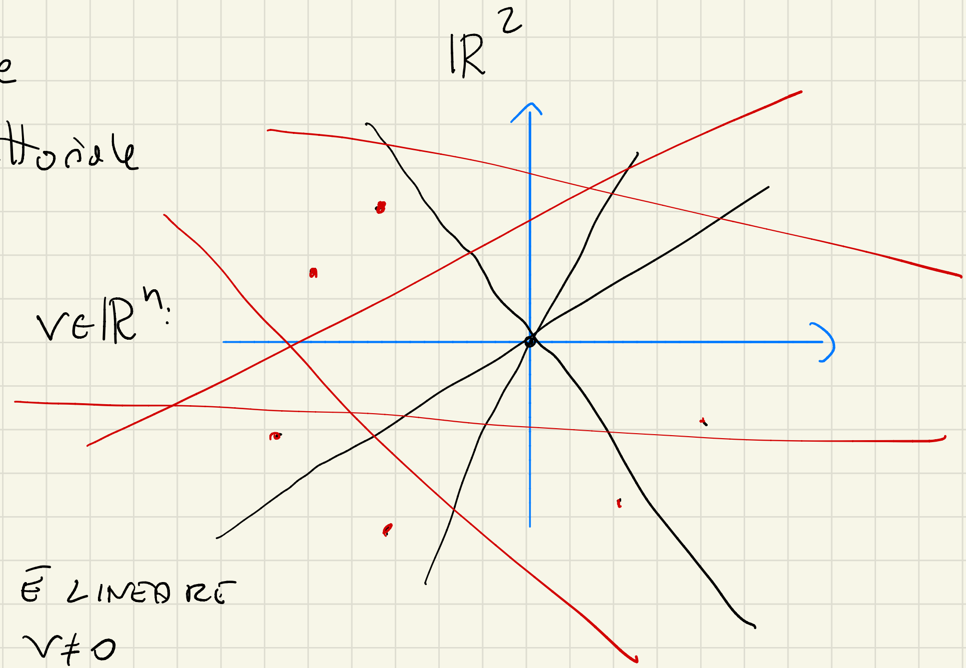
Se $\cos\vartheta = 0$: $\begin{pmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$ $\quad m_a(0) = 2$, $m_g(0) = 1$ $\Rightarrow$ non diag.

Se $\cos\vartheta = 1$: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ è diag.bile.

SOTTOSPAZI AFFINI

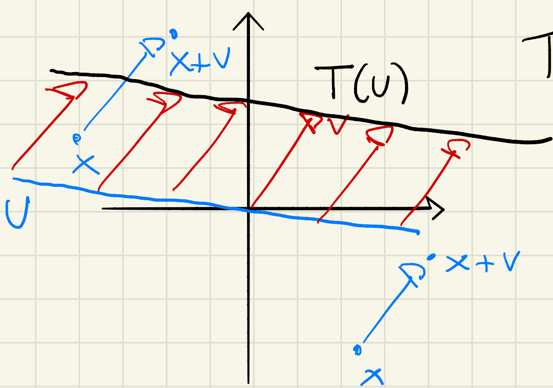
Def:

Un sottospazio affine si ottiene traslando un sottospazio vettoriale in $\mathbb{R}^n$



TRASLAZIONE di un vettore $v \in \mathbb{R}^n$:

