ESERCIZI SU AFFINITÀ

Titolo nota 23/05/2018

4.13 f, g rotazioni di $\pi/2$ intorno a 2 punti del piano P, Q. Dimostrare che $f \circ g$ è una riflessione

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} x - x_P \\ y - y_P \end{pmatrix} + \begin{pmatrix} x_P \\ y_P \end{pmatrix} = \begin{pmatrix} -y + x_P + y_P \\ x - x_P + y_P \end{pmatrix}$$

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}\begin{pmatrix} x - x_Q \\ y - y_Q \end{pmatrix} + \begin{pmatrix} x_Q \\ y_Q \end{pmatrix} = \begin{pmatrix} -y + x_Q + y_Q \\ x - x_Q + y_Q \end{pmatrix}$$

$$(f \circ g)\begin{pmatrix} x \\ y \end{pmatrix} = f\begin{pmatrix} -y + x_Q + y_Q \\ x - x_Q + y_Q \end{pmatrix} = \begin{pmatrix} -x + x_Q - y_Q + x_P + y_P \\ -y + x_Q + y_Q - x_P + y_P \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_P + x_Q + y_P - y_Q \\ -x_P + x_Q + y_P + y_Q \end{pmatrix}} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\frac{x + x'}{2} = x_R \qquad \frac{y + y'}{2} = y_R$$

$$x' = 2x_R - x \qquad y' = 2y_R - y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \boxed{\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2x_R \\ 2y_R \end{pmatrix}}$$

$$x_R = \frac{x_P + x_Q + y_P - y_Q}{2} \qquad y_R = \frac{-x_P + x_Q + y_P + y_Q}{2}$$

In questo modo abbiamo anche determinato R

$$f\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & b \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad g\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A'} & b' \\ \hline 0 & 0 & 1 \end{array}\right)$$

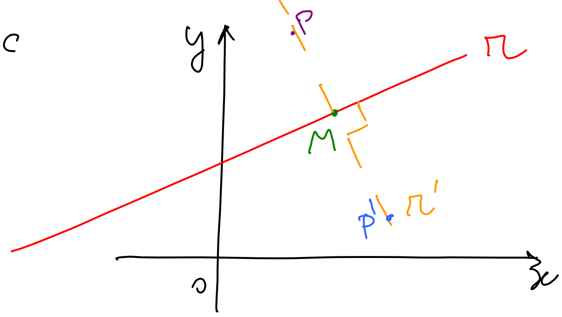
$$(f \circ g)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & b \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} \multicolumn{2}{c|}{A'} & b' \\ \hline 0 & 0 & 1 \end{array}\right)\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{AA'} & ? \\ \hline 0 & 0 & 1 \end{array}\right)$$

$$A = A' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad AA' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

simmetria rispetto a 0
oppure rotazione di π

4.14 $\quad r: ax+by+c$



$r' \perp r$ e passante per P

$M = r \cap r' \quad (\exists !)$

$P' \in r'$ tale che $MP' = PM$

$r': \; b(x-x_P) - a(y-y_P) = 0$

$r \cap r' = \begin{cases} ax+by = -c \\ -bx+ay = ay_P - bx_P \end{cases} \qquad \Delta = a^2+b^2 \neq 0$

$$x_M = \frac{1}{\Delta}\begin{vmatrix} -c & b \\ ay_P - bx_P & a \end{vmatrix} = \frac{-ac - aby_P + b^2 x_P}{a^2+b^2}$$

$$y_M = \frac{1}{\Delta}\begin{vmatrix} a & -c \\ -b & ay_P - bx_P \end{vmatrix} = \frac{-bc - abx_P + a^2 y_P}{a^2+b^2}$$

$$x_{P'} = 2x_M - x_P = \frac{(-a^2+b^2)x_P - 2aby_P - 2ac}{a^2+b^2}$$

$$y_{P'} = 2y_M - y_P = \frac{-2abx_P + (a^2-b^2)y_P - 2bc}{a^2+b^2}$$

$$\begin{pmatrix} x_{P'} \\ y_{P'} \end{pmatrix} = \frac{1}{a^2+b^2}\left[\begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix}\begin{pmatrix} x_P \\ y_P \end{pmatrix} - 2c\begin{pmatrix} a \\ b \end{pmatrix}\right] \qquad R_r$$

