ESERCIZI SU JORDAN

Titolo nota 21/03/2017

2.3 2 matrici A e $B \in M(2)$ stesso p diverso m

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$m(\lambda) = \lambda - 1$

$A - I = 0$

$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad p(\lambda) = (\lambda-1)^2$

$m(\lambda) = p(\lambda) = (\lambda-1)^2$

$(B-I)^2 = 0 \qquad B^2 = 2B - I$

$B^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =$

$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \longleftarrow$ OK!

$2B - I = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

2.4 $\begin{pmatrix} 10 & 4 \\ -5 & 10 \end{pmatrix}$

$\det\begin{pmatrix} 10-\lambda & 4 \\ -5 & 10-\lambda \end{pmatrix} = \lambda^2 - 20\lambda + 120$

$\lambda_1 = 10 - 2i\sqrt{5} \quad \lambda_2 = 10 + 2i\sqrt{5}$

complessi e distinti

in $\mathbb{R}$ non si fa niente, in $\mathbb{C}$ si diagonalizza "normalmente"

$\begin{pmatrix} 5 & -4 \\ 9 & -7 \end{pmatrix} \qquad \det\begin{pmatrix} 5-\lambda & -4 \\ 9 & -7-\lambda \end{pmatrix} = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2$

$\lambda_1 = \lambda_2 = -1$

$ma(-1) = 2 \qquad mg(-1) = 2 - rk\begin{pmatrix} 6 & -4 \\ 9 & -6 \end{pmatrix} = 1 \qquad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ non si diagonalizza

$(A+I)^2 = 0 \qquad A^2 = -2A - I \qquad A^4 = (A^2)^2 =$

$= (-2A - I)^2 = 4A^2 + I + 4A = 4(-2A - I) + I + 4A$

$= -4A - 3I \qquad A^3 = +3A + 2I$

$A^3 = A \cdot A^2 = A \cdot (-2A - I) = -2A^2 - A = 4A + 2I - A$
$= 3A + 2I$

$A^n = (-1)^{n+1}[nA + (n-1)I]$ dim. per induzione

$\begin{pmatrix} 4 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 4 \end{pmatrix}$ Si vedono "al volo" $\lambda_1 = 1$, $\lambda_2 = 4$

$\lambda_3 = 4$ per motivi di traccia

$$ma(4) = 2 \qquad mg(4) = 3 - rk\begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 3 \\ 5 & 0 & 0 \end{pmatrix} = 3-2 = 1$$

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 4 & 1 \\ 0 & 0 & 4 \end{array}\right)$$

$$m_0 = 0$$
$$m_1 = 1$$
$$m_2 = 3 - rk\begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 3 \\ 5 & 0 & 0 \end{pmatrix}^2 = 3-1 = 2$$
$$m_3 = 2$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 3 \\ 5 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 2 & -3 & 3 \\ 5 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 9 & +9 & -9 \\ 0 & 0 & 0 \end{pmatrix}$$

1 blocco di taglia ≥ 1

1 blocco di taglia ≥ 2

$$\begin{matrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{matrix}$$

$$\det\begin{pmatrix} 5-\lambda & 4 & 3 \\ -1 & -\lambda & -3 \\ 1 & -2 & 1-\lambda \end{pmatrix} = \det\begin{pmatrix} 5-\lambda & 4 & 3 \\ 0 & -2-\lambda & -2-\lambda \\ 1 & -2 & 1-\lambda \end{pmatrix}$$

$$= \det\begin{pmatrix} 5-\lambda & 1 & 3 \\ 0 & 0 & -2-\lambda \\ 1 & -3+\lambda & 1-\lambda \end{pmatrix} = (2+\lambda)\det\begin{pmatrix} 5-\lambda & 1 \\ 1 & -3+\lambda \end{pmatrix}$$

$$= (2+\lambda)(-\lambda^2+8\lambda-16) = -(2+\lambda)(\lambda-4)^2$$

$$\lambda_1 = -2 \qquad \lambda_2 = \lambda_3 = 4$$

$$ma(4) = 2$$
$$mg(4) = 3 - rk\begin{pmatrix} 1 & \boxed{4 \quad 3} \\ 2 & \boxed{-3 \quad 3} \\ 5 & 0 \quad 0 \end{pmatrix} = 3-2 = 1$$

$$\left(\begin{array}{c|cc} -2 & 0 & 0 \\ \hline 0 & 4 & 1 \\ 0 & 0 & 4 \end{array}\right)$$