$$T(x) = x + v$$



T NON È LINEARE se $v \neq 0$

$$T(0) = v$$

$$T(U) = \{x + v \mid x \in U\}$$

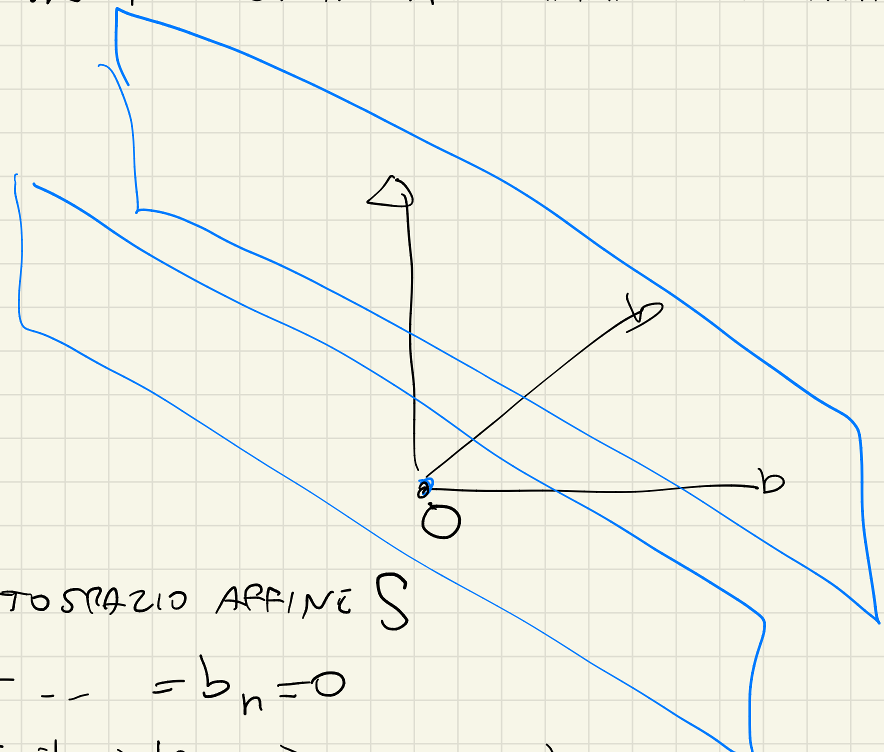
U è la GIACITURA di $T(U)$

I sottospazi affini si derivano in FORMA CARTESIANA o PARAMETRICA

CARTESIANA:

sistema lineare

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n = b_m \end{cases}$$



Le soluzioni formano un SOTTOSPAZIO AFFINE S

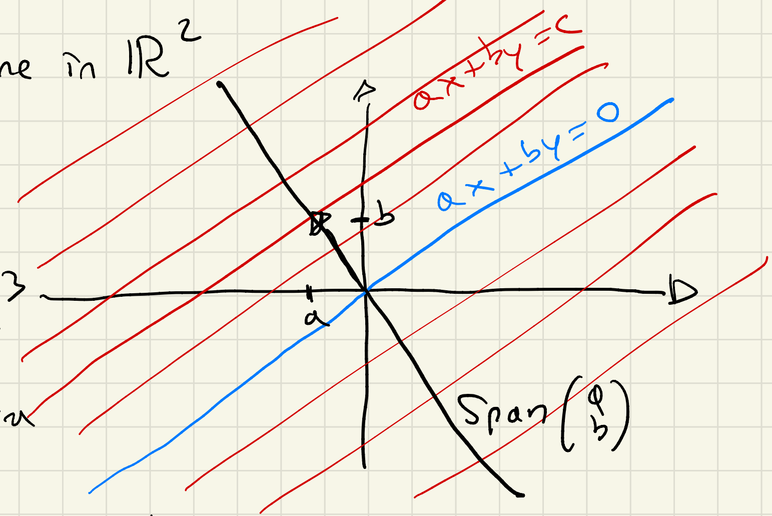
S contiene $O \iff b_1 = \dots = b_n = 0$

(cioè il sistema è omogeneo)

In generale, se pongo $b_1 = \dots = b_n = 0$ ottengo la GIACITURA di S

Esempio: $ax + by = c$ retta affine in $\mathbb{R}^2$

$ax + by = 0$



$ax + by + cz = d$ piano affine in $\mathbb{R}^3$

$ax + by + cz = 0$ è la sua giacitura

Cambiando i termini noti si ottengono altri sottospazi affini paralleli

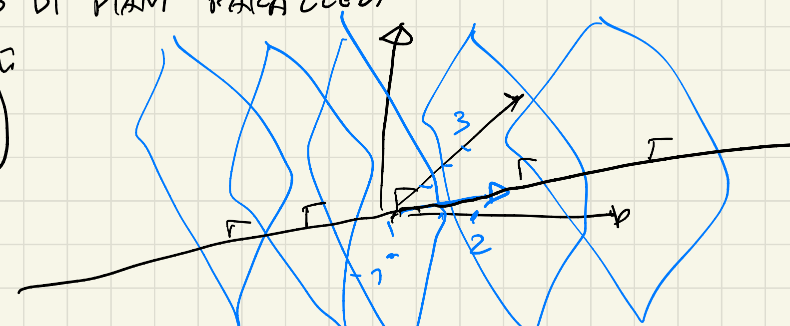
$2x + 3y - z = \cancel{5}\ 3$

$\{2x + 3y - z = k\}_{k \in \mathbb{R}}$

FASCIO DI PIANI PARALLELI

$2x + 3y - z = 0$ ortogonali a $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$k = 0$ $\quad 2x + 3y - z = 0$