$s: \; a'x+b'y+c'=0$

$$\begin{pmatrix} x_{P'} \\ y_{P'} \end{pmatrix} = \frac{1}{a'^2+b'^2}\left[\begin{pmatrix} b'^2-a'^2 & -2a'b' \\ -2a'b' & a'^2-b'^2 \end{pmatrix}\begin{pmatrix} x_P \\ y_P \end{pmatrix} - 2c'\begin{pmatrix} a' \\ b' \end{pmatrix}\right] \qquad R_s$$

Se $\begin{pmatrix} a' \\ b' \end{pmatrix} = k\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow$ il prodotto delle matrici dà $\mathbb{1}$

altrimenti dà una matrice di rotazione "vera"

$$\frac{1}{(a^2+b^2)(a'^2+b'^2)}\begin{pmatrix} b^2-a^2 & -2ab \\ -2ab & a^2-b^2 \end{pmatrix}\begin{pmatrix} b'^2-a'^2 & -2a'b' \\ -2a'b' & a'^2-b'^2 \end{pmatrix} =$$

$$\frac{1}{\Delta\Delta'}\begin{pmatrix} (b^2-a^2)(b'^2-a'^2)+4aa'bb' & -2a'b'(b^2-a^2)-2ab(a'^2-b'^2) \\ -2ab(b'^2-a'^2)-2a'b'(a^2-b^2) & +4aa'bb'+(a^2-b^2)(a'^2-b'^2) \end{pmatrix}$$

$$= \frac{1}{\Delta\Delta'}\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

$$A = (b^2-a^2)(b'^2-a'^2) + 4aa'bb'$$

$$B = 2[ab(b'^2-a'^2) + a'b'(a^2-b^2)]$$

Va dimostrato svolgendo il calcolo

$$\boxed{\frac{A^2+B^2}{(\Delta\Delta')^2} = 1} \qquad \cos\Phi = \frac{A}{\Delta\Delta'} \qquad \operatorname{sen}\Phi = \frac{B}{\Delta\Delta'}$$

Se $a' = ka$ e $b' = kb$ $\qquad A = k^2(b^2-a^2)^2 + 4k^2a^2b^2$

$$B = 2[ab(b^2-a^2) + ab(a^2-b^2)] = 0$$

$$A = k^2[b^4+a^4-2a^2b^2+4a^2b^2] = k^2(a^2+b^2)^2 = k^2\Delta^2$$

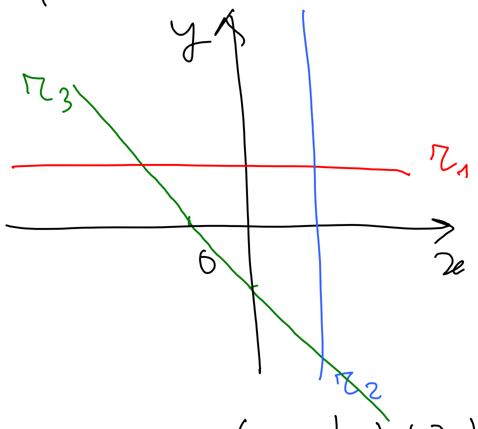
$$\cos\phi = \frac{k^2\Delta^2}{\Delta\cdot(k^2\Delta)} = 1$$

Il prodotto è $\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

4.15 $\quad f(x) = Ax+b$ di $\mathbb{R}^2$ $\qquad r_1 = \{y=1\}$

$f(r_1)=r_2 \qquad f(r_2)=r_3 \qquad f(r_3)=r_1 \qquad r_2\ \{x=1\}$

$r_3 = \{x+y=-1\}$



$r_1: \begin{cases}x=t\\ y=1\end{cases} \qquad r_2: \begin{cases}x=1 & t\in\mathbb{R}\\ y=u & u\in\mathbb{R}\end{cases}$

$\forall t\in\mathbb{R}, \exists u\in\mathbb{R}$ tale che

$f\begin{pmatrix}t\\1\end{pmatrix} = \begin{pmatrix}1\\u\end{pmatrix} \qquad f(r_1)=r_2$

$$f\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}e\\f\end{pmatrix} \qquad \begin{cases}a\cdot t+b\cdot 1+e=1\\ c\cdot t+d\cdot 1+f=u\end{cases}$$

a,b,c,d,e,f costanti (per ora incognite) $\qquad u = ct+d+f$

$at+(b+e-1)=0\ \forall t \Rightarrow \boxed{a=0,\ b+e=1}$

$\forall u$ devo trovare s tale che valga

$r_3: \begin{cases}x=s\\ y=-1-s\end{cases}$ $s\in\mathbb{R}$ $\qquad f\begin{pmatrix}1\\u\end{pmatrix} = \begin{pmatrix}s\\-1-s\end{pmatrix}$