$$\begin{pmatrix} 9 & 7 & 3 \\ -9 & -7 & -4 \\ 4 & 4 & 4 \end{pmatrix}$$

$$\det\begin{pmatrix} 9-\lambda & 7 & 3 \\ -9 & -7-\lambda & -4 \\ 4 & 4 & 4-\lambda \end{pmatrix} = \det\begin{pmatrix} 2-\lambda & 7 & 3 \\ -2+\lambda & -7-\lambda & -4 \\ 0 & 4 & 4-\lambda \end{pmatrix} =$$

$$= \det\begin{pmatrix} 2-\lambda & 7 & 3 \\ 0 & -\lambda & -1 \\ 0 & 4 & 4-\lambda \end{pmatrix} = (2-\lambda)\det\begin{pmatrix} -\lambda & -1 \\ 4 & 4-\lambda \end{pmatrix} =$$

$$= (2-\lambda)(\lambda^2-4\lambda+4) = (2-\lambda)^3$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 2$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$ma(2) = 3$$

$$m_0 = 0$$
$$m_1 = 3 - rk\begin{pmatrix} 7 & 7 & 3 \\ -9 & \boxed{-9 \quad -4} \\ 4 & \boxed{4 \quad 2} \end{pmatrix} = 1$$
$$m_2 = 2$$
$$m_3 = 3 \qquad m_4 = 3$$

$$m_2 = 3 - rk \begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & 4 \\ 4 & 4 & 2 \end{pmatrix}^2 = 3-1 = 2$$

$$\begin{array}{r} -63 \\ +81 \\ -16 \\ \hline +2 \end{array}$$

$$\begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & -4 \\ 4 & 4 & 2 \end{pmatrix}\begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & -4 \\ 4 & 4 & 2 \end{pmatrix} = \begin{pmatrix} -2 & -2 & -1 \\ 2 & 2 & +1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$m_3 = 3 - rk \begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & -4 \\ 4 & 4 & 2 \end{pmatrix}^3 = 3-0=3 \qquad \begin{pmatrix} 7 & 7 & 3 \\ -9 & -9 & -4 \\ 4 & 4 & 2 \end{pmatrix}\begin{pmatrix} -2 & -2 & -1 \\ 2 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$T(e_i) = \sum_{k=1}^{i} a_{ik} e_k$$

$$\begin{pmatrix} 2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{pmatrix}$$

$$\det \begin{pmatrix} 2-\lambda & 2 & -1 \\ -1 & -1-\lambda & 1 \\ -1 & -2 & 2-\lambda \end{pmatrix} = \det \begin{pmatrix} 1-\lambda & 1-\lambda & 0 \\ -1 & -1-\lambda & 1 \\ -1 & -2 & 2-\lambda \end{pmatrix} =$$

$$\det \begin{pmatrix} 1-\lambda & 0 & 0 \\ -1 & -\lambda & 1 \\ -1 & -1 & 2-\lambda \end{pmatrix} = (1-\lambda)\det \begin{pmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} =$$

$$(1-\lambda)(\lambda^2-2\lambda+1) = (1-\lambda)^3$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1$$

$$ma(1) = 3$$

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

$$mg(1) = 3 - rk \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \\ -1 & -2 & 1 \end{pmatrix} = 3-1=2$$

$$(\lambda-1)^2 \qquad A^2 = 2A - I$$

2.5 $\quad p(\lambda) = (\lambda-1)^2(\lambda+2)^3 \qquad \lambda_1 = \lambda_2 = 1$

$$\lambda_3 = \lambda_4 = \lambda_5 = -2$$

2 possibilità

$$\left(\begin{array}{cc|ccc} \boxed{\begin{matrix} 1 & ? \\ 0 & 1 \end{matrix}} & & & 0 & \\ \hline & & & & \\ & 0 & & \boxed{\begin{matrix} -2 & ? & 0 \\ 0 & -2 & ? \\ 0 & 0 & -2 \end{matrix}} & \end{array}\right)$$

3 possibilità

$2 \times 3 = 6$ tipi possibili (forme)

$$m(\lambda) = (\lambda-1)^1(\lambda+2)^2$$

Quale forma di J, corrisponde a $m(A)$?

$$\begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & -2 & 0 & 0 \\ & & 0 & -2 & +1 \\ & & 0 & 0 & -2 \end{pmatrix}$$

2,8 matrice 4×4

λ_0 unico autovalore $ma(\lambda_0)=4$

Trovare tutte le possibili forme di J.

$$\begin{pmatrix} \lambda_0 & & & ? \\ 0 & \lambda_0 & & \\ 0 & 0 & \lambda_0 & \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix}$$