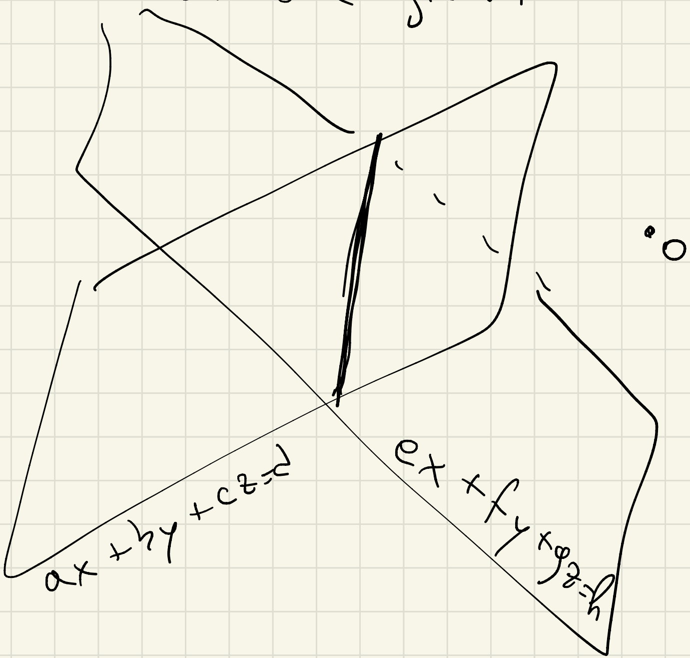
Una retta in $\mathbb{R}^3$

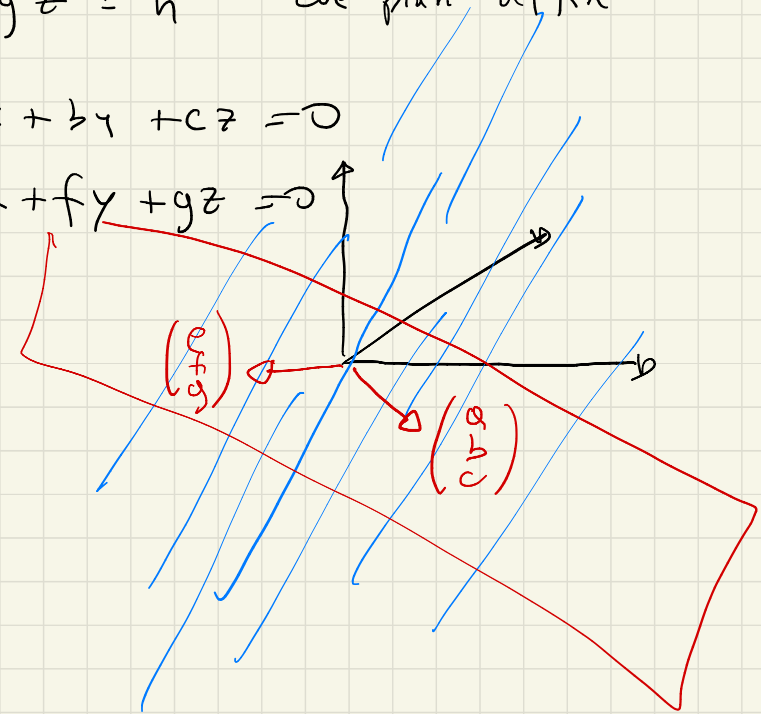
$$\begin{cases} ax + by + cz = d \\ ex + fy + gz = h \end{cases}$$

intersezione di due piani affini

La sua giacitura è

$$\begin{cases} ax + by + cz = 0 \\ ex + fy + gz = 0 \end{cases}$$





Forma parametrica

$U \subseteq \mathbb{R}^n$ sottosp. vett. $\qquad U = \mathrm{Span}(v_1, \dots, v_k)$