$\begin{cases}a\cdot1+b\cdot u+e=s\\ c\cdot1+d\cdot u+f=-1-s\end{cases} \qquad c+du+f = -1-a-bu-e \quad \forall u$

$(d+b)u+c+f+a+e+1=0\ \forall u \qquad \boxed{\begin{cases}d+b=0\\ c+f+a+e+1=0\end{cases}}$

3ª $\begin{cases}as+b(-1-s)+e=t\\ cs+d(-1-s)+f=1\end{cases}$ $\quad \forall s$ devo trovare t

$(c-d)s-d+f-1=0\ \forall s \qquad \boxed{c-d=0,\ f-d=1}$

$$\begin{cases} a=0 \\ b+e=1 \\ b+d=0 \\ c+f+a+e=-1 \\ c-d=0 \\ f-d=1 \end{cases} \qquad \begin{cases} a=0 \\ c=d \\ f=1+d \\ b=-d \\ e=1+d \\ d+1+d+0+1+d=-1 \end{cases} \qquad \begin{cases} a=0 \\ b=1 \\ c=-1 \\ d=-1 \\ e=0 \\ f=0 \end{cases}$$

$$3d=-3 \rightarrow d=-1$$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \cancel{+\begin{pmatrix} 0 \\ 0 \end{pmatrix}} = \begin{pmatrix} y \\ -x-y \end{pmatrix}$$

$$f\begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1-t \end{pmatrix} \quad u \qquad f\begin{pmatrix} 1 \\ u \end{pmatrix} = \begin{pmatrix} u \\ -1-u \end{pmatrix} \quad =s$$

$\in r_1$ $\quad \in r_2$ $\qquad \in r_2$ $\quad \in r_3$

$$f\begin{pmatrix} s \\ -1-s \end{pmatrix} = \begin{pmatrix} -1-s \\ -s+1+s \end{pmatrix} = \begin{pmatrix} -1-s \\ 1 \end{pmatrix} \quad t$$

$\in r_3$ $\qquad \in r_1$

4.16 $\quad f(x) = Ax+b \quad$ isometria senza punti fissi tale che $f(r)=r'$

$r=\{y=x\} \quad r'=\{y=-x\}$

$P\in r \quad P=\begin{pmatrix} t \\ t \end{pmatrix} \qquad P'\in r' \quad P'=\begin{pmatrix} u \\ -u \end{pmatrix}$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

$\forall t\in\mathbb{R},\ \exists u\in\mathbb{R}$ tale che $\begin{cases} at+bt+e=u \\ ct+dt+f=-u \end{cases}$

$(a+b+c+d)t+(e+f)=0 \ \forall t \implies \begin{matrix} e+f=0 \\ a+b+c+d=0 \end{matrix}$

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & -(a+b+c) \end{pmatrix}}_{\det=\pm1} + \begin{pmatrix} e \\ -e \end{pmatrix}$$

non basta

$-a(a+b+c)-bc=\pm 1$ (ISOMETRIA)

F punto fisso di f $\qquad f\begin{pmatrix} x_F \\ y_F \end{pmatrix} = \begin{pmatrix} x_F \\ y_F \end{pmatrix}$

$\begin{cases} ax_F+by_F+e=x_F \\ cx_F+dy_F+f=y_F \end{cases}$ NON DEVE AVERE SOLUZ. se non vogliamo avere punti fissi

$$\begin{cases} (a-1)x_F + b y_F = -e \\ c\, x_F + (d-1) y_F = -f \end{cases}$$

1) $rk\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} = 1 \qquad rk\begin{pmatrix} a-1 & b & -e \\ c & d-1 & -f \end{pmatrix} = 2$

2) $rk\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} = 0 \qquad rk\begin{pmatrix} a-1 & b & -e \\ c & d-1 & -f \end{pmatrix} = 1$

IMPOSSIBILE

1) $(a-1)(d-1) - bc = 0 \qquad \det\begin{pmatrix} a-1 & -e \\ c & -f \end{pmatrix} \neq 0$

$\wedge$

$d = -(a+b+c)$

$$\begin{cases} -(a-1)(a+b+c+1) - bc = 0 \\ -a(a+b+c) - bc = \pm 1 \\ (a-1)e + ce \neq 0 \end{cases}$$