- 4 blocchi da 1
- 2 blocchi da 1 e 1 da 2
- 1 blocco da 1 e 1 da 3
- 2 blocchi da 2
- 1 blocco da 4

sono 5

$$\begin{pmatrix} \lambda_0 & & & \\ & \lambda_0 & & \\ & & \lambda_0 & \\ & & & \lambda_0 \end{pmatrix}$$

$m(\lambda)=\lambda-\lambda_0$

$$\begin{pmatrix} \lambda_0 & 0 & & \\ 0 & \lambda_0 & & \\ & & \lambda_0 & 1 \\ & & 0 & \lambda_0 \end{pmatrix}$$

$m(\lambda)=(\lambda-\lambda_0)^2$

$$\begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix}$$

$m(\lambda)=(\lambda_0-\lambda)^3$

$$\begin{pmatrix} \lambda_0 & 1 & & \\ 0 & \lambda_0 & & \\ & & \lambda_0 & 1 \\ & & 0 & \lambda_0 \end{pmatrix}$$

$m(\lambda)=(\lambda-\lambda_0)^2$

$$\begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ & \lambda_0 & 1 & 0 \\ & & \lambda_0 & 1 \\ & & & \lambda_0 \end{pmatrix}$$

$m(\lambda)=(\lambda-\lambda_0)^4$

2,9 A matrice 7×7 $(A-I)^3=0$

$rk(A-I)^2=2$

J_A

l'unico λ è 1

$m_A=(x-1)^d$ $d=1,2,3$

$m_0=0$

$m_1=?$

$m_2=7-2=5$

$m_3=7$

$m_3 - m_2 = 2$

$m_2 - m_1 = 5 - r$

$m_1 - m_0 = r$

n° di blocchi con dim ≥ 3 è 2

n° di blocchi con dim ≥ 2 è $5-r$

n° di blocchi con dim ≥ 1 è r

$r=1$	$r=2$	$r=3$
~~2~~	~~2~~	2
~~4~~	~~3~~	2
~~1~~	~~2~~	3

3 3 1

3 2 2 non può succedere

2.10 tutte le matrici complesse 2×2 nilpotenti

Brutale $\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} = \begin{cases} z_1^2 + z_2 z_3 = 0 \\ z_1 z_2 + z_2 z_4 = 0 \\ z_1 z_3 + z_3 z_4 = 0 \\ z_2 z_3 + z_4^2 = 0 \end{cases} \qquad \begin{cases} z_1^2 + z_2 z_3 = 0 \\ z_2 (z_1 + z_4) = 0 \\ z_3 (z_1 + z_4) = 0 \\ z_4^2 + z_2 z_3 = 0 \end{cases}$$

$\begin{cases} z_1 = -z_4 \end{cases}$ $\quad z_1^2 + z_2 z_3 = 0 \quad z_2 \neq 0$

$$\boxed{\begin{pmatrix} z_1 & z_2 \\ \frac{z_1^2}{z_2} & -z_1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ z_3 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & z_2 \\ 0 & 0 \end{pmatrix}}$$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad z_2 = 0 \quad z_1 = 0$

$\begin{cases} z_3 z_4 = 0 \\ z_4^2 = 0 \end{cases}$

$$\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} \qquad \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}\overset{M}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = (\text{generica})$$

2.12 $\quad A^2-2A+I=0 \quad (*)$

A è diagonalizzabile se e solo se $A=I$

I soddisfa $I^2-2I+I=0$ e quindi è nella classe di matrici $(*)$, poi è già diagonale e non devo dimostrare nulla.

Se A è diag., allora $\exists M$ tale che $M^{-1}AM=D$

$$A=\frac{A^2+I}{2} \quad \text{dalle ipotesi } A^2-2A+I=0$$

$$M^{-1}AM = M^{-1}\left(\frac{A^2+I}{2}\right)M = \frac{M^{-1}A^2M+I}{2}$$

$$M^{-1}A^2M = 2D-I \qquad M^{-1}AM=D \text{ per ipotesi}$$

$$(M^{-1}AMM^{-1}AM) = D' \qquad D^2=2D-I$$

$$(M^{-1}AM)^2 = 2D-I$$

$$D=\begin{pmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn}\end{pmatrix} \qquad D^2=\begin{pmatrix} d_{11}^2 & & 0 \\ & \ddots & \\ 0 & & d_{nn}^2\end{pmatrix}$$

$$\forall i=1\ldots n \qquad d_{ii}^2 = 2d_{ii}-1$$

$$d_{ii}^2-2d_{ii}+1=0 \qquad (d_{ii}-1)^2=0$$

$$d_{ii}=1 \qquad D=I \rightarrow A=I$$