$$= \{ t_1 v_1 + \dots + t_k v_k \mid t_1, \dots, t_k \in \mathbb{R} \}$$

$S \subseteq \mathbb{R}^n$ sottospazio AFFINE

$$S = \{ P + t_1 v_1 + \dots + t_k v_k \mid t_1, \dots, t_k \in \mathbb{R} \}$$

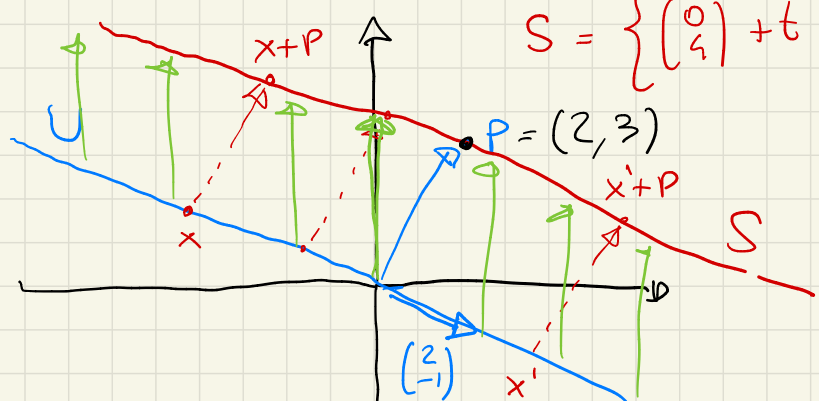
$$S = T(U) \qquad T(x) = x + P$$

Esempio:

$$U = \mathrm{Span}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

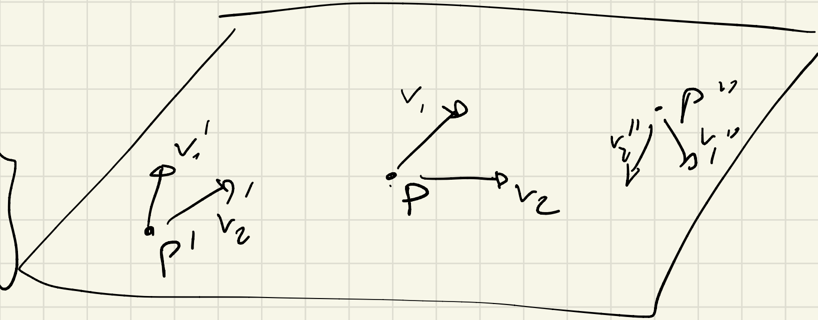
$$\boxed{S = \left\{ \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}}$$

$$S = \left\{ \begin{pmatrix} 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$



$U = \text{Span}\left(\begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\3\end{pmatrix}\right)$ piano vettoriale

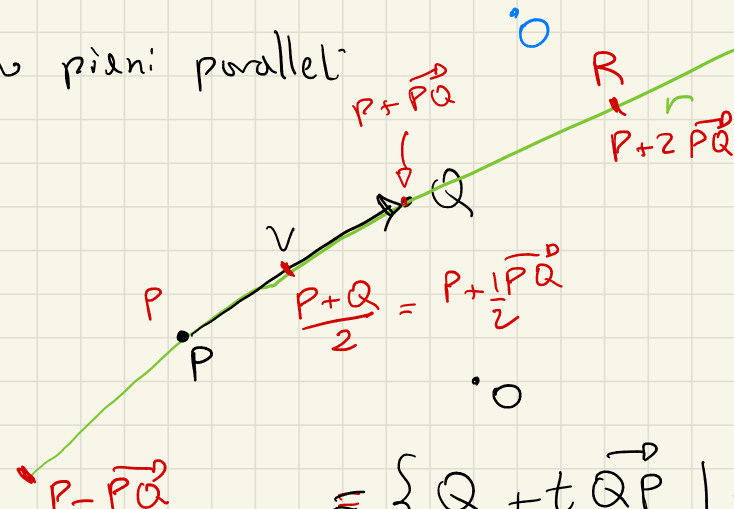
$$S = \left\{ \underset{P}{\begin{pmatrix}1\\0\\1\end{pmatrix}} + t\underset{v_1}{\begin{pmatrix}1\\-1\\1\end{pmatrix}} + u\underset{v_2}{\begin{pmatrix}2\\0\\3\end{pmatrix}} \right\}$$