$e \neq 0$

$a+c \neq 1$

$b+c = -2$

$c = -2-b$

$$\begin{cases} a(a+b+c) + bc = -1 \\ (a-1)(a+b+c+1) + bc = 0 \end{cases}$$

$\cancel{a^2} + \cancel{ab} + \cancel{ac} - \cancel{a^2} - \cancel{ab} - \cancel{ac} - \cancel{a} + \cancel{a} + b + c + 1 = -1$

$a(a-2) + b(-2-b) = -1$

$a^2 - 2a - b^2 - 2b = -1$

$(a+b)(a-b) - 2(a+b) = -1$

$(a+b)(a-b-2) = -1$

$$\begin{cases} a+b = 1 \\ a-b = 1 \end{cases}$$

$a = 1$

$b = 0$

$c = -2$

$d = 1$

$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & -(a+b+c) \end{pmatrix}\begin{pmatrix} a & c \\ b & -(a+b+c) \end{pmatrix} = \mathbb{1}$

$$\begin{cases} a^2 + b^2 = 1 \\ c^2 + (a+b+c)^2 = 1 \longrightarrow c^2 + 1 + c^2 + 2ab + 2ac + 2bc = 1 \\ ac - b(a+b+c) = 0 \end{cases}$$

$c^2 + ab + \boxed{ac + bc} = 0$

$\boxed{ac} - ab - b^2 \boxed{+bc} = 0$

$ab + ac + bc = -c^2$

$-b^2-c^2-ab-ab=0$

$$\begin{cases} b^2+c^2=-2ab \\ a^2+b^2=1 \\ ab+ac+bc=-c^2 \\ (a-1)(a+b+c+1)+bc=0 \end{cases}$$

$b=\operatorname{sen}\alpha$

$c=-\operatorname{sen}\alpha$

$-(\cos\alpha)=\cos\alpha$ $\quad\cos\alpha=0$

$\alpha=\pi/2$ $\quad$ o $3\pi/2$

$a=\cos\alpha$

$-(a+b+c)=\cos\alpha$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{cases} -x_F+y_F=-e \\ -x_F-y_F=e \end{cases}$$

$(a-1)(a+b+c+1)+bc=0$

$-1\cdot 1+(-1)=-2\neq 0$

$$\frac{1}{\Delta^2}\begin{pmatrix} \beta^2-\alpha^2 & -2\alpha\beta \\ -2\alpha\beta & \alpha^2-\beta^2 \end{pmatrix}$$

$a+b+c+d=0 \Rightarrow -4\alpha\beta=0$ $\quad\alpha=0$ oppure $\beta=0$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

$(a-1)(a+b+c+1)+bc=$

$0\,(2)-0=0$

$-2\cdot 0=0$

Vanno bene entrambe!

$$\begin{cases} \cancel{x_F}+e=\cancel{x_F} \quad e\neq 0 \\ -y_F+f=y_F \end{cases}$$

non deve avere soluzioni

$$f\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} 1 \\ -1 \end{pmatrix}=\begin{pmatrix} x+1 \\ -y-1 \end{pmatrix}$$

$x=t$

$y=t$

$\begin{pmatrix} t \\ t \end{pmatrix} \to \begin{pmatrix} t+1 \\ -(t+1) \end{pmatrix}$

$u=t+1$ $\quad$ OK

$x=x+1$ impossibile! Non ci sono punti fissi!

4.17 rotaz. d'asse

$$r: \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{tale che } f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha & 0 \\ \operatorname{sen}\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x-1 \\ y+1 \\ z-2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

$$= \begin{pmatrix} (x-1)\cos\alpha - (y+1)\operatorname{sen}\alpha + 1 \\ (x-1)\operatorname{sen}\alpha + (y+1)\cos\alpha - 1 \\ z+b \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\operatorname{sen}\alpha & 0 \\ \operatorname{sen}\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 1-\cos\alpha-\operatorname{sen}\alpha \\ -1-\operatorname{sen}\alpha+\cos\alpha \\ b \end{pmatrix}$$

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1-\cos\alpha-\operatorname{sen}\alpha \\ -1-\operatorname{sen}\alpha+\cos\alpha \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$$

$$\begin{cases} \cos\alpha + \operatorname{sen}\alpha = -1 \\ \cos\alpha - \operatorname{sen}\alpha = -1 \\ b = 1 \end{cases} \qquad \begin{array}{l} \cos\alpha = -1 \\ \operatorname{sen}\alpha = 0 \\ b = 1 \end{array} \qquad \alpha = \pi$$

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ +1 \end{pmatrix}$$