Cambiando P si ottengono piani paralleli

⊙ Retta per due punti:

Forma parametrica



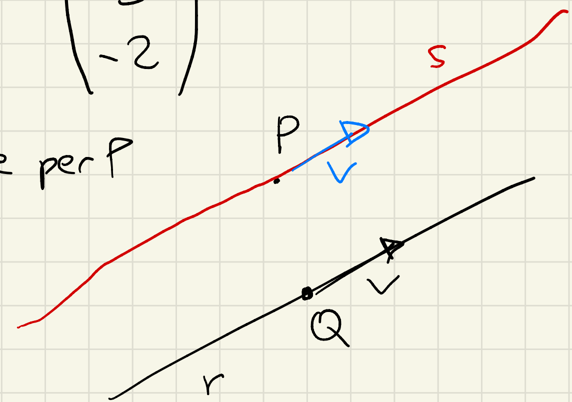
$$r = \{P + t\vec{PQ} \mid t \in \mathbb{R}\}$$

$$= \{Q + t\vec{QP} \mid t \in \mathbb{R}\}$$

Esempio: $P=\begin{pmatrix}1\\0\\7\end{pmatrix}$ $\quad Q=\begin{pmatrix}-2\\3\\5\end{pmatrix}$ $\quad r=\left\{\begin{pmatrix}1\\0\\7\end{pmatrix}+t\begin{pmatrix}-3\\3\\-2\end{pmatrix}\,\middle|\, t\in\mathbb{R}\right\}$

$$\vec{PQ}=Q-P=\begin{pmatrix}-3\\3\\-2\end{pmatrix}$$

- Retta parallela a r passante per P

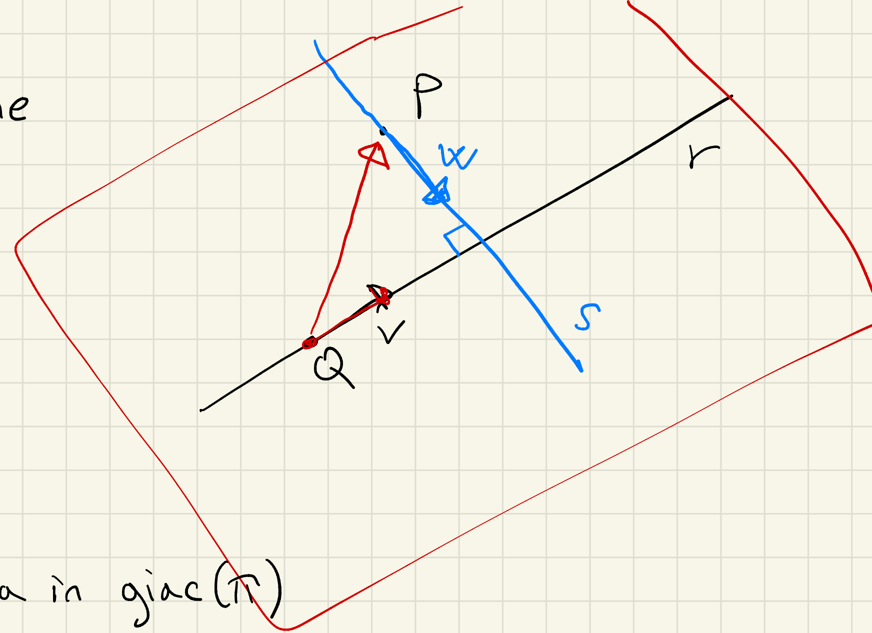


$$r=\{Q+tv\}$$

$$s=\{P+tv\}$$

- Retta perpendicolare a r passante per P

1) Trovo il piano π che contiene sia P che r



$$r = \{Q + tv\}$$

$$\pi = \{Q + tv + u\overrightarrow{QP}\}$$

Cerco w direzione contenuta in giac(π) perpendicolare a v

$$P = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad r = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad P \notin r$$

$$\pi = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \text{giac}\,\pi = \left\{ t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= \text{Span}\left(\underset{v}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \right)$$

$v = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ Cerco $w \in$ giac π ortogonale a v

G.S. $\Rightarrow$ $v_2 \longrightarrow w$ ortog. a v

$$s = \{ P + tw \} = \left\{ \underset{P}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} + t \underset{w}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \right\}$$

uguali per caso